Polyhedral Theory for the Asymmetric Traveling Salesman Problem (Book Chapter)

Egon Balas
GSIA, Carnegie Mellon University
eb17@andrew.cmu.edu

Matteo Fischetti
DEI, University of Padova
fisch@dei.unipd.it

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0 Introduction

The application of polyhedral methods to the TSP started in the mid-1970's (see [15] and [16] for the first major breakthroughs). For more than a decade, until the late 1980's, the main emphasis was on the symmetric TSP (STSP). There were several reasons for this: the traveling salesman paradigm suggests a geometric interpretation in which the costs are symmetric; some of the important real world applications, like in chip manufacturing, are symmetric; the polyhedral formulation of the symmetric TSP connects nicely to matching theory and borrows from the latter the family of facet defining 2-matching inequalities; finally, the asymmetric TSP (ATSP) can be reduced to a STSP on an undirected graph with twice as many nodes.

Nevertheless, there are good reasons to study the asymmetric TSP on its own. First, the asymmetric TSP is the more general one, which subsumes the symmetric TSP as a special case.

Second, the structure of the ATS polytope – i.e. the convex hull of incidence vectors of TS tours in a digraph – is much richer than that of the STS polytope. Every facet F of the STS polytope defined on a complete undirected graph G corresponds (trivially) to a face F' of the ATS polytope defined on the directed graph G' obtained from G by replacing every edge with a pair of antiparallel arcs. But the symmetric inequalities obtained this way define only a tiny fraction of the multitude of facets of the ATS polytope, the vast majority of the latter being defined by asymmetric inequalities that have no counterpart for the STS polytope defined on G.

Third, there are many important real-world problems that are naturally modeled as asymmetric TSP's. In industrial scheduling, the optimal sequencing of jobs on machines with setup times is an ATSP; more generally, the optimal ordering of any set of tasks or operations with sequence dependent changeover costs is an ATSP or one of its generalizations.

Fourth, the study of the ATS polyhedron provides certain insights into the structure of the STS polyhedron. While it is true that an asymmetric facet inducing inequality $\alpha' x \leq \alpha_0$ for the ATS polytope defined on the digraph G' typically has no counterpart for the STS polytope defined on G, it is also true that such an inequality gives rise to a family of facet inducing inequalities for a symmetric TS polytope defined on another, larger undirected graph, as will be discussed later in this survey.

Finally, while it is true that an asymmetric TSP can be reduced to a symmetric one, this symmetric TSP has a very special structure and is defined on an undirected graph with twice as many nodes as the digraph of the ATSP, and so this transformation is not without a price.

We next introduce the main notation used in the sequel.

Let $G = (V, A)$ be the complete (loop-free) digraph on n nodes, with $n \geq 5$. We associate a variable x_{ij} with every arc $(i,j) \in A$. For $S, T \subseteq V$, we denote $x(S,T) := \sum(x_{ij} : i \in S,\ j \in T,\ (i,j) \in A)$ and write $x(i,S)$ and $x(S,j)$ for $x(\{i\},S)$ and $x(S,\{j\})$ respectively. Further, for any $S \subseteq V$ we denote by $\delta^+(S)$ the set of arcs with their tail in S and their head in $V \backslash S$, and by $\delta^-(S)$ the set of arcs with their tail in $V \backslash S$ and their head in S. Also, we write $\delta^+(v), \delta^-(v)$ for $\delta^+(\{v\}), \delta^-(\{v\})$, respectively. Finally, we denote $\delta(S) := \delta^+(S) \cup \delta^-(S)$.

When G has costs (usually restricted to nonnegative values) on its arcs, the traveling salesman problem defined on G is the problem of finding a minimum-cost directed Hamiltonian cycle in G. It is also called the *Asymmetric Traveling Salesman Problem* (ATSP), to distinguish it from its *symmetric* counterpart (STSP) defined on an undirected graph.

There are several known formulations of the ATSP. We will use the standard one, due to Dantzig, Fulkerson and Johnson [10]:

$$\text{minimize } c\,x$$

s.t.

$$x(\delta^+(v)) = 1 \qquad v \in V \tag{0.1}$$

$$x(\delta^-(v)) = 1 \qquad v \in V \tag{0.2}$$

$$x(S,S) \leq |S| - 1 \qquad S \subset V, \quad |S| \geq 2 \tag{0.3}$$

$$x \in \{0,1\}^A \tag{0.4}$$

Here $c = (c_{ij})$ is the vector of costs, the equations (0.1) and (0.2) are the *outdegree* and *indegree* constraints, respectively (briefly, the degree constraints), while (0.3) are the *subtour elimination constraints* (SEC, for short). This formulation has $n^2 - n$ variables and $O(2^n)$ constraints. There are more compact formulations, but this one has proved so far the most useful.

