Controllability and Reachability of 2-D Positive Systems: A Graph Theoretic Approach

Ettore Fornasini and Maria Elena Valcher, *Senior Member, IEEE*

***Abstract*—When dealing with two-dimensional (2-D) discrete state-space models, controllability properties are introduced in two different forms: a local form, which refers to single local states, and a global form, which instead pertains the infinite set of local states lying on a separation set. In this paper, these concepts are investigated in the context of 2-D positive systems by means of a graph theoretic approach. For all these properties, necessary and sufficient conditions, which refer to the structure of the digraph, are provided. While the global reachability index is bounded by the system dimension n, the local reachability index may far exceed the system dimension. Upper bounds on the local reachability index for some special classes of positive systems are finally derived.**

***Index Terms*—Controllability, finite memory systems, influence digraph, reachability, strong connectedness, two-dimensional (2-D) positive systems, zero controllability.**

I. Introduction

TWO-DIMENSIONAL (2-D) positive system theory is concerned with 2-D state-space models whose input, state, and output variables take positive (or at least nonnegative) values. Research interests in this topic have been stimulated by a series of contributions dealing with river pollution modeling [5], modeling of a single-carriageway traffic flow [11], gas absorption, and water stream heating [19], diffusion of a tracer into a blood vessel [22], etc. These contributions share two common features. On the one hand, all "internal" variables are intrinsically nonnegative, as they represent concentrations, pressures, numbers of vehicles, etc., on the other hand, the dynamics is well described by a (quarter plane causal) 2-D state-space model, as the system variables depend on a time and a space coordinate and obey a quarter plane causality law.

The results on 2-D positive system theory, aiming at providing a theoretical framework for the aforementioned models, have grown consistently in the last decade. The first contributions were oriented to extend positive matrix theory to matrix pairs, thus leading to a fairly complete analysis of the free state evolution [9], [10], [12] of 2-D positive systems and to a characterization of their asymptotic stability [21]. More recently, research in 2-D positive systems has concentrated on the analysis of their structural properties, and some results about reachability and controllability have been presented in [13]–[17].

When dealing with 2-D systems, the concepts of reachability, controllability and zero controllability are naturally introduced in two different forms: a weak (local) form, which refers to single "local states", and a strong (global) form, which pertains the infinite set of local states lying on some "separation set" [2], [6]. In this paper, these concepts are introduced and investigated in the context of 2-D positive systems described by the following first-order state-updating equation [6]:

$$\begin{aligned}\mathbf{x}(h+1,k+1) = A_1\mathbf{x}(h,k+1) + A_2\mathbf{x}(h+1,k)\\ +B_1\mathbf{u}(h,k+1) + B_2\mathbf{u}(h+1,k) \quad (1)\end{aligned}$$

where the n-dimensional **local states** $\mathbf{x}(\cdot,\cdot)$ and the m-dimensional inputs $\mathbf{u}(\cdot,\cdot)$ take nonnegative values, A_1 and A_2 are nonnegative $n \times n$ matrices, B_1 and B_2 are nonnegative $n \times m$ matrices, and the initial conditions are assigned by specifying the (nonnegative) values of the state vectors on the **separation set** $\mathcal{C}_0 := \{(h,k) : h,k \in \mathbb{Z}, h+k=0\}$, namely by assigning all local states of the initial **global state** $\mathcal{X}_0 := \{\mathbf{x}(h,k) : (h,k) \in \mathcal{C}_0\}$.

As in the one-dimensional (1-D) case, the positivity of the input sequence represents a tight constraint, as it may prevent, for instance, local/global reachability of nonnegative states that yet could be reached by resorting to unconstrained input sequences. Also, under the positivity assumption, structural properties exhibit a combinatorial nature, which motivates a graph theoretic approach to their analysis. Indeed, to every 2-D positive state-space model of dimension n with m inputs one can associate a 2-D influence digraph [10], [12], [13] with n vertices, m sources and two types of arcs interconnecting the sources and the vertices, and every structural property admits both algebraic and graph-theoretic characterizations.

The paper is organized as follows. Section II introduces some notations and provides both local and global definitions. In Section III, local and global zero controllability are addressed. Both properties turn out to be equivalent to finite memory, a property which has been investigated in detail in [8] and [9]. Local and global reachability, as well as the corresponding indices, $I_{\rm LR}$ and $I_{\rm GR}$, are fully characterized in Sections IV and V, where it is shown that the global reachability index is bounded by the system dimension n, while the local reachability index may far exceed the system dimension and even reach $(n+1)^2/4$. Local and global reachability criteria are applied, in Section VI, to a 2-D positive system describing the self-purification process of a polluted river. Though a general upperbound on the local reachability index seems a nontrivial goal to pursue, upperbounds on $I_{\rm LR}$ are presented in Section VII, for special classes of 2-D systems with scalar inputs.

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The authors are with the Faculty of Engineering, University of Padova, 35131 Padova, Italy.

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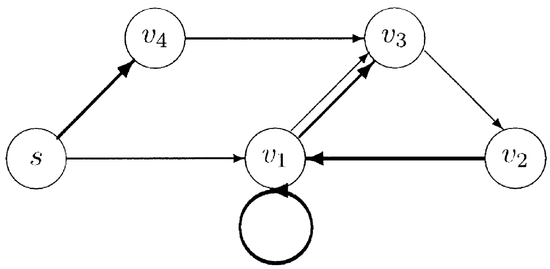

Fig. 1. 2-D influence digraph corresponding to (2).

II. Notations and Preliminary Definitions

Before proceeding, it is convenient to introduce some basic definitions and preliminary concepts that will be used in the paper. The Hurwitz products of two $n \times n$ matrices A_1 and A_2 are inductively defined [6] as

$$\begin{aligned}
A_1{}^{i}\sqcup\!\sqcup^{j} A_2 &= 0, \qquad \text{when either } i \text{ or } j \text{ is negative}\\
A_1{}^{i}\sqcup\!\sqcup^{0} A_2 &= A_1^{i}, \qquad \text{if } i \geq 0\\
A_1{}^{0}\sqcup\!\sqcup^{j} A_2 &= A_2^{j}, \qquad \text{if } j \geq 0\\
A_1{}^{i}\sqcup\!\sqcup^{j} A_2 &= A_1\left(A_1{}^{i-1}\sqcup\!\sqcup^{j} A_2\right)\\
&\quad + A_2\left(A_1{}^{i}\sqcup\!\sqcup^{j-1} A_2\right), \qquad \text{if } i, j > 0.
\end{aligned}$$

Notice that $\sum_{i+j=\ell} A_1{}^{i}\sqcup\!\sqcup^{j} A_2 = (A_1 + A_2)^{\ell}$.

A **2-D influence digraph** $\mathcal{D}^{(2)}$ is a directed graph which exhibits two types of arcs and input flows [10], [12], [13]. In detail, it is a sextuple $(S, V, \mathcal{A}_1, \mathcal{A}_2, \mathcal{B}_1, \mathcal{B}_2)$, where $S = \{s_1, s_2, \ldots, s_m\}$ is the set of **sources**, $V = \{v_1, v_2, \ldots, v_n\}$ is the set of **vertices**, $\mathcal{A}_1$ and $\mathcal{A}_2$ are subsets of $V \times V$ whose elements are called $\mathcal{A}_1$**-arcs** and $\mathcal{A}_2$**-arcs**, respectively, while $\mathcal{B}_1$ and $\mathcal{B}_2$ are subsets of $S \times V$ whose elements are called $\mathcal{B}_1$**-arcs** and $\mathcal{B}_2$**-arcs**, respectively.

To every 2-D positive system (1), we associate a 2-D influence digraph $\mathcal{D}^{(2)}(A_1, A_2, B_1, B_2)$ with n vertices, $v_1, v_2, \ldots, v_n$ and m sources $s_1, s_2, \ldots, s_m$. There is an $\mathcal{A}_1$ arc (an $\mathcal{A}_2$ arc) from v_j to v_i if and only if the (i, j)th entry of A_1 (of A_2) is nonzero. There is a $\mathcal{B}_1$ arc (a $\mathcal{B}_2$ arc) from s_j to v_i if and only if the (i, j)th entry of B_1 (of B_2) is nonzero.

