Game theory for information engineering

Leonardo Badia

leonardo.badia@gmail.com

Duopolies

An interesting application of NE

Leonardo Badia - leonardo.badia@gmail.com

Cournot duopoly

- Cournot (1838) anticipated Nash's results in a particular context: a special duopoly.
- In the Cournot model, we have two firms (called 1 and 2) producing a good in quantities q_1 and q_2. Let $Q = q_1 + q_2$.
- The cost to produce q is the same for both firms and equals $C(q) = c\,q$ (with constant c)
- When the good is sold on the market, its price is $P(Q) = a - Q$. (with constant $a > c$)
- More precisely, $P(Q) = (a - Q)\, h\,[a - Q]$.

Cournot duopoly

- If the firms chooses q_1 and q_2 simultaneously, can we predict their optimal production?
- I.e., is there a Nash equilibrium of the game?
- Both firms $i = 1,2$ have a single-move strategy represented by q_i and $S_i = [0, \infty)$; actually, any $q_i > a$ is pointless, we can put $S_i = [0, a)$.
- The payoff of a firm is simply its profit (revenue minus cost):

$$u_i(q_i, q_j) = q_i [P(q_i + q_j) - c] = q_i (a - q_i - q_j - c)$$

NE of a Cournot duopoly

- Is there any NE (q_1^*, q_2^*)?
- For each player i, q_i^* must satisfy:

$$q_i^* = \max_{q_i} u_i\,(q_i, q_j^*)$$

- We solve for $q_i \in [0,\infty)$: $\max_{q_i} q_i\,(a - q_i - q_j^* - c\,)$

NE of a Cournot duopoly

$q_i^* = \max_{q_i} q_i (a - q_i - q_j^* - c)$

- The solution for both is $q_1^* = q_2^* = (a - c)/3$
  - The profit for both is $u_1^* = u_2^* = (a - c)^2/9$

Leonardo Badia - leonardo.badia@gmail.com

Monopoly solution

- In case of a single firm (monopoly) the optimum production would be (set $q_2^* = 0$) :

$$\max_{q_1} q_1 (a - q_1 - c)$$

$$q_m = (a - c)/2$$

- In which case the profit is

$$u_m = (a - c)^2/4$$

Trust case

- The two firms could compare their NE, which achieves profit $u^* = (a - c)^2/9$, with the following alternate solution.
- They could cooperate as it were a monopoly.
- The produce half of q_m, so they could share $u_m = (a - c)^2/4$. Hence, the profit is higher.
- In other words, produce less than the equilibrium so the price is higher and therefore the revenue is increased.

Why is it not a NE?

- Each firm has an incentive to deviate from such a strategy ($q_1 = q_m/2$ is not best response to $q_2 = q_m/2$ and vice versa)
- As the price is high, unilaterally increasing the production level will raise the revenue (while at the same time decreasing that of the other firm).
- At the same time, this decreases the price, so this deviation goes on as long as there is no longer incentive in betraying the trust.

Bertrand duopoly

- Bertrand (1883) argued against Cournot model that firms choose prices, not q_j s.
- Now, we have an **entirely different** game. Strategies are prices p_i and $p_i \in S_i = [0, \infty)$
- E.g., assume people buy $q_i = a - p_i$ from the firm with cheaper price and 0 from the other (if the p_i s are equal, share q_i between them)
- Cost is $C(q) = c\,q$ (as in Cournot case, $a > c$)
- Competition leads to lowering the price.
- NE of this game is $p_1{}^* = p_2{}^* = c$

Bertrand duopoly

- Similarly to Cournot's, Bertrand equilibrium is clearly not the best outcome for the firms.
- In fact, they could agree on a higher price and share the market. The price can be pushed up to $(a + c)/2 > c$.
- However, this is not a NE as each of the firm has a (selfish) incentive to deviate, i.e., decrease price, so as to conquer the market.
- This process is indefinitely repeated as long as the price is c .

