Quantum Communications: An introduction

DEI
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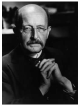
Purpose of this lecture

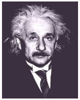
To introduce the fundamentals of Quantum Mechanic (QM), and related mathematical tools, with the objective of understanding elementary Quantum (optical) Communication Systems and prove their superiority with respect to classical (optical) Communication Systems.

Outline

1. Brief History
2. Postulates of QM and related mathematical tools
3. Quantum Communication Systems
4. Optimum Quantum Detection
5. Performance of elementary Communication Systems
6. Implementation of QTLC systems
7. Our research program and teaching experience

The beginning


Planck


Einstein

The maturity


Dirac


Von Neumann

Quantum communications


Helstrom


Glauber


Forney

Labs Researching on QC

- Massachusetts Institute of Technology (MIT)
- Jet Propulsion Laboratory (JPL)
- Tamagawa University, Machida, Tokyo, Japan
- Japan Posts and Telecommunications, Koganei, Tokyo, Japan
- IBM

Several other institutes are already working on QC and many others are going to.

Results mainly appear in the following journals:

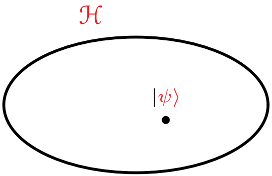
- IEEE Trans. on Information Theory
- IEEE Trans. on Communications
- J. Applied Physics A

The Four Postulates of Quantum Mechanics

- Postulate I: formulation of quantum states in a Hilbert space
  - Hilbert space $\mathcal{H}$ over the field of complex numbers $\mathbb{C}$
  - Dirac notation
- Postulate II: Quantum system evolution
  - Operators on $\mathcal{H}$: essentially unitary and Hermitian operators
- Postulate III: Quantum measurements (extraction of information from a Quantum system)
  - Positive definite Hermitian operators
  - Resolution of the identity
  - Projectors (idempotent Hermitian operators)
  - Eigendecomposition of Hermitian operators
- Postulate IV: Composite Quantum systems
  - Tensorial product of several Hilbert spaces and operators

Postulate I of Quantum Mechanics

Every closed physical system is represented by a state $|\psi\rangle$ (ket) in a Hilbert space $\mathcal{H}$ of suitable dimension *n*, $2 \leq n \leq \infty$, over the field of complex numbers $\mathbb{C}$



$\mathcal{H}$: Hilbert space over $\mathbb{C}$

$|\psi\rangle$: state of the system

For an elementary formulation of QM, the Hilbert space $\mathcal{H}$ can be regarded as an inner-product vector space where a point of $\mathcal{H}$ is represented by a (column) vector

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad x^* = [x_1^*, \ldots, x_n^*]$$

- The inner product (in matrix notation) is given by

$$(x, y) = y^* x \in \mathbb{C}$$

- The length of the vector is

$$\|x\| = (x, x)^{1/2} = (x^* x)^{1/2} = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}$$

DIRAC'S BRA-KET NOTATION

- The ket

$$|x\rangle \rightarrow \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

- The bra

$$\langle x| \rightarrow [\, x_1^* \; x_2^* \; \ldots \; x_n^* \,]$$

This notation is very useful in QM to represent states and operations in a very compact form.

INNER PRODUCT

- The inner product of the bra $\langle y| = [\, y_1^* \; y_2^* \; \ldots \; y_n^* \,]$ with the ket $|x\rangle$

$$\langle y|x\rangle = \sum_{i=1}^{n} x_i y_i^* \qquad , \qquad \langle x|y\rangle = \langle y|x\rangle^*$$

- The norm (length) $\| |x\rangle \|$ of $|x\rangle$ is

$$\| \, |x\rangle \, \|^2 = (\, \langle x|x\rangle \,)^{1/2} = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}$$

Remark: In QM all the states in the Hilbert space, $|x\rangle \in \mathcal{H}$, are normalized to unit length $\langle x|x\rangle = 1$.