Example 1: The positive system with a single input described by the matrices

$$(A_1, A_2, B_1, B_2) = \left(\begin{bmatrix} 1 & 5 & 0 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 4 & 0\\ 2 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0\\ 0\\ 0\\ 1 \end{bmatrix}, \begin{bmatrix} 2\\ 0\\ 0\\ 0 \end{bmatrix}\right) \tag{2}$$

corresponds to the 2-D digraph, with 4 vertices and a single source, of Fig. 1. We have represented $\mathcal{A}_1$ arcs and $\mathcal{B}_1$ arcs by means of thick lines, while $\mathcal{A}_2$ arcs and $\mathcal{B}_2$ arcs by means of thin lines. This will be a steady assumption throughout the paper.

A **path** p in $\mathcal{D}^{(2)}(A_1, A_2, B_1, B_2)$ is a sequence of adjacent arcs and, in particular, an s_j**-path** is a path which originates from the source s_j. A path p is specified by assigning its vertices and the type of arcs they are connected by.

If we denote by $|p|_1$ the number of $\mathcal{A}_1$ arcs and $\mathcal{B}_1$ arcs and by $|p|_2$ the number of $\mathcal{A}_2$ arcs and $\mathcal{B}_2$ arcs occurring in p, then $[|p|_1\ |p|_2]$ is the **composition** of p and $|p| = |p|_1 + |p|_2$ its **length**. A path whose extreme vertices coincide is a **cycle**. In particular, if each vertex appears exactly once as the first vertex of an arc, the cycle is a **circuit**.

An n-dimensional vector $\mathbf{v}$ is said to be an i**th monomial vector** if it can be expressed as $\alpha \mathbf{e}_i$, $\mathbf{e}_i$ being the ith vector of the standard basis in $\mathbb{R}^n$ and α is some positive coefficient. A **monomial matrix** is a nonsingular (square) matrix whose columns are monomial vectors. Given a $p \times m$ matrix M, by $\mathrm{Cone} M$ we mean the set of nonnegative combinations of the columns of M, i.e., the (polyhedral) cone generated in $\mathbb{R}^p$ by the columns of M. $\mathbf{1}_n$ is the vector of size n with all entries equal to 1, while the symbol $*$ represents the Hadamard product (entry by entry) of two matrices.

As previously recalled, two distinct definitions of reachability can be considered [6] for state-space models (1): local and global reachability. Local reachability refers to the possibility of "reaching" an arbitrary local state $\mathbf{x}^* \in \mathbb{R}^n$ in some point $(h, k) \in \mathbb{Z} \times \mathbb{Z}$, starting from zero initial conditions, while global reachability amounts to the possibility (starting, again, from zero initial conditions) of obtaining an arbitrary sequence of local states $\mathbf{x}(h, k)$ on some separation set

$$\mathcal{C}_t := \{(h, k) : h, k \in \mathbb{Z}, h + k = t\}.$$

Of course, all input sequences involved have supports included in the half-plane $\{(h, k) \in \mathbb{Z} \times \mathbb{Z} : h + k \geq 0\}$. Similarly, controllability and zero controllability properties can be given in a local and in a global form, depending on whether one considers a single local state on the (final) separation set $\mathcal{C}_t$ or the entire **global state**

$$\mathcal{X}_t := \{\mathbf{x}(h, k) : (h, k) \in \mathcal{C}_t\}.$$

We first introduce the local versions of the aforementioned structural properties under the positivity constraint.

Definition 2.1: A 2-D state-space model (1) may exhibit the following structural properties.

- It is **(positively) locally reachable** [6] if, upon assuming $\mathcal{X}_0 = 0$, for every $\mathbf{x}^* \in \mathbb{R}_+^n$ there exist $(h, k) \in \mathbb{Z} \times \mathbb{Z}$, $h + k > 0$, and a nonnegative input sequence $\mathbf{u}(\cdot, \cdot)$ such that $\mathbf{x}(h, k) = \mathbf{x}^*$. When so, we will say that $\mathbf{x}^*$ is reachable in $h + k$ steps and the smallest number of steps which allows to reach every nonnegative local state represents the local reachability index I_{LR} of the 2-D positive system.
- It is **(positively) locally controllable** if, corresponding to any nonnegative $\mathcal{X}_0$ and any $\mathbf{x}^* \in \mathbb{R}_+^n$, there exist $(h, k) \in \mathbb{Z} \times \mathbb{Z}$, $h + k > 0$, and a nonnegative input sequence $\mathbf{u}(\cdot, \cdot)$ such that $\mathbf{x}(h, k) = \mathbf{x}^*$.
- It is **(positively) locally zero controllable** if, corresponding to any nonnegative $\mathcal{X}_0$, there exist $(h, k) \in \mathbb{Z} \times \mathbb{Z}$, $h + k > 0$, and a nonnegative input sequence $\mathbf{u}(\cdot, \cdot)$ such that $\mathbf{x}(h, k) = 0$.

In the following, the specification "positively" will be omitted. The global versions of the previous properties are introduced in Definition 2.2, below.

Definition 2.2: A 2-D state-space model (1) may exhibit the following structural properties.

- It is **globally reachable** [6], [7] if, upon assuming $\mathcal{X}_0 = 0$, for every global state $\mathcal{X}^*$ with entries in $\mathbb{R}_+^n$, there exist $N \in \mathbb{Z}_+$ and a nonnegative input sequence $\mathbf{u}(\cdot,\cdot)$ such that the global state $\mathcal{X}_N := \{\mathbf{x}(h,k) : h,k \in \mathbb{Z}, h+k = N\}$ coincides with $\mathcal{X}^*$. When so, we will say that $\mathcal{X}^*$ is reachable in N steps. The smallest number of steps which allows to reach every nonnegative global state represents the **global reachability index** I_{GR} of the system.
- It is **globally controllable** if, corresponding to any nonnegative initial global state $\mathcal{X}_0$ and any nonnegative $\mathcal{X}^*$, there exist $N \in \mathbb{Z}_+$ and a nonnegative input sequence $\mathbf{u}(\cdot,\cdot)$ such that the global state $\mathcal{X}_N$ coincides with $\mathcal{X}^*$.
- It is **globally zero controllable** if, corresponding to any nonnegative initial global state $\mathcal{X}_0$, there exist $N \in \mathbb{Z}_+$ and a nonnegative input sequence $\mathbf{u}(\cdot,\cdot)$ such that the global state $\mathcal{X}_N$ is identically zero.

Clearly, each global property ensures the corresponding local one, and a 2-D positive system is locally (globally) controllable if and only if it is both locally (globally) reachable and locally (globally) zero controllable. These results are consistent with the analogous ones for standard 2-D systems.

"Global" properties, as defined in this paper, refer to the straight-line structure of the separation set $\mathcal{C}_0$ which constitutes the support of the initial conditions. Separation sets with different shapes have been also considered in the literature, mainly when dealing with Roesser models or the so called "general" 2-D systems [17]. While local definitions are essentially the same, independently of the shapes of the separation sets, global reachability and controllability definitions for Roesser and "general" 2-D positive systems, by naturally taking into account the different supports of the global states, significantly differ from those investigated in this paper.

III. Zero Controllability and Finite Memory

As a first step, we aim at showing that, when dealing with 2-D positive systems, local zero controllability and global zero controllability are equivalent properties and they both coincide with the finite memory property [2], [9].

A standard (i.e., not necessarily positive) 2-D system is said to be **finite memory** if for every initial global state $\mathcal{X}_0$ there exists $N \in \mathbb{Z}_+$ such that the corresponding free state evolution goes to zero within N separation sets, namely $\mathcal{X}_N = 0$. The finite memory definition for 2-D positive systems is obtained by simply introducing the positivity constraint on the initial global state $\mathcal{X}_0$. Several characterizations of finite memory positive systems have been provided in [9]. In particular, the finite memory property for 2-D positive systems corresponds to the lack of cycles in the associated 2-D digraph.