Bertrand duopoly

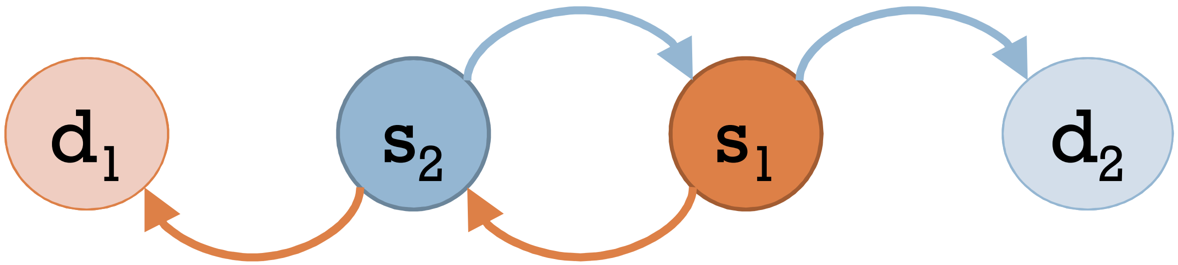
- Economic-wise, Bertrand equilibrium is nice for the customers. But, is it realistic?
- Possible explanation: goods are not perfect substitute.
- Let $q_i = a - p_i + b\,p_j$ (with constant $b < 2$)
- Note: this is **yet another game**!
- b is a sort of exchange rate between goods.

- Again, it can be shown that there is a Nash equilibrium: $p_1{}^* = p_2{}^* = (a + c)/(2 - b)$

Application examples

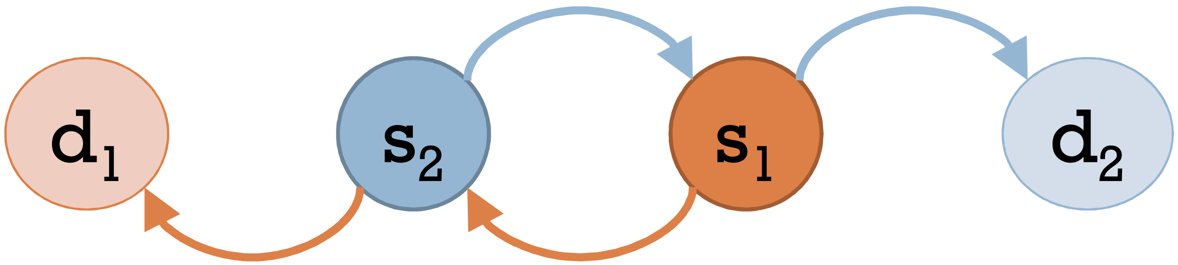
How GT models familiar problems

Wireless multi-hop routing



- Assume sources s_1 and s_2 want to send a packet to destinations d_1 and d_1.
- s_1 and s_2 are the players. d_1 and d_2 are passive.
- d_1 cannot be covered by s_1 , so s_1 must relay the packet through s_2.

Wireless multi-hop routing



- Delivering a packet yields a utility of 1.
- Forwarding a packet implies further cost $c < 1$ (energy and computation expenditure).
- The payoff is utility minus cost.
- Strategies are { (**D**)rop, (**F**)orward }

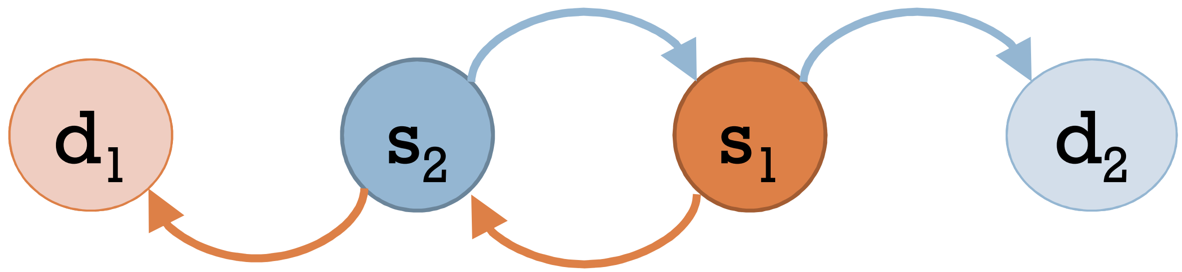
Wireless multi-hop routing

- The same holds for the Forwarder's Dilemma.

		s_2	
		D	[illegible]
s_1	D	0, 0	[illegible]
	F	[illegible]	[illegible]

- Each source is tempted to drop the packet of the other source. Both packets are discarded. Hence the dilemma.