OUTER PRODUCT AND TRACE

- The outer product of $|x\rangle$ with $\langle y|$

$$|x\rangle\langle y| = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \begin{bmatrix} y_1^* & y_2^* & \dots & y_n^* \end{bmatrix} = \begin{bmatrix} x_1 y_1^* & x_1 y_2^* & \dots & x_1 y_n^* \\ x_2 y_1^* & x_2 y_2^* & \dots & x_2 y_n^* \\ \vdots & \vdots & \ddots & \vdots \\ x_n y_1^* & x_n y_2^* & \dots & x_n y_n^* \end{bmatrix}$$

- The trace of a matrix A

$$\mathrm{Tr}\,(A) = \sum_{i=1}^{n} a_{ii}$$

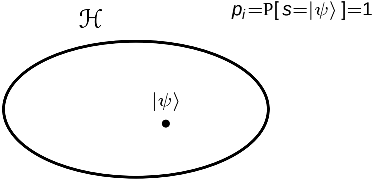
- Relation between inner product and outer product

$$\langle x|y\rangle = \sum_{i=1}^{n} x_i y_i^* = \mathrm{Tr}\,(|y\rangle\langle x|)$$

Pure and Mixed Quantum States

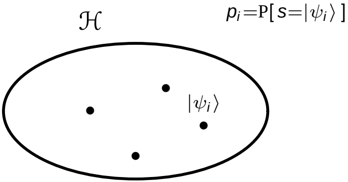
- A pure state is a normalized vector $|\psi\rangle$. It is introduced when the quantum state is exactly known to the observer.
- A mixed state is introduced when the observer has uncertainty on the states: the quantum state of the system is random and may be one of the states $|\Psi_i\rangle$ with probability p_i

pure state



$\boldsymbol{\rho} = |\psi\rangle\langle\psi|$

mixed state



$\boldsymbol{\rho} = \sum_i p_i \, |\psi_i\rangle\langle\psi_i|$

ρ is the density operator

DENSITY OPERATOR

- The density operator is

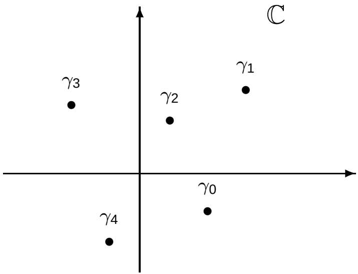
$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \qquad p_i > 0 \quad \text{and} \quad \sum_i p_i = 1$$

- Probabilistic meaning: random state assuming one among the possible states $|\psi_i\rangle$ with probability p_i.
- Density operator of a pure state: $\rho = |\psi\rangle\langle\psi|$
- In Quantum Communications
  a pure state to represent a system neglecting thermal noise (only shot noise)
  a mixed state to represent a system considering also thermal noise

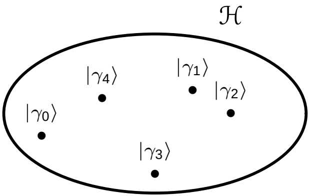
Example 1: the Qubit

- The qubit (Quantum Bit) is a pure state in a two-dimensional $n = 2$ Hilbert space.
- Given a base of two vectors $|0\rangle$ e $|1\rangle$, the general state has the form
$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \qquad , \qquad |\alpha|^2 + |\beta|^2 = 1$$
- Model for two–level physical systems: spin of a particle, polarization of a photon

Constellation of complex numbers



Constellation of Glauber states



Roy J. Glauber the Nobel Prize in Physics in 2005 for his work dated 1963

THE SPACE OF COHERENT STATES

The Glauber space is a Hilbert space with infinite dimensions. Given the base made by the number states $|n\rangle$, $n = 0, 1, \ldots$, and a complex number α, the expression of a coherent state is

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

This space represents the model for a coherent laser beam.

Probabilistic interpretation: the number of photons in the beam is a Poisson random variable with average number of photons $N = |\alpha|^2$.