It is immediately apparent that, when dealing with positive systems, both local and global zero controllability are properties which just pertain the free state evolution, as nonnegative inputs could not make the task of obtaining a zero local or global state easier! Based on this simple remark, which holds true also for 1-D positive systems, the proof of the following proposition becomes almost straightforward.

Proposition 3.1: Given a 2-D positive system (1), of dimension n, the following facts are equivalent.

i) The system is locally zero controllable.
ii) The system is finite memory.
iii) The system is globally zero controllable.

Proof:

i) $\Rightarrow$ ii) Suppose that the system is locally zero controllable and choose as $\mathcal{X}_0$ the positive global state whose local states $\mathbf{x}(i,-i)$, $i \in \mathbb{Z}$, are all equal to the vector $\mathbf{1}_n$. For every $(h,k) \in \mathbb{Z} \times \mathbb{Z}$, with $h+k > 0$, we have $\mathbf{x}(h,k) = (A_1 + A_2)^{h+k}\mathbf{1}_n$. Since there exists (h,k) such that $\mathbf{x}(h,k) = 0$, we have also $(A_1 + A_2)^{h+k} = 0$, which ensures [9] the finite memory property of the 2-D system described by the positive matrix pair (A_1, A_2).

ii) $\Rightarrow$ iii) For every nonnegative $\mathcal{X}_0$, just leave the system evolve freely.

iii) $\Rightarrow$ i) Obvious. ■

At this point, it is clear that a 2-D positive system is locally (globally) controllable if and only if it is both finite memory and locally (globally) reachable. Since finite memory property is easy to check, by either algebraic means or graph inspection, our interest will focus on local and global reachability properties. Characterizations of such properties will immediately lead to characterizations of local and global controllability.

Remark: Controllability (to zero) of positive 2-D systems has been defined and investigated, for various classes of 2-D positive state-space models, in Chapter 6 of [17]. Controllability to zero of 2-D positive systems in "general" form is strictly related to local zero controllability discussed in this section, the main difference stemming from the slightly different state-updating equation the two models adopt. Theorem 6.9 in [17] shows that also for 2-D positive systems in "general" form controllability to zero is equivalent to the finite memory property. Proposition 3.1 in this paper shows, in addition, that when dealing with 2-D positive systems (1) local zero controllability is strong enough to ensure the corresponding global property. This result could be easily extended to the class of 2-D positive systems in "general" form.

IV. Local Reachability

When dealing with standard 2-D systems, local reachability is easily tested by evaluating the column span of the **reachability matrix in** k **steps** [6], i.e.,

$$\begin{aligned}\mathcal{R}_k &= \Big[B_1\ B_2\ A_1B_1\ A_1B_2+A_2B_1\ A_2B_2\ A_1^2B_1 \\ &\qquad \left(A_1{}^1{\sqcup\!\sqcup}{}^1A_2\right)B_1 + A_1^2B_2 \ldots A_2^{k-1}B_2\Big] \\ &= \left[(A_1{}^{i-1}{\sqcup\!\sqcup}{}^{j}A_2)B_1 + (A_1{}^{i}{\sqcup\!\sqcup}{}^{j-1}A_2)B_2\right]_{i,j\geq 0,\ 0<i+j\leq k}\end{aligned}$$

as k varies over the set $\mathbb{N}$ of positive integers. Indeed, reachable states in k steps, i.e., local states that can be reached in any assigned position of the separation set $\mathcal{C}_k$, starting from $\mathcal{X}_0 = 0$, constitute a linear subspace $X_k \subseteq \mathbb{R}^n$, spanned by the columns of $\mathcal{R}_k$. Clearly, the ascending chain $X_1 \subseteq X_2 \subseteq X_3 \subseteq \cdots$

eventually reaches stationarity and this necessarily happens, by the 2-D Cayley–Hamilton theorem [8], in no more than n steps. As a consequence, if the system is locally reachable, the point (h,k) where $\mathbf{x}(h,k)$ can reach the desired value $\mathbf{x}^*$ (see Definition 2.1) can always be chosen on the separation set $\mathcal{C}_n$.

Once we constrain the input sequence to be nonnegative, the reachability subspaces X_k, $k \in \mathbb{N}$, are replaced by the **reachability cones** X_k^+, $k \in \mathbb{N}$. In fact, the set X_k^+ of all local states that can be reached in any assigned position of the separation set $\mathcal{C}_k$, by means of nonnegative inputs and starting from initial zero conditions ($\mathcal{X}_0 = 0$), obviously coincides with the set of all nonnegative combinations of the columns of $\mathcal{R}_k$, namely $X_k^+ = \mathrm{Cone}\mathcal{R}_k$. Consequently, a system is locally reachable if and only if there exists $k \in \mathbb{N}$ such that $\mathrm{Cone}\mathcal{R}_k = \mathbb{R}_+^n$. When so, the smallest such k represents the reachability index I_{LR} of the (locally reachable) 2-D positive system. It is worth remarking, however, that as in the case of 1-D positive systems [18], the chain of reachability cones does not necessarily reach stationarity and, indeed, certain positive states can be reached only asymptotically.

Positive local reachability is trivially equivalent to the possibility of reaching (starting from zero initial conditions) every vector of the standard basis in $\mathbb{R}^n$ by means of nonnegative inputs, which in turn amounts to saying that there exists $k \in \mathbb{N}$ such that the reachability matrix in k steps, $\mathcal{R}_k$, includes an $n \times n$ monomial submatrix [1], [4] (similar results hold for Roesser and "general" 2-D positive models [17]). Keeping in mind the structure of the columns of $\mathcal{R}_k$, the previous condition can be equivalently stated by saying that a 2-D system is locally reachable if and only if there exist n pairs $(h_i,k_i) \in \mathbb{Z}_+ \times \mathbb{Z}_+$, $i = 1,2,\ldots,n$, and n indices $j = j(i) \in \{1,2,\ldots,m\}$ such that $(A_1^{h_i-1} \sqcup\!\sqcup^{k_i} A_2)B_1\mathbf{e}_j + (A_1^{h_i} \sqcup\!\sqcup^{k_i-1} A_2)B_2\mathbf{e}_j$ is an ith monomial vector. If so

$$I_{\mathrm{LR}} = \max_i \min_{h_i,k_i} \Big\{ h_i + k_i : \exists\, j = j(i) \text{ s.t. } \left(A_1^{h_i-1} \sqcup\!\sqcup^{k_i} A_2\right) B_1\mathbf{e}_j + \left(A_1^{h_i} \sqcup\!\sqcup^{k_i-1} A_2\right) B_2\mathbf{e}_j \text{ is an } i\text{th monomial vector} \Big\}$$

and, for systems with scalar inputs

$$I_{\mathrm{LR}} = \max_i \min_{h_i,k_i} \{ h_i + k_i : \left(A_1^{h_i-1} \sqcup\!\sqcup^{k_i} A_2\right) B_1 + \left(A_1^{h_i} \sqcup\!\sqcup^{k_i-1} A_2\right) B_2 \text{ is an } i\text{th monomial vector}\}.$$

Notice, finally, that all pairs (h_i,k_i) are necessarily distinct, but the case may occur that $h_i + k_i = h_j + k_j$ for $i \neq j$.

As for 1-D positive systems, local reachability of 2-D positive systems is a structural property, by this meaning that it only depends on the nonzero patterns of the system matrices and not on the specific values of their nonzero elements. However, differently from the 1-D case and the standard 2-D case, the reachability index I_{LR} of a (locally reachable) 2-D positive system is not bounded by the system dimension.

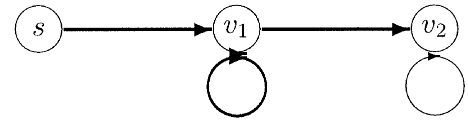

Fig. 2. 2-D influence digraph corresponding to Example 2.

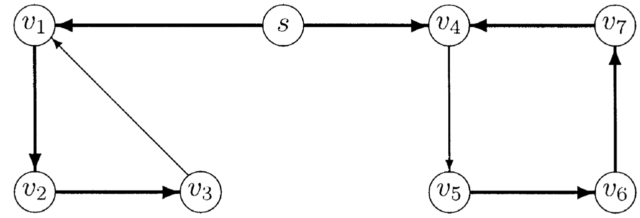

Fig. 3. 2-D influence digraph corresponding to Example 3.