The ATS polytope is then the convex hull of points satisfying (0.1)-(0.4).

The *monotone ATS polytope* $\tilde{P}$ is obtained from P by replacing $=$ with $\leq$ in all the degree constraints. It is well known [16] that P has dimension $n(n-1) - 2n + 1$ whereas $\tilde{P}$ is of full dimension.

Whenever there is a need to specify the graph G on which P or $\tilde{P}$ is defined, we will write $P(G)$ and $\tilde{P}(G)$, respectively. Moreover, we sometimes use the notation P_n for P defined on the complete digraph with n nodes.

The polyhedral approach to combinatorial optimization consists in trying to describe the solution set to the problem studied through a system of linear inequalities (and possibly equations) that define its convex hull. Of particular importance in this attempt is the identification of inequalities that define facets of the convex hull, because these form a minimal system. If the attempt is successful in that all the facets of the convex hull are identified, then the problem becomes a linear program. If the attempt is only partly successful, in that it fails to completely characterize the convex hull but it succeeds in describing large families of facets of the latter, then the partial description of the convex hull obtained this way usually makes it much easier to solve the problem by enumerative methods like branch and bound, branch and cut, etc.

While a complete characterization of the ATS polytope – i.e. a listing of all the inequalities defining it – is only available for $n \leq 6$ [11], several families of valid inequalities have been identified recently, and many of them have been shown to be facet defining for the corresponding polytope. In this survey we focus on ATSP-specific results (asymmetric inequalities, lifting theorems, etc.), and refer the reader to Chapter 4 (Naddef) of the present book for a thorough description of the properties shared with the STSP. Also, the separation problem for ATSP inequalities is not addressed in the present chapter: the interested reader is referred to Chapter 4 for the separation of symmetric inequalities, and to Chapter ? (Fischetti-Toth) and to [6] and [7] for ATSP-specific separation procedures.

Finally, we assume the reader is familiar with the fundamentals of polyhedral theory, as given e.g. in the excellent survey of polyhedral theory for the TSP by Grötschel and Padberg [18].

The present chapter is organized as follows. In Section 1 we review the main classes of inequalities for the ATS polytope (for short: ATS inequalities) known at the time of the writing of [18]. Section 2 addresses the monotone ATS polytope, a widely used relaxation of the ATS polytope. Facet-lifting procedures are analyzed in Section 3. The important question of whether two different facet defining inequalities define the same facet of P, is addressed in Section 4. Sections 5 and 6 are devoted to the study of large classes of ATS-specific facet-defining inequalities introduced recently, namely the odd CAT and the SD inequalities, respectively. Lifted cycle inequalities are finally investigated in Section 7, where a characterization of this class is provided, and some specific subclasses with interesting properties are analyzed.

1 Basic ATS inequalities

Several classes of valid inequalities for both P and $\tilde{P}$ were known at the time of the writing of the Grötschel and Padberg survey [18]. Among them, we will outline next the so-called comb, clique tree, D_k^+, D_k^-, T_k, C2 and C3 inequalities.

1.1 Symmetric Inequalities

Since the STSP is a special case of the ATSP, any given inequality $\bar{\alpha}y := \sum_{e \in E} \bar{\alpha}_e y_e \leq \alpha_0$ defined for the STS polytope associated with a complete undirected graph $G_E = (V, E)$, has an obvious ATS counterpart $\alpha x := \sum_{(i,i) \in A} \alpha_{ij} x_i \leq \alpha_0$ obtained by defining $\alpha_{ij} := \alpha_{ji} := \bar{\alpha}_e$ for each $e = [i, j] \in E$. Conversely, every ATS inequality $\alpha x \leq \alpha_0$ which is *symmetric* (in the sense that $\alpha_{ij} = \alpha_{ji}$ for all $(i, j) \in A$, $i < j$) has an STS counterpart $\bar{\alpha}y := \sum_{e \in E} \bar{\alpha}_e y_e \leq \alpha_0$ obtained by setting $\bar{\alpha}_e := \alpha_{ij}$ $(= \alpha_{ji})$ for each $e = [i, j] \in E$. It can easily be shown that the above mapping between STS and symmetric ATS inequalities preserves validity (but not necessarily the facet-defining property, as discussed later in this subsection). As a consequence, the ATS polytope inherits from its undirected version all classes of valid STS inequalities.