Wireless multi-hop routing



		s_2: D	s_2: F
s_1	D	0, 0	1, -c
	F	-c, 1	1-c, 1-c

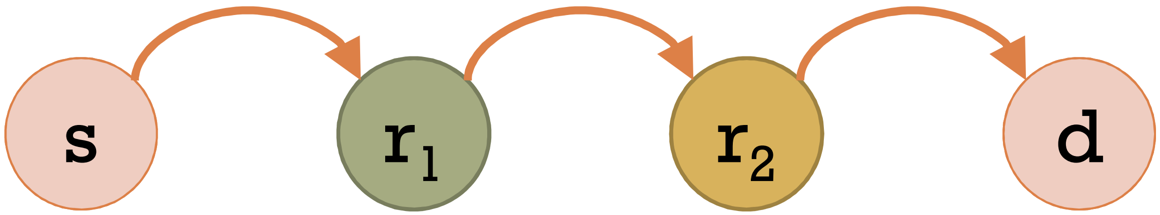
- As in the Prisoner's Dilemma, the Wireless multi-hop problem has a NE where both users do not cooperate.

The Forwarder's Dilemma

		s_2 D	F
s_1	D	0, 0	1, -c
	F	-c, 1	1-c, 1-c

- The resulting bi-matrix is very similar to the Prisoner's Dilemma.
- Hence the name "The Forwarder's Dilemma."

Joint forwarding



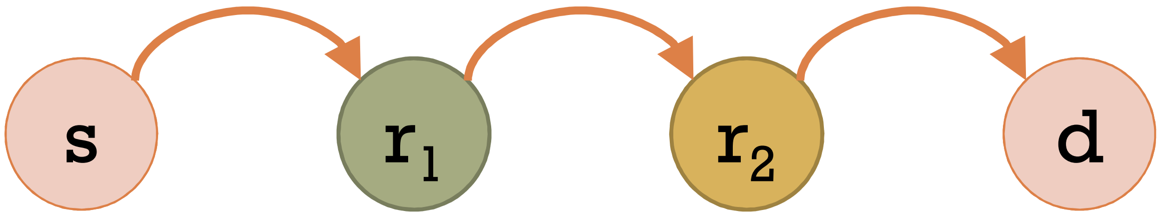
- In this game, a single source s sends a packet toward destination d, through relays r_1 and r_2.
- To correctly receive the packet at d, both r_1 and r_2 must forward. If so, they gain payoff 1.
- Again, strategies are { (**D**)rop, (**F**)orward }. The latter has cost c.

Joint forwarding

r_1 \ r_2	D	F
D	0, 0	0, 0
F	-c, 0	1-c, 1-c

- Here, cooperation may have an incentive.
- r_1 can have non-zero payoff only if chooses F.
- Also F seems to be a good choice for r_2 .
- Is this the only option?

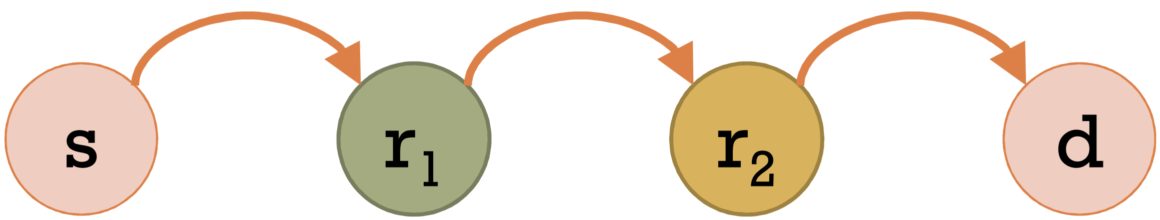
Joint forwarding



r_1 \ r_2	D	F
D	0, 0	0, 0
F	-c, 0	1-c, 1-c

- Here, it seems logical that both nodes cooperate to achieve a common goal.
- However, no **strict** dominance can be found.