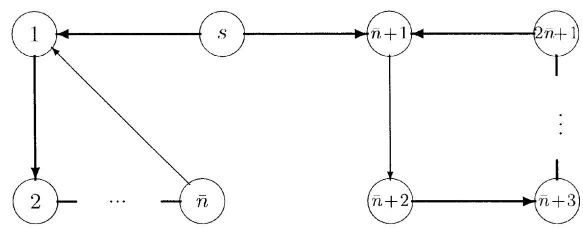

Fig. 4. 2-D influence digraph generalizing Example 3.

Example 2: The positive system described by the matrices

$$(A_1,A_2,B_1,B_2) = \left(\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right)$$

corresponds to the 2-D digraph of Fig. 2.

It is easy to verify that the system is locally reachable and $I_{\mathrm{LR}} = 3 > 2 = n$, as

$$\mathcal{R}_1 = [B_1 \quad B_2] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$
$$\mathcal{R}_2 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$
$$\mathcal{R}_3 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

Example 3: In the 2-D positive system (A_1,A_2,B_1,B_2)

$$A_1 = [\mathbf{e}_2 \quad \mathbf{e}_3 \quad 0 \quad 0 \quad \mathbf{e}_6 \quad \mathbf{e}_7 \quad \mathbf{e}_4]$$
$$A_2 = [0 \quad 0 \quad \mathbf{e}_1 \quad \mathbf{e}_5 \quad 0 \quad 0 \quad 0]$$
$$B_1 = [\mathbf{e}_1 + \mathbf{e}_4]$$
$$B_2 = [0]$$

which corresponds to the 2-D digraph of Fig. 3, the local reachability index is 16 while the system dimension is $n = 7$. The above example can be generalized to 2-D influence digraphs consisting of two loops including $\bar{n}$ and $\bar{n}+1$ vertices, respectively, connected by arcs of type 1 and 2 as indicated in Fig. 4. The reachability index turns out to be of the same order as $\bar{n} \cdot (\bar{n}+1) + (\bar{n}+1)$, namely of the same order as $\left(\frac{n-1}{2}\right) \cdot \left(\frac{n+1}{2}\right) + \left(\frac{n+1}{2}\right) = \left(\frac{n+1}{2}\right)^2$, since $n = \bar{n} + (\bar{n}+1)$.

A necessary condition for local reachability is the following one, which extends a similar result obtained for 1-D positive systems [20].

Proposition 4.1: If the positive system (1) is locally reachable, then $[A_1 \quad A_2 \quad B_1 \quad B_2]$ includes an $n \times n$ monomial submatrix.

Proof: If the system is locally reachable, then there exist n nonnegative pairs $(h_i, k_i) \in \mathbb{Z}_+ \times \mathbb{Z}_+$, $i = 1, 2, \ldots, n$, and n indices $j = j(i) \in \{1, 2, \ldots, m\}$ such that

$$\left(A_1{}^{h_i-1} \sqcup\!\sqcup^{k_i} A_2\right) B_1\mathbf{e}_j + \left(A_1{}^{h_i} \sqcup\!\sqcup^{k_i-1} A_2\right) B_2\mathbf{e}_j$$

is an ith monomial vector. If $h_i + k_i = 1$, the ith monomial vector is a column either of B_1 or of B_2. If $h_i + k_i > 1$, the ith monomial vector is a column either of A_1 or of A_2 (possibly both). ■

As for 1-D positive systems, local reachability admits an interesting and useful characterization in terms of the 2-D influence digraph associated with the system. Indeed, saying that $(A_1{}^{h_i-1} \sqcup\!\sqcup^{k_i} A_2)B_1\mathbf{e}_j + (A_1{}^{h_i} \sqcup\!\sqcup^{k_i-1} A_2)B_2\mathbf{e}_j$ is an ith monomial vector just means that the set of s_j-paths p of composition $[|p|_1 \ |p|_2] = [h_i \ k_i]$ is not empty and each of them reaches the vertex v_i alone. If so, we will say that the vertex v_i is **deterministically reached** by all s_j-paths of composition $[h_i \ k_i]$.

As a consequence, the 2-D system (1) is locally reachable if and only if for every $i \in \{1, 2, \ldots, n\}$ there exists $j = j(i)$ such that the vertex v_i is deterministically reached by all s_j-paths of some composition $[h_i \ k_i]$. Moreover

$$\begin{aligned} I_{\mathrm{LR}} = \max_i \min_{h_i, k_i} \{ & h_i + k_i : \exists j = j(i) \text{ s.t. all } s_j - \text{paths} \\ & \text{of composition } [h_i \ k_i] \text{ deterministically reach } v_i\}. \end{aligned} \quad (3)$$

V. Global Reachability

When addressing global reachability, it suffices to focus on the reachability of global states consisting of all zero (local) states except for one of them, which coincides with $\mathbf{e}_i$, $i \in \{1, 2, \ldots, n\}$. Indeed, if the system is globally reachable then, in particular, all such global states must be reachable. On the other hand, if all such global states are reachable, each of them can be reached by means of a suitable finite support nonnegative input sequence. So, by superposing nonnegative combinations of such finite support input sequences, one can reach every nonnegative global state. Obviously, if N_i denotes the minimum number of steps required to reach any global state consisting of all zero local states except for one, which coincides with $\mathbf{e}_i$, then it is easily seen that the global reachability index, I_{GR}, coincides with $\max_i N_i$.

As for the local case, global reachability may be characterized in terms of the columns of the reachability matrix.

Proposition 5.1: A 2-D system (1) is globally reachable if and only if there exist n pairs $(h_i, k_i) \in \mathbb{Z}_+ \times \mathbb{Z}_+$, $i = 1, 2, \ldots, n$, and n indices $j = j(i) \in \{1, 2, \ldots, m\}$ such that

$$\left(A_1{}^{h_i-1} \sqcup\!\sqcup^{k_i} A_2\right) B_1\mathbf{e}_j + (A_1{}^{h_i} \sqcup\!\sqcup^{k_i-1} A_2)B_2\mathbf{e}_j \text{ is an } i\text{th monomial vector} \quad (4)$$

$$\left(A_1{}^{h-1} \sqcup\!\sqcup^{k} A_2\right) B_1\mathbf{e}_j + \left(A_1{}^{h} \sqcup\!\sqcup^{k-1} A_2\right) B_2\mathbf{e}_j = 0 \quad \forall\, (h, k) \neq (h_i, k_i) \text{ with } h + k = h_i + k_i. \quad (5)$$

Proof: Let $\mathcal{X}^{(i)}$ denote a global state consisting of all zero local states except for one of them, located e.g., in $(\tilde{h}_i, \tilde{k}_i)$, which coincides with $\mathbf{e}_i$, and suppose that $\mathcal{X}^{(i)}$ is globally reachable after $N_i := \tilde{h}_i + \tilde{k}_i$ steps. By the system nonnegativity, the support of any input sequence that allows reaching $\mathcal{X}^{(i)}$ can be restricted to the triangular region

$$\mathcal{T}_{(\tilde{h}_i, \tilde{k}_i)} := \{(h, k) \in \mathbb{Z} \times \mathbb{Z} : h + k \geq 0, h \leq \tilde{h}_i, k \leq \tilde{k}_i\}.$$

In particular, there exists at least one input sequence which is identically zero, except in some point $(\bar{h}_i, \bar{k}_i)$ of $\mathcal{T}_{(\tilde{h}_i, \tilde{k}_i)}$, where it coincides with some monomial (say jth) vector.