Note however that symmetric inequalities are just a small fraction of the whole set of (facet-defining) ATS inequalities. For example, Euler and Le Verge [11] showed that only 2 out of the 287 classes of inequalities representing a complete and irredundant description of the ATS polytope on 6 nodes, are symmetric (namely, the classes of SEC's (0.3) with $|S| = 2, 3$). This implies that only 35 out of the 319,015 facets of P_6 are defined by symmetric inequalities.

We next present the two most basic families of symmetric inequalities, the comb and clique-tree inequalities, and refer the reader to Chapter 4 of the present book for other large classes of STS inequalities.

Proposition 1.1 ([17]). *Let $H \subset V$ be a "handle" and $T_1, \dots, T_s \subset V$ be pairwise disjoint "teeth" satisfying: (i) $|H \cap T_i| \geq 1$ and $|T_i \setminus H| \geq 1$ for all $i = 1, \dots, s$, and (ii) $s \geq 3$ and odd. The following* comb inequality *is valid for both P and $\tilde{P}$:*

$$x(H,H) + \sum_{i=1}^{s} x(T_i, T_i) \leq |H| + \sum_{i=1}^{s} (|T_i| - 1) - (s+1)/2.$$

Theorem 1.2 ([12]). *Comb inequalities are facet inducing for P_n for $n \geq 7$, and for $\tilde{P}_n$ for $n \geq 6$.*

Note that, unlike in the symmetric case, a comb inequality does not define a facet of P_n when $n = 6$, as in this case the corresponding face turns out to be the intersection of two distinct facets of P_n defined by asymmetric inequalities having no counterpart in the symmetric case – a pathological situation that will be discussed later in this subsection.

Proposition 1.3 ([19]). *Let $C := \{H_1, \dots, H_r, T_1, \dots, T_s\}$, with $s \geq 1$ and odd, be a family of nonempty subsets of V where the H_i ($i = 1, \dots, r$) are called* handles *and the T_j ($j = 1, \dots, s$) are called* teeth, *which satisfy (see Figure 1.1 for an illustration): (i) $T_i \cap T_j = \emptyset$, for each $i, j \in \{1, \dots, s\}$, $i \neq j$; (ii) $H_i \cap H_j = \emptyset$, for each $i, j \in \{1, \dots, r\}$, $i \neq j$; (iii) $2 \leq |T_j| \leq n-2$ and $T_j \setminus (\bigcup_{i=1}^{r} H_i) \neq \emptyset$, for $j = 1, \dots, s$; (iv) the number, h_i, of teeth overlapping H_i is odd and at least 3, for $i = 1, \dots, r$; (v) the* intersection graph *of C (i.e., the undirected graph having one node for each subset belonging to C and one edge for each pair of overlapping subsets) is a tree.*

Then the following clique tree inequality *is valid for both P and $\tilde{P}$:*

$$\sum_{i=1}^{r} x(H_i, H_i) + \sum_{j=1}^{s} x(T_j, T_j) \leq s(C),$$

where t_j denotes the number of handles intersecting tooth T_j ($j = 1, \dots, s$), and $s(C) := \sum_{i=1}^{r} |H_i| + \sum_{j=1}^{s} (|T_j| - t_j) - (s+1)/2$ is the size *of C.*

Observe that clique tree inequalities with just one handle coincide with the comb inequalities introduced previously.

Clique tree inequalities have been proved by Grötschel and Pulleyblank [19] to be valid and facet-inducing (in their undirected form) for the STS counterparts, Q and $\tilde{Q}$, of the ATS polytopes P and $\tilde{P}$. As already observed, this implies that they are valid (but not necessarily facet-defining) for P and $\tilde{P}$. Later, Fischetti [14] proved the following:

Theorem 1.4. *All clique tree inequalities (except for combs when $n = 6$) define facets of both P and $\tilde{P}$.*