The state of a closed system evolves according to the Schrödinger's equation

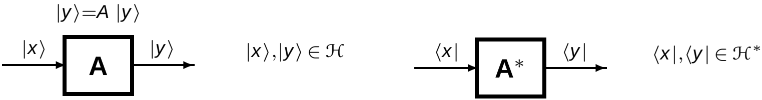
$$\frac{ih}{2\pi}\frac{d|\psi(t)\rangle}{dt} = H|\psi(t)\rangle$$

with h Planck constant and H linear Hermitian operator called the Hamiltonian.

Alternatively (and equivalently)

$$|\psi(t)\rangle = U(t, t_0)|\psi(t_0)\rangle$$

with $U(t, t_0)$ unitary operator.



Linear operator $\quad A : \mathcal{H} \to \mathcal{H}$ such that

$$A(\alpha|x_1\rangle + \beta|x_2\rangle) = \alpha A|x_1\rangle + \beta A|x_2\rangle$$

In QM only Hermitian and unitary operators are used:
Hermitian: $\quad A = A^*$ $\qquad$ (A^* adjoint of A)
Unitary: $\quad AA^* = I_{\mathcal{H}}$ $\qquad$ $I_{\mathcal{H}}$ identity operator on $\mathcal{H}$

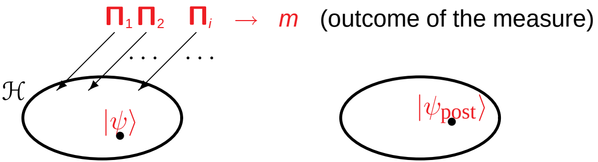
In practice, linear operators are treated as square complex matrices (possibly with infinite dimensions).

Given the system quantum state $|\psi\rangle$ and a POVM (Positive Operator Valued Measurement) system $\{\Pi_i, i \in \mathcal{M}\}$, the probability that the measure outcome m is equal to i is given by

$$\mathrm{P}\left[m = i \mid |\psi\rangle\right] = \langle\psi|\Pi_i|\psi\rangle \ .$$

If the outcome is $m = i$, the state collapses into the new state

$$|\psi_{\text{post}}\rangle = \frac{\Pi_i|\psi\rangle}{\sqrt{\langle\psi|\Pi_i|\psi\rangle}}$$



EXTENSION TO MIXED STATES

Given a quantum system in a mixed state with density operator ρ and a POVM system $\{\Pi_i, i \in \mathcal{M}\}$, the probability that the measure outcome is equal to i is given by

$$\mathrm{P}\left[m = i \mid \rho\right] = \mathrm{Tr}\left(\rho \Pi_i\right) .$$

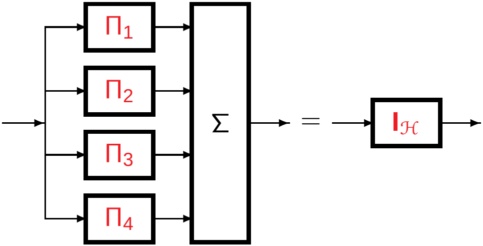
This extension is easily obtained from the previous formulation.

POVM=Positive Operator Valued Measure

POVM are operators $\{\Pi_i, i \in \mathcal{M}\}$ with the following properties

1. Π_i are Hermitian ($\Pi_i^* = \Pi_i$)
2. Π_i are positive semidefinite ($\Pi_i \geq 0$)
3. They give a resolution of the identity

$$\sum_{i \in \mathcal{M}} \Pi_i = I_{\mathcal{H}}$$



Motivation of the properties: to assure correct probabilities in the quantum measurement.

Orthogonal Projectors

This is Noble subclass of POVM (the corresponding measurements are called von Neumann measures).

Orthogonal projectors:

1. Idempotent: $\Pi_i^2 = \Pi_i$
2. $\Pi_i \, \Pi_j = 0, \; i \neq j$

Example of projector :
Let $|b\rangle$ be a unit norm vector. The operator $B = |b\rangle\langle b|$, given by the outer product of a ket by itself, is (rank 1) projector.

How to get a POVM system?