But then, it is immediately seen that the value of each local state $\mathbf{x}(h, k)$ generated by such input on the separation set $\mathcal{C}_{N_i}$ coincides with the (nonnegative multiple of the) jth column of the block matrix $(A_1{}^{(h-\bar{h}_i)-1} \sqcup\!\sqcup^{k-\bar{k}_i} A_2)B_1 + (A_1^{h-h_i} \sqcup\!\sqcup^{(k-\bar{k}_i)-1} A_2)B_2$. As a consequence, the global state $\mathcal{X}^{(i)}$ can be reached if and only if there exists an integer pair $(h_i, k_i) = (\tilde{h}_i - \bar{h}_i, \tilde{k}_i - \bar{k}_i)$ and some $j = j(i)$ such that (4) and (5) hold. Since this condition must be verified for every $i \in \{1, 2, \ldots, n\}$, the proposition is proved. ■

Not unexpectedly, global reachability is stronger than local reachability. This clearly arises by comparing the results of Section IV with Proposition 5.1, but it is also shown by means of the simple Example 2.

Proposition 5.1 can be interpreted in graph theoretic terms: the 2-D system (1) is globally reachable if and only if for every $i \in \{1, 2, \ldots, n\}$ there exists $j = j(i) \in \{1, 2, \ldots, m\}$ such that the vertex v_i is deterministically reached by all s_j-paths of a given composition $[h_i \ k_i]$, and no s_j-path exists, having the same length $h_i + k_i$ and different composition. Moreover, $I_{\mathrm{GR}} = \max_i \min_{h_i, k_i}\{h_i + k_i : \exists j = j(i)$ s.t. all s_j-paths of composition $[h_i \ k_i]$ deterministically reach v_i and there is no s_j-path of length $h_i + k_i$ and different composition $\}$.

The following lemma leads the way to further characterizations of global reachability.

Lemma 5.2: If the 2-D system (1) is globally reachable, then the 1-D positive system described by the pair $(A_1 + A_2, B_1 + B_2)$ is (positively) reachable.

Proof: From Proposition 5.1 it follows that if the 2-D system (1) is globally reachable there exist n pairs $(h_i, k_i) \in \mathbb{Z}_+ \times \mathbb{Z}_+$, $i = 1, 2, \ldots, n$, and n indices $j = j(i) \in \{1, 2, \ldots, m\}$ such that

$$\begin{aligned} \alpha\mathbf{e}_i = & \sum_{\substack{h,k \in \mathbb{Z} \\ h+k = h_i + k_i}} \left(A_1{}^{h-1} \sqcup\!\sqcup^{k} A_2\right) B_1\mathbf{e}_j \\ & + \left(A_1{}^{h} \sqcup\!\sqcup^{k-1} A_2\right) B_2\mathbf{e}_j \\ = & (A_1 + A_2)^{h_i + k_i - 1}(B_1 + B_2)\mathbf{e}_j \end{aligned}$$

for some $\alpha > 0$. This ensures [4] that $(A_1 + A_2, B_1 + B_2)$ is (positively) reachable. ■

For systems with scalar inputs, Lemma 5.2 leads to a "canonical" global reachability form.

Proposition 5.3: For a 2-D positive system (1) with scalar inputs the following facts are equivalent.

i) The system is globally reachable.

ii) There exist n pairs $(h_i, k_i) \in \mathbb{Z}_+ \times \mathbb{Z}_+$, $i = 1, 2, \ldots, n$, such that iia) $(A_1{}^{h_i-1}{}_{\sqcup\!\sqcup}{}^{k_i}A_2)B_1 + (A_1{}^{h_i}{}_{\sqcup\!\sqcup}{}^{k_i-1}A_2)B_2$ is an ith monomial vector;

iib) $(A_1{}^{h-1}{}_{\sqcup\!\sqcup}{}^{k}A_2)B_1 + (A_1{}^{h}{}_{\sqcup\!\sqcup}{}^{k-1}A_2)B_2 = 0$, $\forall\,(h,k) \neq (h_i,k_i)$ with $h+k = h_i+k_i$.

iii) There exists a permutation matrix P such that

$$P^T(A_1+A_2)P = \begin{bmatrix} \star & + & & & 0 \\ \star & 0 & + & & 0 \\ \star & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & + \\ \star & 0 & \dots & 0 & 0 \end{bmatrix}$$
$$P^T(B_1+B_2) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ + \end{bmatrix} \tag{6}$$

where $\star$ and $+$ represent a nonnegative and a positive entry, respectively, and

$$P^T(A_1 * A_2)P = \left[\begin{array}{c|c} \star & \\ \star & \\ \vdots & 0_{n\times(n-1)} \\ \star & \end{array}\right] \quad P^T(B_1 * B_2) = 0. \tag{7}$$

Proof: For the sake of simplicity, as positive (either 1-D or 2-D global) reachability does not depend on the values of the nonzero entries of all matrices involved, within the proof all nonzero entries will be assumed unitary.

1) i)$\Leftrightarrow$ ii) is obvious from Proposition 5.1.

2) ii) $\Rightarrow$ iii) If the system is gloibally reachable, by the first part of the proof, conditions iia)–iib) hold. Also, by Lemma 5.2, the pair (A_1+A_2, B_1+B_2) is (positively) reachable and hence [4], [18] there exists a permutation matrix P such that (6) holds. The generality of the proof will not be affected by assuming $P = I_n$. We first show that only one among B_1 and B_2 is nonzero. If not, by (6), both of them should coincide with $\mathbf{e}_n$, but this would ensure that for every $i \geq 1$ either

$$(A_1{}^{i-\ell-1}{}_{\sqcup\!\sqcup}{}^{\ell}A_2)B_1 + (A_1{}^{i-\ell}{}_{\sqcup\!\sqcup}{}^{\ell-1}A_2)B_2 = 0$$

for every ℓ or there would be two consecutive nonnegative values of ℓ such that

$$(A_1{}^{i-\ell-1}{}_{\sqcup\!\sqcup}{}^{\ell}A_2)B_1 + (A_1{}^{i-\ell}{}_{\sqcup\!\sqcup}{}^{\ell-1}A_2)B_2 \neq 0$$

thus contradicting iia)–iib). This proves the first identity in (7). Suppose, now, w.l.o.g., $B_1 = \mathbf{e}_n$ and $B_2 = 0$. By applying the same reasoning to

$$(A_1{}^{1}{}_{\sqcup\!\sqcup}{}^{0}A_2)B_1 + (A_1{}^{0}{}_{\sqcup\!\sqcup}{}^{1}A_2)B_1 = (A_1+A_2)B_1 = \mathbf{e}_{n-1}$$

and making use of conditions iia)–iib), it can be shown that only one among $(A_1{}^{1}{}_{\sqcup\!\sqcup}{}^{0}A_2)B_1$ and $(A_1{}^{0}{}_{\sqcup\!\sqcup}{}^{1}A_2)\,B_1$ coincides with $\mathbf{e}_{n-1}$, while the other is zero. This means that only one among $A_1\mathbf{e}_n$ and $A_2\mathbf{e}_n$ is $\mathbf{e}_{n-1}$, while the other is zero. By proceeding in this way, we show that only one among $A_1\mathbf{e}_i$ and $A_2\mathbf{e}_i$ is an $(i-1)$th monomial vector, $i = 2,\dots,n-1,n$, while the other is zero.

3) iii)$\Rightarrow$ ii) Conditions (6) and (7) easily imply that there exist n pairs $(h_i,k_i) \in \mathbb{Z}_+ \times \mathbb{Z}_+$, $i = 1,2,\dots,n$, such that iia) and iib) hold. ■

Remarks: As a consequence of the previous proposition, all pairs (h_i,k_i), that make iia)–iib) satisfied, sum up to n distinct integers h_i+k_i none of them exceeding n. This means that the set of all such h_i+k_i, $i = 1,2,\dots,n$, coincides with the set $\{1,2,\dots,n\}$ and hence, in particular, the global reachability index for 2-D (globally reachable) systems with scalar inputs coincides with n. This situation is quite different from the one arising when local reachability is concerned, since the local reachability index can far exceed the system dimension.

For systems with scalar inputs, Proposition 5.1 can be restated in terms of the reachability matrix in n steps. Indeed, $\mathcal{R}_n$ can be block-partitioned as $\mathcal{R}_n = [R_1 \mid R_2 \mid \dots \mid R_n]$, where R_ℓ represents the block matrix including all columns $(A_1{}^{i-1}{}_{\sqcup\!\sqcup}{}^{j}A_2)B_1 + (A_1{}^{i}{}_{\sqcup\!\sqcup}{}^{j-1}A_2)B_2$, with $i+j = \ell$. Equations (4) and (5) (and hence global reachability) hold if and only if, for every $i = 1,2,\dots,n$, there exists $\ell_i \in \{1,2,\dots,n\}$ such that R_{ℓ_i} consists of all zero columns except for one which is an ith monomial vector.