We next address the correspondence between facets of the STS and ATS polytopes. It was observed in [12] that the STS counterpart of every symmetric ATS inequality, say $\alpha x \leq \alpha_0$, defining a facet of P (resp., $\tilde{P}$) necessarily defines a facet of Q (resp., $\tilde{Q}$). Indeed, consider the polytope P (our reasoning trivially applies to $\tilde{P}$ as well). Let $\bar{\alpha} y \leq \alpha_0$ be the undirected counterpart of $\alpha x \leq \alpha_0$. This inequality clearly defines a proper face of Q. Assume now that

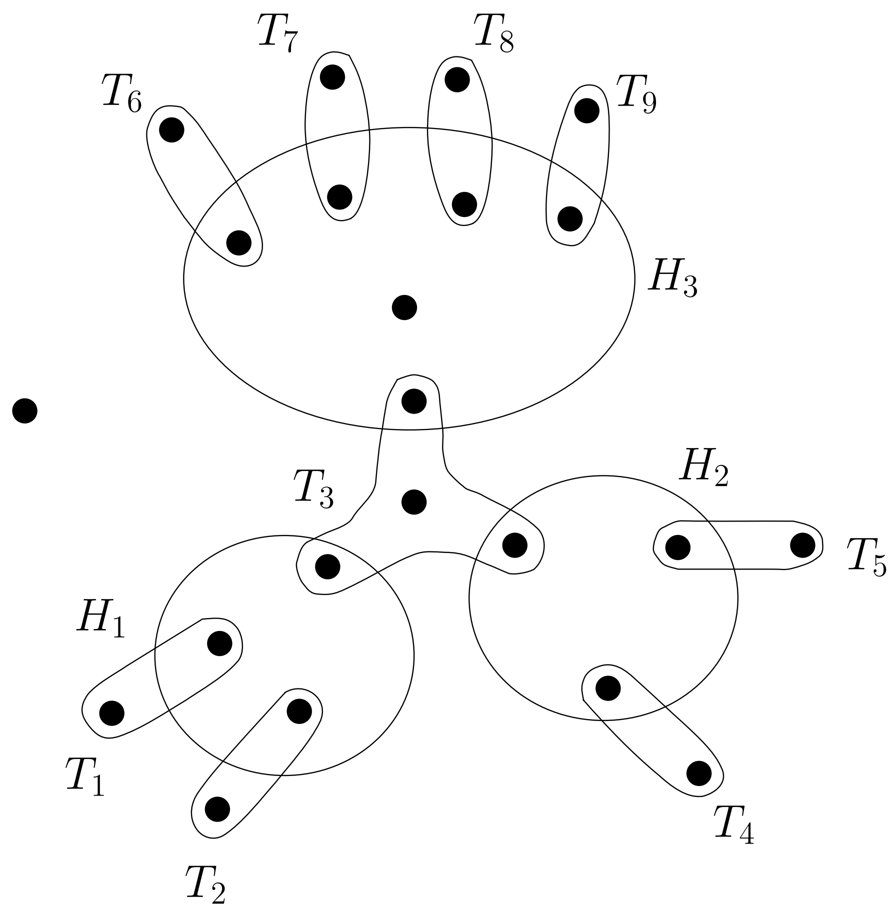


Figure 1.1: A clique tree with $r = 3$ handles and $s = 9$ teeth

it is not facet defining for Q. Then there must exist a facet defining inequality $\bar{\beta}y \leq \beta_0$, valid for Q and satisfying $\{y \in Q : \bar{\alpha}y = \alpha_0\} \subset \{y \in Q : \bar{\beta}y = \beta_0\} \subset Q$. But this implies that the directed counterpart $\beta x \leq \beta_0$ of $\bar{\beta}y \leq \beta_0$ is valid for P and induces a proper face which strictly contains the face $\{x \in P : \alpha x = \alpha_0\}$, impossible since $\alpha x \leq \alpha_0$ is facet-defining by assumption.

An important question raised in [18] is whether or not the directed version of a facet-defining inequality for the STS polytope is also facet defining for the ATS polytope. This issue has been investigated by Queyranne and Wang [31], who showed that several known classes of facet-defining inequalities for the STS polytope, including the path, wheelbarrow, chain, and ladder inequalities (see Chapter 4 (Naddef) on these inequalities), do define facets of P (with the exception of some pathological cases). Moreover, [31] proposed an operation called tree-composition and used it to produce a new large class of symmetric inequalities that define facets of the ATS (and hence of the STS) polytope.