1) An orthonormal base of the Hilbert space $\mathcal{H}$, $\{|b_i\rangle\}$ with $|b_i\rangle\langle b_j| = \delta_{i,j}$, gives a system of orthogonal projectors

$$\Pi_i = |b_i\rangle\langle b_i|$$

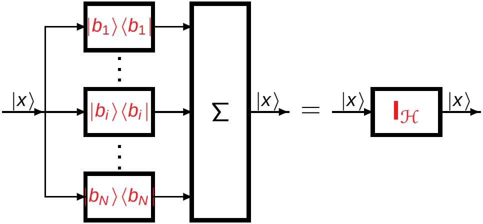


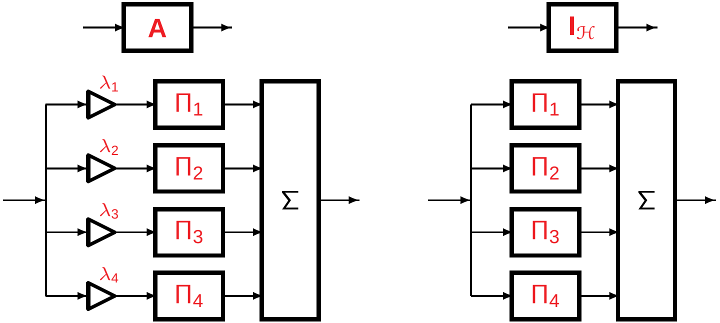
Fig. 8 – The elementary operators $|b_i\rangle\langle b_i|$ give *a resolution of the identity* $\mathbf{I}_{\mathcal{H}}$.

2) For a Hermitian operator A we have the unique spectral decomposition

$$A = \sum_{i=1}^{r} \lambda_i \Pi_i$$

where

- λ_i are real numbers
- Π_i form a system of orthogonal projectors



Example: Measure on a qubit

A qubit in the state

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

is applied to a quantum measurement by means of two projectors

$$\Pi_0 = |0\rangle\langle 0| \qquad , \qquad \Pi_1 = |1\rangle\langle 1|$$

The result may be $m = 0$ with probability

$$\langle\psi|\Pi_0|\psi\rangle = \langle\psi|0\rangle\langle 0|\psi\rangle = |\langle\psi|0\rangle|^2 = |\alpha|^2$$

and the state collapses into the state $|0\rangle$.
The result may be $m = 1$ with probability

$$\langle\psi|\Pi_1|\psi\rangle = \langle\psi|1\rangle\langle 1|\psi\rangle = |\langle\psi|1\rangle|^2 = |\beta|^2$$

and the state collapses into the state $|1\rangle$.

EXAMPLE: THE PHOTON COUNTER

A coherent state

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

in the Glauber space is subject to a measurement by means of the elementary projectors

$$\Pi_i = |i\rangle\langle i| \qquad i = 0, 1, 2, \ldots$$

The result is $m = i$ with probability

$$\langle\alpha|\Pi_i|\alpha\rangle = \langle\alpha|i\rangle\langle i|\alpha\rangle = |\langle\alpha|i\rangle|^2 = e^{-|\alpha|^2} |\frac{\alpha|^{2i}}{i!}$$

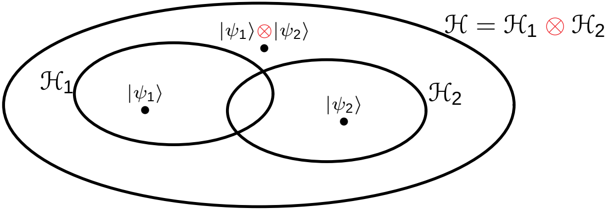
Given two isolated quantum systems described by the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, the composite system is described within the Hilbert space

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$$

where $\otimes$ denotes the tensorial product.