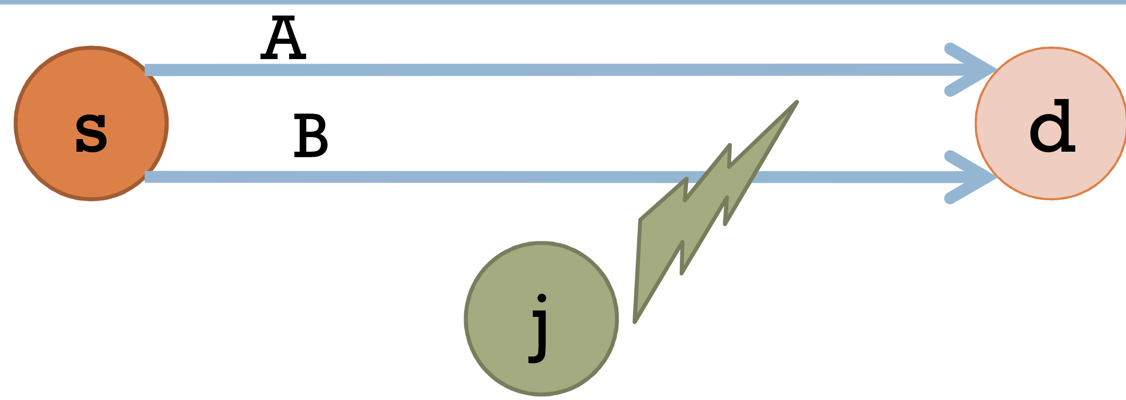
back to Joint forwarding



r_1 \ r_2	D	F
D	0, 0	0, 0
F	-c, 0	1-c, 1-c

- (F,F) is not the result of IES, but it is a NE (thus: the users have an incentive to cooperate).
- But also (D,D) is a NE. So, what do they do?

Jammin'



- Source s wants to access some resource (transmission opportunity, computation) available at destination d (passive).
- Jammer j is only interested in disturbing s.
- There are two accesses (A,B) to this resource.
- Both players can access only one at a time.

Jammin'

		j: A	j: B
s	A	-P, P	P, -P
s	B	P, -P	-P, P

- Assume they both have the same positive payoff P if they succeed, -P if they fail.
- This game becomes identical to the "Odd/Even" game.
- Unfortunately, it means also no clear solution.

Dominance, efficiency

further comparisons

Leonardo Badia - leonardo.badia@gmail.com

Strict/weak dominance

- For brevity, we write thereafter

  $s_{-i} = (s_j)_{j \neq i} = (s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$

- Recall that s_i' **strictly dominates** s_i if

  $u_i(s_i', s_{-i}) > u_i(s_i, s_{-i})$ for every s_{-i}

- We say that s_i' **weakly dominates** s_i if

  $u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i})$ for **every** s_{-i}

  $u_i(s_i', s_{-i}) > u_i(s_i, s_{-i})$ for **some** s_{-i} **(*)**

- Without **(*)**, we say that s_i' **dominates** s_i

Dominance/Nash equilibrium

- A strategy that (strictly, weakly) dominates every other strategy of a user is said to be **(strictly, weakly) dominant**.
- **Lemma**

  If every user i has a dominant strategy s_i* then (s_1*,…,s_i*,…,s_n*) is a Nash equilibrium.
- It directly follows from the definition of NE
- The reverse statement is false (only sufficient condition, not necessary)

Do not eliminate weakly dom.

- Enlarge the Odd/Even game with a third strategy "Punch the opponent" (P).
- P is weakly dominated, yet it is a NE.
- If we eliminate it, we lost the only NE.

(this strange NE will be clearer later)

Even

		0	1	[illegible]
Odd	0	-5, 5	5, -5	[illegible]
	1	5, -5	-5, 5	[illegible]
	[illegible]	[illegible]	[illegible]	[illegible]

Pareto efficiency

- A joint strategy s is **Pareto dominated** by another joint strategy s' if

  $u_i(s') \geq u_i(s)$ for **every** player i

  $u_i(s') > u_i(s)$ for **some** player i

- A joint strategy s not Pareto dominated by any joint strategy s', is said to be **Pareto efficient**.

- There may be more than one Pareto efficient strategy, none of which dominates the others.

NE vs. Pareto efficiency

- Pareto efficiency is different from NE:
  - Pareto efficiency: no way (in the **whole game**) a user can improve without somebody else being worse
  - Nash equilibrium: no way a user can improve **with a unilateral change**

- The outcome of the Prisoner's Dilemma is not "efficient!"