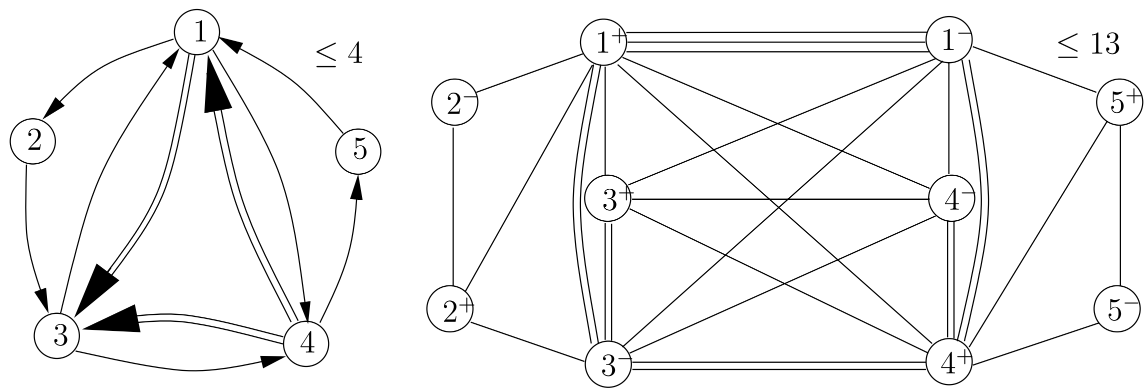


Figure 1.2: Curtain inequality for the ATSP and corresponding inequality for the STSP.

A more intriguing correspondence between STS and ATS polytopes (defined on graphs with a different number of nodes) derives from the known fact [23] that every ATSP defined on the complete digraph $G = (V, A)$ can be restated as a STSP defined on an undirected bipartite graph $G^*_B := (V^*, E^*_B)$, where V^* has a pair of nodes i^+, i^- for every node $i \in V$, and E^*_B has an edge (i^+, j^-) for every arc $(i, j) \in A$, and an edge (i^+, i^-) for every node $i \in V$, with the condition that the only admissible tours in G^*_B are those that contain every edge (i^+, i^-), $i = 1, \ldots, n$. (Here the subscript B stands for "bipartite.") This transformation has been used in [22] for solving hard instances of the ATSP by means of efficient computer codes designed for the STSP. (See Chapter ? (Fischetti-Lodi-Toth) for a computational assessment of this approach). However, the same transformation has important polyhedral implications; namely, it can be used to find new facets of the STS polytope, as shown in [6].

To this end, denote by $G^* := (V^*, E^*)$ the complete graph on V^*. Clearly, $E^*_B \subset E^*$ and G^*_B is a proper subgraph of G^*, defined on the same node set V^*. Also, denote by $Q(G^*)$ and $Q(G^*_B)$ the STS polytopes defined on G^* and G^*_B, respectively, with $y_{k\ell}$ the variable associated with edge $(k, \ell) \in E^*$. To the incidence vector $x \in \{0, 1\}^A$ of any tour in G, we

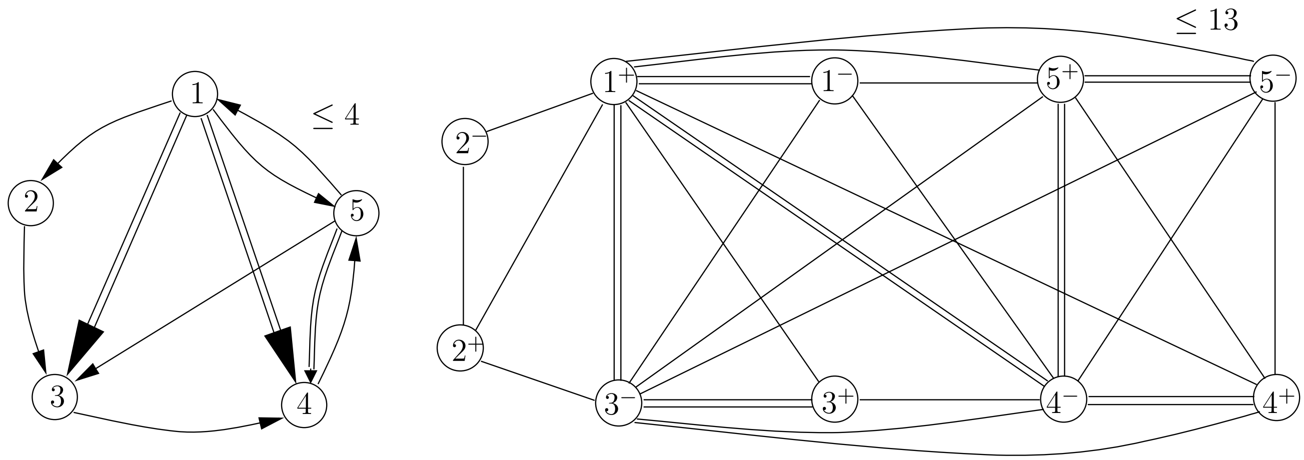


Figure 1.3: Fork inequality for the ATSP and corresponding inequality for the STSP.