Example 1 (Continued): The 2-D positive system of Example 1 is locally reachable with local reachability index $I_{\mathrm{LR}} = 3$, however it is not globally reachable. It is easily seen, however, that if B_2 is replaced by the zero column vector, the system is just in the canonical form (6) and (7), for $P = I_4$, and hence is globally reachable (as well as locally reachable with $I_{\mathrm{LR}} = 4$).

When dealing with systems with several inputs, Lemma 5.2 leads to a characterization of global reachability similar to the one given in Proposition 5.3. This requires, however, to consider the canonical forms available for reachable 1-D positive systems with several inputs [3], [20]. As such forms are rather complicate, except when the 1-D system matrix (A_1+A_2, in this case) is devoid of zero columns, we restrict ourselves to this special case.

Proposition 5.4: For a 2-D positive system (1) with m inputs and A_1+A_2 devoid of zero columns, the following facts are equivalent.

i) The system is globally reachable.

ii) $\exists r \in \mathbb{N}$, $r \leq m$, and suitable permutation matrices, P and Q, such that the equation at the bottom of the next page is true, where

$$F_{ii} = \begin{bmatrix} \star & + & & & 0 \\ \star & 0 & + & & 0 \\ \star & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & + \\ \star & 0 & \dots & 0 & 0 \end{bmatrix} \in \mathbb{R}_+^{n_i\times n_i}$$
$$F_{ij} = \begin{bmatrix} \star & 0 & & & 0 \\ \star & 0 & & & 0 \\ \star & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \\ \star & 0 & \dots & 0 & 0 \end{bmatrix} \in \mathbb{R}_+^{n_i\times n_j}, \text{ for } i \neq j$$

where $\star$ and $+$ represent a nonnegative and a positive entry, respectively, and G_{rem} collects the “un-necessary" column of G. Moreover, $P^T(A_1 * A_2)P$ has all zero columns except, possibly, for those corresponding to the first columns of the blocks (namely the columns of indices $1, n_1+1, n_1+n_2+1, \ldots$) and $P^T(B_1 * B_2)\begin{bmatrix} I_r \\ 0 \end{bmatrix} = 0$.

Remarks: The proof of the previous proposition can be obtained by resorting to the canonical form given in [20] and to the same reasonings adopted within the proof of Proposition 5.3. Moreover, from the structure of the canonical form one can deduce that the global reachability index I_{GR} coincides with $\max\{n_1, n_2, \ldots, n_r\}$.

VI. Example: Pollutant Diffusion in A River

In modeling the self-purification process of a polluted river [5], we may introduce the following assumptions:

a) The variety of pollutants dissolved in the river can be reduced to a single class of oxidizable substances, whose concentration is measured by the amount of oxygen (BOD = biological oxygen demand) needed for their complete biochemical oxidation. The selfpurification process is essentially due to dissolved oxygen.

b) A (spatially) one-dimensional model is assumed for the river. The stream and the diffusion velocities are supposed constant and the diffusion wavefront progresses with a velocity which is smaller than the river velocity (actually, a half). The river is divided into elementary reaches of length L. The time step T and the length of the elementary reach L are connected through the stream velocity V by the equation $T = 2L/V$, so that the water element centered in ℓL at time tT will be centered in $(\ell+2)L$ at time $(t+1)T$.

We denote by $\beta(\ell, t)$ the concentration of BOD in the elementary reach centered in ℓL at time tT. The evolution of the BOD concentration is expressed on the basis of a discretized balance equation accounting for different contributions. In fact:

- *Diffusion* is modeled by assuming that the BOD content of the elementary water volume, centered in ℓL at time tT, undergoes in $[tT, (t+1)T)$ a variation proportional to the differences $\beta(\ell-1, t) - \beta(\ell, t)$ and $\beta(\ell+1, t) - \beta(\ell, t)$.
- *Self-purification*: in the time interval $[tT, (t+1)T)$ the BOD concentration in the ℓth river reach decreases by an amount $a\beta(\ell, t)$, $a > 0$.
- *BOD sources*, increasing the BOD concentration, determine a nonnegative exogenous input to the system, which is denoted by $u_\beta(\cdot, \cdot)$.

By making the above assumptions, we obtain the following 2-D recursive equation:

$$\begin{aligned}\beta(\ell+2, t+1) = (1-a)\beta(\ell, t) + d\,[\beta(\ell-1, t) - \beta(\ell, t)] \\ +d\ [\beta(\ell+1, t) - \beta(\ell, t)] + bu_\beta(\ell, t) \quad (8)\end{aligned}$$

where b and d are suitable positive coefficients. To get a 2-D state-space model (1), we introduce first a coordinate transformation in the discrete grid $(h, k) := (\ell - t, t)$, and rewrite (8) as

$$\begin{aligned}\beta(h+1, k+1) = (1-a)\beta(h, k) + d\,[\beta(h-1, k) - \beta(h, k)] \\ + d\,[\beta(h+1, k) - \beta(h, k)] + bu_\beta(h, k).\end{aligned}$$

Next, we assume the following local state vector $\mathbf{x}(h, k) = [\beta(h-1, k) \quad \beta(h, k) \quad \beta(h+1, k)]^T$ and obtain model (1) with $B_1 = 0$ and

$$A_1 := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$
$$A_2 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ d & 1-a-2d & d \end{bmatrix}$$
$$B_2 := \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix}.$$

If the elementary river reach and the time steps are sufficiently small, the positive coefficients a and d are much smaller than 1 and, consequently, the previous model is positive.

It is easily seen, by applying the previous criteria, that this system is locally but not globally reachable. This result is consistent with physical intuition as, indeed, the components of the local state vector $\mathbf{x}(h, k)$ represent the BOD concentrations at the same time instant in three consecutive elementary reaches, and hence they can be arbitrarily assigned by acting on the specific local BOD sources. On the contrary, due to the physical relationship between the values of $\mathbf{x}$ on the points belonging to the same separation set, it is not possible to reach, starting from zero initial conditions, all positive global states.

VII. Local Reachability/controllability Analysis: Some Special Cases

In this section, 2-D positive systems (1) with scalar inputs, having one of the two input-to-state matrices which is zero, are considered. This condition is, of course, restrictive. For certain classes of systems, however, e.g., globally reachable systems with scalar inputs, it becomes a necessary one. Upon setting $B_2 = 0$ and $B := B_1$, (1) becomes

$$\begin{aligned}\mathbf{x}(h+1, k+1) = A_1\mathbf{x}(h, k+1) + A_2\mathbf{x}(h+1, k) \\ +Bu(h, k+1) \quad (9)\end{aligned}$$

where A_1, A_2 are in $\mathbb{R}_+^{n\times n}$ and B is in $\mathbb{R}_+^n$. Our goal is that of providing some bounds on the local reachability index I_{LR} for special classes of systems of this type.

A. 2-D Influence Digraphs Devoid of Cycles

2-D positive systems (9) whose 2-D influence digraph is devoid of cycles are finite memory, or equivalently zero control-

$$[P^T(A_1+A_2)P \mid P^T(B_1+B_2)Q] = \left[\begin{array}{cccc|ccccc} F_{11} & F_{12} & \ldots & F_{1r} & \mathbf{e}_{n_1} & 0 & \ldots & 0 & \\ F_{21} & F_{22} & \ldots & F_{2r} & 0 & \mathbf{e}_{n_2} & \ldots & 0 & \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots & G_{rem} \\ F_{r1} & F_{r2} & \ldots & F_{rr} & 0 & 0 & \ldots & \mathbf{e}_{n_r} & \end{array}\right]$$

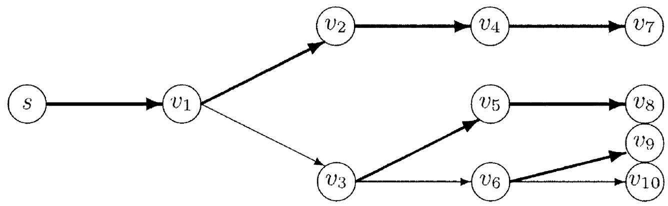

Fig. 5. 2-D influence digraph with minimum $I_{\rm LR}$.

lable. So, they are locally reachable if and only if they are locally controllable. The proof of the following result concerning $I_{\rm LR}$ strongly relies on the graph-theoretic interpretation of local reachability, described at the end of Section IV.