If $|\psi_1\rangle$ and $|\psi_2\rangle$ are the states in the two systems, the state of the global (composite) system is

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$$



POSTULATE IV OF QM: APPLICATION

1. The tensorial product of Hilbert Spaces permits the formulation of entangled states
2. The quantum PPM modulation must be formulated in the tensorial product of the form:

$$\mathcal{H} = \mathcal{H}_0 \otimes \mathcal{H}_0 \cdots \otimes \mathcal{H}_0$$

Quantum Communication Scheme

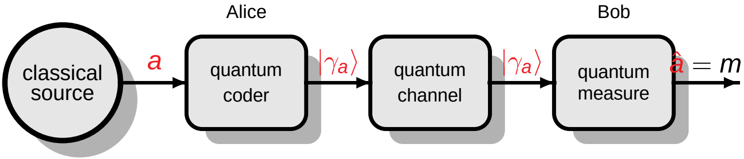


Fig. 11 – Quantum communication system.

- Alice prepares the system in one among m quantum states (m-ary transmission)

$$|\gamma_0\rangle, \ldots, |\gamma_{m-1}\rangle$$

- Bob makes a quantum measure by using a POVM system Π_i to guess the state prepared by Alice

Problem statement

- Classical source with alphabet $\mathcal{A} = \{0, \dots, M-1\}$ and a-priori probability
$$q_i = \mathrm{P}[\, a = i \,]$$
- Quantum coding (modulation) $a \to |\gamma_a\rangle$ with quantum states
$$\{|\gamma_0\rangle, \dots, |\gamma_{M-1}\rangle\}$$
- Ideal quantum channel
- Quantum detection by a POVM system
$$\{\Pi_0, \dots, \Pi_{M-1}\}$$
with outcome $m = \hat{a}$

Transition probabilities (from Postulate III, m outcome of measurement)

$$p(j|i) = \mathrm{P}\left[\hat{a} = j \mid a = i\right] = \mathrm{P}\left[m = j \mid |\gamma_i\rangle\right] = \langle\gamma_i|\Pi_j|\gamma_i\rangle$$

Correct decision probability

$$P_c = \sum_{i=0}^{m-1} q_i \langle\gamma_i|\Pi_i|\gamma_i\rangle$$

Error probability

$$P_e = 1 - P_c$$

- Given the a-priori probabilities q_i
- the state constellation

$$\{|\gamma_0\rangle, |\gamma_1\rangle, \ldots, |\gamma_{m-1}\rangle\}$$

and in general

$$\{\rho_0, \rho_1), \ldots, \rho_{m-1})\}$$

find the optimum POVM

$$\Pi_0, \ldots, \Pi_{m-1}$$

such that

$$max_{\{\Pi_i\}} \mathrm{Tr}\left(\Pi_i \rho_i\right)$$

General Results

A. S. Holevo, "Statistical decision theory for quantum system", *J. Multivar. Anal.*, vol. 3, pp. 337–394, 1973.

H. P. Yuen, R. S. Kennedy eM. Lax, "Optimum Testing of Multiple Hypotheses in Quantum Detection Theory", *IEEE Transaction on Information Theory*, vol. IT-21, no. 2, pp. 125-134, marzo 1975.

Explicit Results

- Helmstron for the binary case
- For multilevel pure states in the presence of GUS (PSK, PPM)
  GUS=Geometric Uniform Symmetry

Numerical approach

through convex linear programming

ERROR PROBABILITY

An ideal transmission system with $P_c = 1$ could be implemented by using a qubit with two orthogonal states

$$|\gamma_0\rangle = |0\rangle \qquad , \qquad |\gamma_1\rangle = |1\rangle$$

between which Alice chooses, and by a system of orthogonal projectors used by Bob as the measure at the receiver

$$\Pi_0 = |0\rangle\langle 0| \qquad , \qquad \Pi_1 = |1\rangle\langle 1|$$

$P_e = 0$ cannot be achieved, because

- In the cases of practical interest, it is not possible to generate orthogonal states. In particular two Glauber states $|\alpha\rangle$ e $|\beta\rangle$ cannot be orthogonal.
- The quantum channel is noisy