Al \ Bob	M	F
M	-1, -1	-21, 0
F	0, -21	-20, -20

These strategies are Pareto efficient

(F,F) is the only Nash equilibrium

NE vs. Pareto efficiency

- Pareto inefficient Nash equilibria arise as we assume players are only driven by egoism.
- To estimate the inefficiency of being selfish (or distributed) one can compare Nash equilibria with Pareto efficient strategies.
- To this end, assume that a joint strategy s has a social cost $K(s)$.
  - For example, $K(s) = \sum_j s_j$, $K(s) = \max_j s_j$

Price of anarchy

- The **price of anarchy** is the ratio between the social costs in the worst NE s^* and in the best Pareto efficient strategy (i.e., social optimum)

  $$A = K(s^*) \, / \, (\max K(s))$$

- If the best NE is considered, it is sometimes spoken of **price of stability**.

- For certain classes of problems, there are theoretical results on the price of anarchy.

Minmax choices

A useful approach for optimization

Leonardo Badia - leonardo.badia@gmail.com

Maxmin

- Consider a "two-"player game (i vs $-i$)
- We define $f_i : S_i \rightarrow \mathbb{R}$ as $f_i\,(s_i) = \min_{s_{-i} \in S_{-i}} u_i(s_i\,, s_{-i})$
- s_i* = arg $\max_{s_i \in S_i} f_i\,(s_i)$ is a **security strategy** (maxminimizer) for i (may not be unique)
- We say that $w_i = \max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} u_i(s_i\,, s_{-i})$ is the **maxmin** or the security payoff of i.
- A security strategy is a conservative approach allowing i to achieve the highest payoff in case of the worst move by $-i$.

Minmax

- Similarly, $F_i : S_{-i} \to \mathbb{R}$ as $F_i(s_{-i}) = \max_{s_i \in S_i} u_i(s_i, s_{-i})$
- Value $z_i = \min_{s_{-i} \in S_{-i}} F_i(s_{-i}) = \min_{s_{-i} \in S_{-i}} \max_{s_i \in S_i} u_i(s_i, s_{-i})$ is called the **minmax** for player *i* .
- If *i* could move after -*i*, the minmax would be the minimum payoff which is guaranteed to player *i*.

Example 7

player B

player A		L	C	R	*f* (min)
	T	5, -	3, -	4, -	3
	D	2, -	6, -	1, -	1
F (max)		5	6	4	

- $\text{maxmin}_A = 3$. Player A can secure this payoff by playing the security strategy T.
- $\text{minmax}_A = 4$. Knowing with certainty what B will play guarantees at least this payoff to A.

Minmax, maxmin, NE

- We can prove:

(1) For every player i , $\text{maxmin}_i \leq \text{minmax}_i$

(2) If joint strategy s is a Nash equilibrium, then for every player i, $\text{minmax}_i \leq u_i(s)$

- The first relationship is obvious. The second follows from every player not desiring to deviate from the NE.

Example 7

player B

player A	L	C	R
T	5, 6	3, 2	4, 1
D	2, 0	6, 8	1, 2

- As previously observed, $\text{maxmin}_A < \text{minmax}_A$.
- Moreover, there are two Nash equilibria:
  - (T,L) where $u_A = 5 > \text{minmax}_A$
  - (D,C) where $u_A = 6 > \text{minmax}_A$
- Check for B!

Example 8

player B

player A	L	C	R
T	3, 4	5, 0	3, 1
D	5, 4	6, 2	7, 2

- Here, there is one NE (D, L). For both players, maxmin = payoff at the NE, so it must be:

$$\text{maxmin}_i = \text{minmax}_i = u_i\,(\text{NE})$$

Example 9

player B

player A	L	C	R		
T	4, 0	3, 1	3, 0	3	1
M	3, 0	4, 0	2, 1	2	1
D	2, 0	1, 0	0, 0	0	0
	4	4	3		
	0	0	0		

- In general, the Lemma does not guarantee a NE.
- Here, $\text{maxmin}_i = \text{minmax}_i$ for each player i

Example 9

player B

player A	L	C	R
T	4, 0	3, 1	3, 0
M	3, 0	4, 0	2, 1
D	2, 0	1, 0	0, 0

- However, there is no NE.