Proposition 7.1: If a 2-D positive system (9), with 2-D influence digraph $\mathcal{D}^{(2)}(A_1, A_2, B, 0)$ devoid of cycles, is locally reachable, then B is a monomial vector and

$$\min\left\{k \in \mathbb{N} : \sum_{i=1}^{k} i \geq n\right\} \leq I_{\rm LR} \leq n. \tag{10}$$

Proof: Since $A_1 + A_2$ is (positive and) nilpotent, it entails no loss of generality [9] assuming that $A_1 + A_2$ (and hence A_1 and A_2, separately) is in upper triangular form with zero diagonal. If the system is locally reachable, then, by Proposition 4.1, the columns of $[A_1\ A_2\ B\ 0]$ must include also an nth monomial vector, which is necessarily B.

Since $(A_1 + A_2)^n = 0$, all Hurwitz products $A_1{}^i{\sqcup\!\sqcup}^j A_2$ are zero whenever $i + j \geq n$ and the reachability cones satisfy $X_k^+ = X_n^+, \forall\, k \geq n$. If B is an nth monomial vector, the only outgoing arc from the source reaches vertex v_n. On the other hand, due to the fact that only two types of arcs are available, s-paths of length $\nu \geq 2$ with a common initial arc and distinct compositions may reach deterministically at most ν vertices. This means that the minimum number of steps required to deterministically reach each vertex is the smallest positive integer k such that $1+2+3+\cdots+k \geq n$. The upperbound is obvious. ■

The lowerbound provided in (10) is tight and examples of 2-D positive systems (9) of order n, with A_1+A_2 nilpotent and index $I_{\rm LR}$ that takes the minimum value given in (10), can be easily constructed for every $n \in \mathbb{N}$. Actually, if we assume $B = \mathbf{e}_1$, we construct a 2-D influence digraph with the structure of a binary tree, having at the νth level exactly ν vertices for all $\nu < k$. The outgoing arcs from each vertex have to be suitably chosen in order to guarantee that all s-paths of the same length (i.e., reaching vertices of the same level) have distinct compositions (see Fig. 5).

The value $I_{\rm LR} = n$ can be obtained by connecting the source and all vertices along a single path.

B. 2-D Influence Digraphs Consisting of Either One or Two (Disjoint) Circuits

Consider, first, single input systems (9) with 2-D influence digraphs consisting of a single circuit (by this meaning that all vertices $v_1, v_2, \ldots, v_n$ belong to a circuit and each pair of adjacent vertices is connected by a single arc). This assumption amounts to saying that $A_1 * A_2 = 0$ and $A_1 + A_2$ is a permutation matrix, which thus can be reduced to the following form

$$A_1 + A_2 = \begin{bmatrix} 0 & + & 0 & & 0 \\ 0 & 0 & + & & 0 \\ & & \ddots & \ddots & \\ & & & \ddots & + \\ + & 0 & & & 0 \end{bmatrix}. \tag{11}$$

When $\mathcal{D}^{(2)}(A_1, A_2, B, 0)$ consists of a single circuit, every monomial vector B makes $(A_1, A_2, B, 0)$ locally reachable with $I_{\rm LR} = n$. However, differently from the 1-D case, the local reachability of such a system does not require B to be a monomial vector [13]. When B is the sum of k distinct monomial vectors and the system is locally reachable, the local reachability index may take quite smaller values.

Proposition 7.2: Let $(A_1, A_2, B, 0)$ be a 2-D positive system such that $\mathcal{D}^{(2)}(A_1, A_2, B, 0)$ consists of a single circuit and assume w.l.o.g. that $A_1 + A_2$ is expressed as in (11) with $A_1 * A_2 = 0$. If the system is locally reachable and B has $k > 1$ nonzero entries, of indices $i_1 < i_2 < \cdots < i_k$, then $I_{\rm LR} \geq \max\{i_2 - i_1, i_3 - i_2, \ldots, i_k - i_{k-1}, n - i_k + i_1\} + 1$.

Proof: Suppose, for the sake of simplicity, that $\max\{i_2 - i_1, i_3 - i_2, \ldots, i_k - i_{k-1}, n - i_k + i_1\} = i_2 - i_1$. Keeping in mind the structure of $\mathcal{D}^{(2)}(A_1, A_2, B, 0)$ and, in particular, the fact that B is not monomial, it is clear that $\min\{h_1 + k_1 :$ all s-paths of composition $[h_1\ k_1]$ deterministically reach $v_{i_1}\}$ cannot be smaller than the length $i_2 - i_1 + 1$ of the s-path that, starting from the source, reaches v_{i_2} at the first step and later enters v_{i_1} without passing through the other vertices v_{i_ℓ} for $\ell \neq 1, 2$. The identity (3) completes the proof. ■

When $k = 2$, the previous bound becomes $I_{\rm LR} \geq \max\{(i_2 - i_1), n - (i_2 - i_1)\} + 1 \geq (n/2) + 1$.

Consider, now, the case of a system (9) with 2-D influence digraph consisting of two disjoint circuits. We have the following result.

Proposition 7.3: Let $(A_1, A_2, B, 0)$ be a 2-D positive system such that $\mathcal{D}^{(2)}(A_1, A_2, 0, 0)$ consists of two disjoint circuits γ and γ' of length n and n', respectively. If the system is locally reachable and B has only two nonzero entries, one for each cycle, then

$$I_{\rm LR} \leq \text{l.c.m}\{n, n'\} + \max\{n, n'\}. \tag{12}$$

Proof: Assume that the vertices in γ are (ordinately) $v_1, v_2, \ldots, v_n$ while the vertices in γ' are (ordinately) $v'_1, v'_2, \ldots, v'_{n'}$. Suppose, also, that the two nonzero entries in B correspond to the vertices v_1 and v'_1. The situation is depicted in Fig. 6.

In this situation, any vertex $v_j \in \gamma$ $(v'_j \in \gamma')$ is periodically visited after $j, j+n, j+2n, \ldots$ steps $(j, j+n', j+2n', \ldots$ steps, respectively). Moreover, for every $k \in \mathbb{N}$ there exist exactly two s-paths of length k in $\mathcal{D}^{(2)}(A_1, A_2, B, 0)$, and they reach vertex $v_{k \bmod n}$ in γ and vertex $v'_{k \bmod n'}$ in γ', respectively. Such vertices are reached deterministically if and only if the two s-paths have distinct compositions. Let N be the l.c.m. of n and n' and suppose, by contradiction, that none of the paths of length $j, j+n, \ldots, j+N$ deterministically reaches v_j. Since after $j+N$ steps we reach, at the same time and with the same composition, the two vertices v_j and v'_j just like after j steps, the subsequent

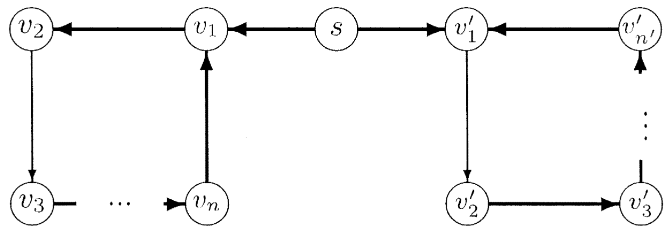

Fig. 6. 2-D influence digraph of Proposition 7.3.