The Binary Quantum Channel

- Classical source with alphabet $\mathcal{B} = \{0, 1\}$ and a-priori probabilities q_0, $q_1 = 1 - q_0$
- Quantum coding $0 \to \rho_0$ and $1 \to \rho_1$ with mixed quantum states ρ_0 and ρ_1
- Quantum measure with a POVM system Π_0 and Π_1. Conditions on POVM

$$\Pi_0 \geq 0 \qquad \Pi_1 \geq 0 \qquad \Pi_0 + \Pi_1 = I$$

Error probability

Transition probability: $p(j|i) = \mathrm{Tr}\left(\rho_i q_j\right)$

Correct decision probability

$$P_c = q_0 \mathrm{Tr}\left(\rho_0 \Pi_0\right) + q_1 \mathrm{Tr}\left(\rho_1 \Pi_1\right) = \mathrm{Tr}\left(q_0 \rho_0 \Pi_0 + q_1 \rho_1 \Pi_1\right)$$

Optimization problem: Determine the positive semi-definite operators Π_0 and Π_1 that maximize the correct decision probability.

HELSTROM'S SOLUTION

Correct decision probability

$$P_c = q_0 \mathrm{Tr}\left(\rho_0(I - \Pi_1)\right) + q_1 \mathrm{Tr}\left(\rho_1 \Pi_1\right) = q_0 + \mathrm{Tr}\left((q_1\rho_1 - q_0\rho_0)\Pi_1\right)$$

The optimum decision is based on the spectral decomposition of the Hermitian operator $D = q_1\rho_1 - q_0\rho_0$

$$D = \sum_k \eta_k |\eta_k\rangle\langle\eta_k|$$

The maximum is

$$P_c = q_0 + \sum_{\eta_k > 0} \eta_k$$

with projectors

$$\Pi_1 = \sum_{\eta_k > 0} |\eta_k\rangle\langle\eta_k| \qquad , \qquad \Pi_0 = I - \Pi_1 = \sum_{\eta_k < 0} |\eta_k\rangle\langle\eta_k|$$

With Pure States the Hermitian operators is

$$D = q_1|\gamma_1\rangle\langle\gamma_1| - q_1|\gamma_0\rangle\langle\gamma_0|$$

The eigenvalues are

$$\eta_{0,1} = \frac{1}{2}(q_1 - q_0 \mp \sqrt{1 - 4q_0q_1|X^2|})$$

with $X = \langle\gamma_0|\gamma_1\rangle$, called overlap coefficient
($X = 0$ in the ideal orthogonal case)

Error probability (Helstrom's bound)

$$P_e = \frac{1}{2}\left(1 - \sqrt{1 - 4q_0q_1|X|^2}\right)$$

When $|\gamma_0\rangle = |\alpha\rangle$ and $|\gamma_1\rangle = |\beta\rangle$ are Glauber states (laser radiation)

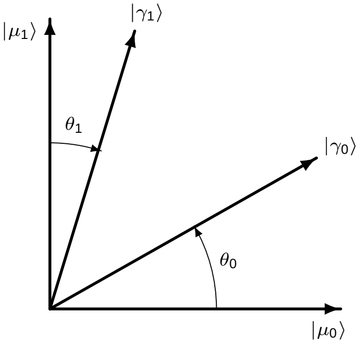
$$|\alpha\rangle = e^{-\alpha^2/2} \sum_{n=0}^{n} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \qquad , \qquad |\beta\rangle = e^{-\beta^2/2} \sum_{n=0}^{n} \frac{\beta^n}{\sqrt{n!}} |n\rangle$$

the quadratic overlap is $|X|^2 = |\langle\alpha|\beta\rangle| = e^{-|\alpha-\beta|^2}$. Hence

$$P_e = \frac{1}{2}\left(1 - \sqrt{1 - e^{-|\alpha-\beta|^2}}\right)$$

Note that $|X|^2 > 0$ for $|\alpha\rangle \neq |\beta\rangle$ and $P_e > 0$

Geometric Interpretation



$|\gamma_0\rangle$ and $|\gamma_1\rangle$ given states
$|\mu_0\rangle$ and $|\mu_1\rangle$ measure vectors to be found by optimization