associate the incidence vector $y \in \{0,1\}^{E^*}$ of a tour in G^*, defined by $y_{i^+j^-} = x_{ij}$ for all $(i,j) \in A$, and

$$\begin{array}{rcll} y_{i^+i^-} & = & 1 & \text{for all } i \in V, \\ y_{i^+j^+} & = & y_{i^-j^-} = 0 & \text{for all } i,j = 1,\ldots,n,\ i \neq j. \end{array} \tag{1.1}$$

The above construction then induces a 1-1 correspondence between tours in G, i.e. vertices of the ATS polytope $P(G)$, and admissible tours in G^*_B, i.e. vertices of the STS polytope $Q(G^*_B)$. Note that $Q(G^*_B)$ is the face of $Q(G^*)$ defined byt he equations (1.1). This implies that to every facet inducing inequality $\alpha x \leq \alpha_0$ for the ATS polytope $P(G)$ there corresponds a facet inducing inequality $\beta y \leq \alpha_0$ for the STS polytope $Q(G^*_B)$, where $\beta \in \mathbb{R}^{E^*_B}$ is defined by $\beta_{i^+j^-} = \alpha_{ij}$ for all $(i,j) \in A$ and $\beta_{i^+j^-} = 0$ for all $i \in V$.

Next, the inequality $\beta y \leq \alpha_0$ can be lifted into one or more facet inducing inequalities $\beta y + \gamma y' \leq \beta_0$, where y' is the vector whose components are associated with the edges of $E^* \backslash E^*_B$, and γ is the vector of lifted coefficients. This can be done, for instance, by sequential lifting [26]. The latter consists in choosing an appropriate sequence for the variables fixed at 1 or 0, freeing them one by one, and calculating the corresponding lifting coefficients. Here β_0 typically differs from α_0, because when the coefficient of a variable previously fixed at one is lifted, the right hand side is also changed.

The inequality $\beta y + \gamma y' \leq \beta_0$ resulting from the lifting is facet inducing for the STS polytope $Q(G^*)$ on the complete undirected graph G^*, and is often of a type unknown before. In other words, the study of the polyhedral structure of the ATS polytope provides new insights also into the structure of its symmetric counterpart.

Figures 1.2 and 1.3 show two known inequalities for the ATS polytope (the so-called *curtain* and *fork* inequalities defined in Section 7) and their counterparts for the STS polytope. The arcs in single and in double line have coefficient 1 and 2, respectively; all remaining arcs have coefficient 0. (This rule applies to all subsequent figures too.)

1.2 Asymmetric inequalities

We next introduce the main classes of asymmetric ATS inequalities known prior to the publication of [18].

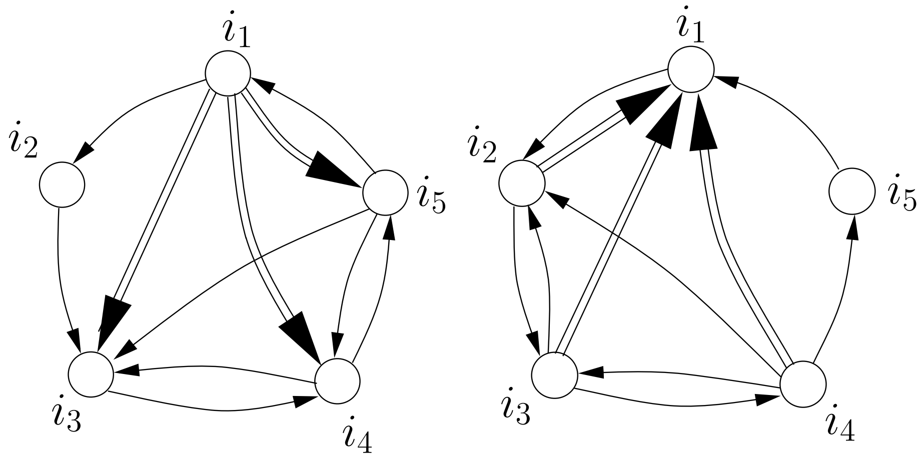


Figure 1.4: The support graph of a D_k^+ (left) and of a D_k^- (right) inequality when $k = 5$.