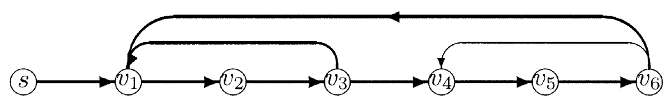

Fig. 7. 2-D influence digraph corresponding to Example 4.

evolution will periodically repeat the same nonzero pattern, thus preventing the possibility of deterministically reaching v_j. By applying the previous reasoning to all vertices of γ and γ' and, in particular, to v_n and $v'_{n'}$, (12) immediately follows. ■

Example 3 shows that the previous bound is tight.

C. Strongly Connected 2-D Influence Digraphs Including Only Two Circuits

A 2-D influence digraph $\mathcal{D}^{(2)}(A_1, A_2, B_1, B_2)$ is strongly connected if for any two vertices v_i and v_j there is a path (of arbitrary composition) connecting v_i to v_j. This ensures that every vertex that can be reached in t steps can also be reached in a larger number of steps, say $t+h$, and that no proper subgraph of $\mathcal{D}^{(2)}(A_1, A_2, B_1, B_2)$ acts as a "trap", by this meaning that once one gets in, there is no way out. This could lead to believe that, due to the increasing number of intersecting paths, at least in the special case when B is a monomial vector, each vertex is either deterministically reached within n steps or it will never be. As a matter of fact, even for a locally reachable 2-D system with a strongly connected 2-D influence digraph, the local reachability index may exceed the system dimension.

Example 4: Consider the 2-D positive system $(A_1, A_2, B, 0)$ with

$$A_1 = [\mathbf{e}_2 \quad \mathbf{e}_3 \quad \mathbf{e}_1 + \mathbf{e}_4 \quad \mathbf{e}_5 \quad \mathbf{e}_6 \quad 0]$$
$$A_2 = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \mathbf{e}_4]$$
$$B = [\mathbf{e}_1]$$

which corresponds to the strongly connected 2-D digraph of Fig. 7.

In this case, $I_{\rm LR} = 9$ while the system dimension is $n = 6$. The above structure can be generalized. If the 2-D influence digraph of a 2-D positive system has the previous structure, by this meaning that it consists of two small loops, each of them including $n/2$ vertices and connected by arcs of type 1 and 2, and a large loop passing through all vertices, as indicated in Fig. 8, then $I_{\rm LR}$ turns out to be $3n/2$.

To conclude, we aim at affording the special class of 2-D positive systems (9) with a strongly connected 2-D influence digraph that includes only two circuits, γ_1 and γ_2. Examples of systems of this type are given in Fig. 9.

The subsequent results depend on the following intuitive graph-theoretic lemma.

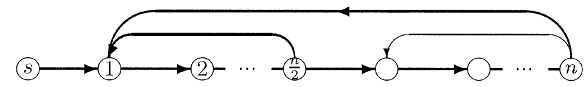

Fig. 8. 2-D influence digraph generalizing Example 4.

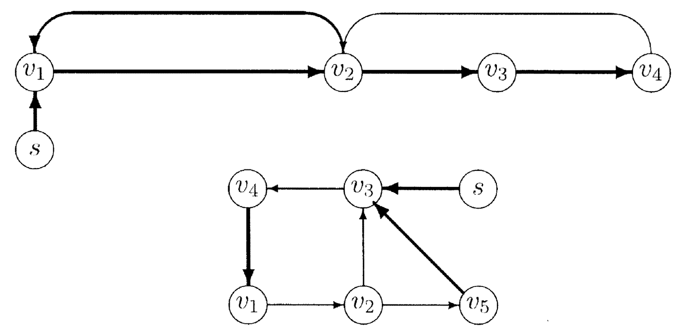

Fig. 9. 2-D influence digraphs strongly connected including 2 circuits.

Lemma 7.4: [13] If $\mathcal{D}^{(2)}(A_1, A_2, B_1, B_2)$ is a strongly connected 2-D influence digraph with n vertices and including only two circuits, say γ_1 and γ_2, then the following hold.

i) Every vertex belongs either to γ_1 or to γ_2 and at least one vertex belongs to both circuits.

ii) Each path p of length $|p| \geq |\gamma_1|$ includes at least one vertex $v_2 \in \gamma_2$ and, conversely, each path p of length $|p| \geq |\gamma_2|$ includes at least one vertex $v_1 \in \gamma_1$.

iii) $|\gamma_1| + |\gamma_2| \geq n + 1$.

Proposition 7.5: Let $(A_1, A_2, B, 0)$ be a 2-D positive system such that $\mathcal{D}^{(2)}(A_1, A_2, B, 0)$ is strongly connected and includes only two circuits γ_1 and γ_2. If $(A_1, A_2, B, 0)$ is locally reachable, then the local reachability index $I_{\rm LR}$ cannot exceed $|\gamma_1| + |\gamma_2|$.

Proof: Set $N := |\gamma_1| + |\gamma_2|$, and note that $N \geq n + 1$. Also, let V_B denote the set of vertices corresponding to the nonzero entries of B. Suppose that there exists a vertex v_r which is deterministically reached from the source s by all s-paths of composition $[h+1\ k]$, with $h + 1 + k \geq N + 1$, but cannot be reached deterministically in a smaller number of steps.

Consider a path p of composition $[h\ k]$ connecting some vertex of V_B to the vertex v_r. As $h + k \geq n + 1$, the path p includes a circuit, e.g., γ_1. Let p' be the path, ending in v_r, obtained by removing γ_1 from p. As v_r cannot be reached deterministically by the paths having the same composition as p', there must be a path q', of that same composition, from some vertex in V_B to some vertex $v_s \neq v_r$. Since $|q'| = (h+k) - |\gamma_1| \geq N - |\gamma_1| = |\gamma_2|$, by the previous lemma at least one vertex of q' belongs to γ_1. But then, a new path q can be obtained by adding γ_1 to q'. So, two paths, p and q, of composition $[h\ k]$ can be found, connecting V_B to the vertices v_r and v_s, respectively. Equivalently, there are two s-paths of composition $[h+1\ k]$ reaching vertices v_r and v_s. This contradicts the original assumption and hence $h + k$ must be smaller than N. ■

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Ettore Fornasini received the laurea degrees in electronic engineering and mathematics from the Universisty of Padova, Padova, Italy.

Since November 1980, he has been a Full Professor of Mathematical System Theory in the Faculty of Engineering, University of Padova, and teaches courses in the areas of Systems Theory, Automatic Control and Ecological Modeling. He has also been a Guest Researcher at the University of Florida, Gainesville, the Center for Mathematical Systems Theory, Massachusetts Institute of Technology, Cambridge, the Technical University, Delft, The Netherlands, at the Institut fuer Mathematik der Universitaet Innsbruck, Innsbruck, Austria , the University of Aveiro, Aveiro, Portugal. His research interests include modeling and realization of nonlinear systems, dissipative systems, the polynomial approach to linear systems, the structure, realization and control of multidimensional systems, modeling of two-dimensional (2-D) ecological systems, multidimensional Markov chains, one-dimensional (1-D) and 2-D convolutional coding theory, 1-D and 2-D positive systems.

He is an Editor of *Multidimensional Systems and Signal Processing* and a reviewer of several technical journals.

Maria Elena Valcher (M'99–SM'03) received the laurea degree in electronic engineering and the Ph.D. in system theory from the University of Padova, Padova, Italy, in 1991 and 1995, respectively.

She was Assistant Professor at the University of Padova from 1994 to 1998, and an Associate Professor of System Theory at the University of Lecce, Lecce, Italy from 1998 to 2001. From November 2001 to December 2004, she was an Associate Professor of Control Theory at the University of Padova, where she is currently a Full Professor. She is author/co-author of more than 40 international journal papers, ten book chapters and 50 contributions at international conferences. She has also co-authored a monography and two textbooks (in Italian). Her research interests include multidimensional systems theory, polynomial matrix theory, behavior theory, convolutional coding, fault detection and observer design, delay-differential systems and positive systems.

Dr. Valches was in the Editorial Board of the IEEE TRANSACTIONS ON AUTOMATIC CONTROL from 1999 to 2002. She is currently on the Editorial Boards of *Multidimensional Systems and Signal Processing* and *Systems and Control Letters*, and an elected Member of the Board of Governors of the IEEE Control Systems Society.