Optimum measure vectors are obtained with equal angles with the orthogonality constraints

2–PSK Modulation

With 2–PSK the quantum state constellation is

$$|\gamma_0\rangle = |\alpha\rangle \qquad , \qquad |\gamma_1\rangle = |-\alpha\rangle$$

where α is real. The quadratic overlap coefficient is

$$X^2 = \langle\alpha|-\alpha\rangle = e^{-4\alpha^2} = e^{-4N_R}$$

Helstrom's bound gives

$$P_e = \frac{1}{2}\left(1 - \sqrt{1 - e^{-4N_R}}\right)$$

where N_R is the average number of photons per bit

Received signal

$$x_R(t) = X_0 \cos(2\pi\nu t + a_0\pi) \qquad a_0 \in \{0, 1\}$$

Local signal

$$x_L(t) = X_L \cos(2\pi\nu t)$$

with $X_L \gg X_0$

Energy over $[0, T]$

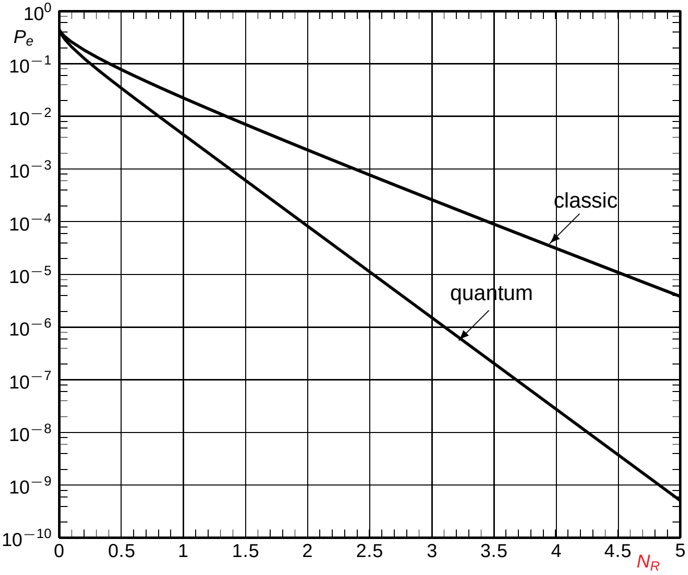
$$K[X_L^2 + X_0^2 + 2X_L X_0 \cos(\pi a_0)]$$

Useful signal $\pm K\,2X_L X_0$ and (Gaussian) noise with variance KX_L^2 The error probability is

$$P_e = Q\left(\sqrt{4N_R}\right)$$

where N_R is the average number of photons per bit

Quantum and Classical 2–PSK: Comparison



Quantum and Classic 64–PPM: Comparison

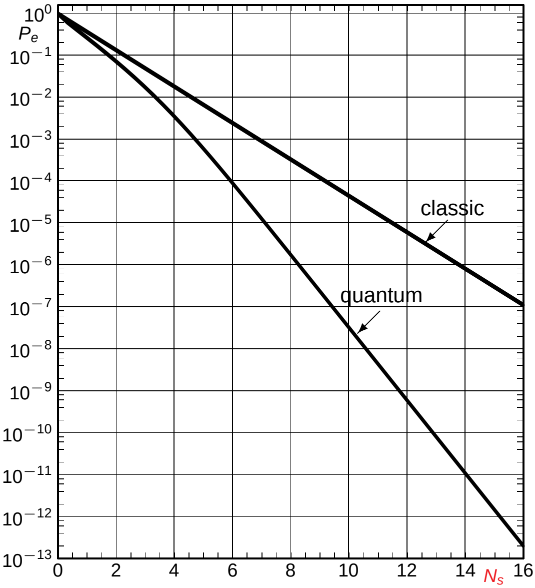
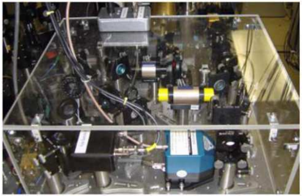


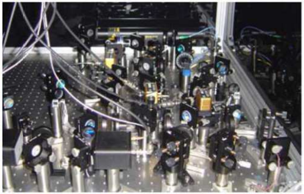
Fig. 14

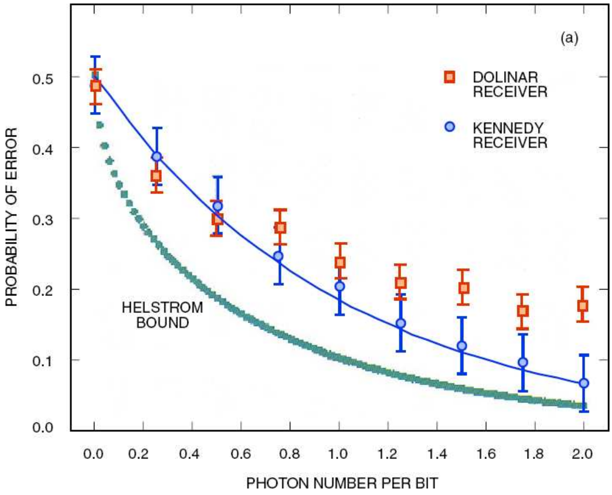
N_s: average number of photons per symbol (1 symbol=8 bits)

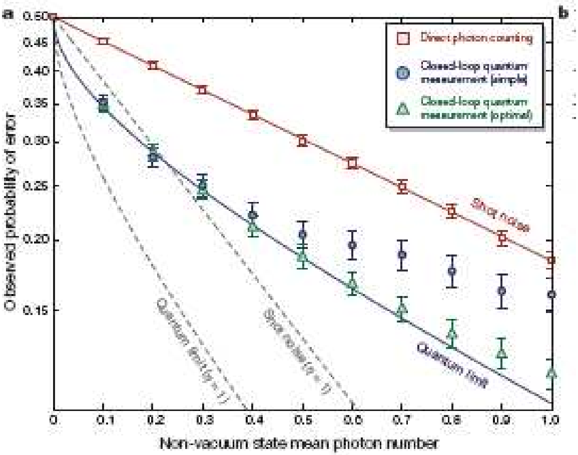
- 1973, January – Kennedy (MIT) suggests his near–optimum receiver
- 1973, October – Dolinar (MIT) suggests his receiver ideally achieving Helstrom bound
- 2006 – Dolinar, Geremia, *et al.* (JPL– CalTech) demonstrate the feasibility of both the Kennedy and the Dolinar receiver (see below)
- 2008 – Geremia *et al.* (New Mexico Univ., Albuquerque) realize a Dolinar receiver overperforming the Kennedy receiver

JPL – Caltech 2006 (1/2)






(a)
DOLINAR RECEIVER
KENNEDY RECEIVER
HELSTROM BOUND
PROBABILITY OF ERROR
0.0
0.1
0.2
0.3
0.4
0.5
0.0 0.2 0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8 2.0
PHOTON NUMBER PER BIT


a
b
Observed probability of error
0.50
0.45
0.40
0.35
0.30
0.25
0.20
0.15
Non-vacuum state mean photon number
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
Direct photon counting
Closed-loop quantum measurement (simple)
Closed-loop quantum measurement (optimal)
Shot noise
Quantum limit

People

Quantum Communication at DEI
A. Assalini, G. Cariolaro, R. Corvaija, G. Pierobon

Optical Physics at DEI
G. Naletto, T. Occhipinti, P. Villoresi

Astronomy
G. Barbieri, F. Tamburini

Alcatel Lucent at Vimercate (MI)
S. Gastaldello, P. Danielutti, R. Valussi

Research Road Map

Implementation of 2–PSK and 2-PPM Quantum Systems
Implementation of 4–PSK and 4-PPM Quantum Systems

G. Cariolaro, Comunicazioni Quantistiche 2009

within the course of Optical Communication (last year of Laurea)

25 hours of lectures to teach the subject presented here in Brixen