Proposition 1.5 ([15, 16]). *Let* $\{i_1, i_2, \ldots, i_k\} \subset V$, $3 \le k \le n-1$*; then the* D_k^+ inequality

$$\sum_{j=1}^{k-1} x_{i_j i_{j+1}} + x_{i_k i_1} + 2\,x(\{i_1\}, \{i_3, \ldots, i_k\}) + \sum_{j=4}^{k} x(\{i_j\}, \{i_3, \ldots, i_{j-1}\}) \le k - 1$$

and the D_k^- inequality

$$\sum_{j=1}^{k-1} x_{i_j i_{j+1}} + x_{i_k i_1} + 2\,x(\{i_2, \ldots, i_{k-1}\}, \{i_1\}) + \sum_{j=3}^{k-1} x(\{i_j\}, \{i_2, \ldots, i_{j-1}\}) \le k - 1$$

are valid for both P *and* $\tilde{P}$ *(see Figure 1.4 for an illustration).*

Notice that D_k^- inequalities can be obtained from D_k^+ inequalities by switching the coefficients of each pair of arcs (i,j) and (j,i) – and by reversing the order of the nodes $i_1, i_2, \ldots, i_k$.

Proposition 1.6 ([15, 16]). *Let* $S \subset V$ *with* $2 \le k := |S| \le n-2$*, and let* $w \in S$*,* $p, q \in V \setminus S$*, and* $p \ne q$*. Then the following* T_k inequality *is valid for both* P *and* $\tilde{P}$ *(see Figure 1.5):*

$$x(S,S) + x_{pw} + x_{wq} + x_{pq} \le k.$$

Fuerthermore, T_k *inequalities define facets of* P *if and only if* $k \ne n-2$*, and of* $\tilde{P}$ *for any* k*.*

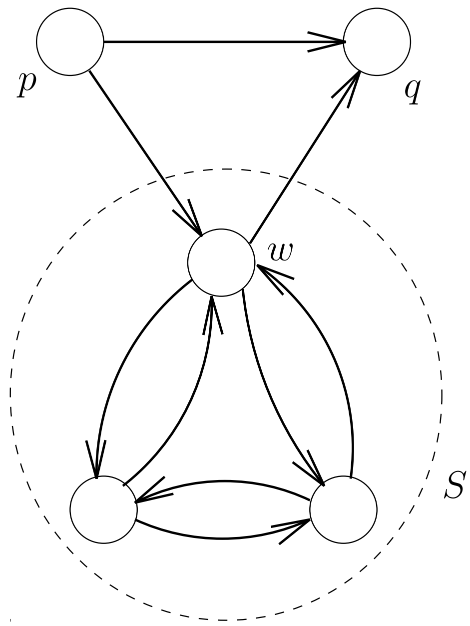


Figure 1.5: The support graph of a T_k inequality ($k = 3$).

T_k inequalities can be generalized by attaching a source p and a sink q to a comb, as follows.

Proposition 1.7 ([15, 16]). *Let $H \subset V$ be a "handle" and $T_1, \ldots, T_s \subset V$ be pairwise disjoint "teeth" satisfying: (i) $|H \cap T_i| \geq 1$ and $|T_i \setminus H| \geq 1$ for all $i = 1, \ldots, s$, and (ii) $s \geq 3$ and odd. For each pair of distinct vertices p and q in $(V \setminus H) \setminus (\bigcup_{i=1}^{s} T_i)$, the following* C2 *inequality is valid for P and $\tilde{P}$ (see Figure 1.6):*

$$x(H, H) + \sum_{i=1}^{s} x(T_i, T_i) + \sum_{v \in H} (x_{pv} + x_{vq}) + x_{pq} \leq s(C) + 1,$$

where $s(C) = |H| + \sum_{i=1}^{s}(|T_i| - 1) - (s + 1)/2$.

Proposition 1.8 ([15, 16]). *Let i_1, i_2, and i_3 be three different vertices, and let $W_1, W_2 \subset V$ such that: (i) $W_1 \cap W_2 = \emptyset$, and (ii) $W_j \cap \{i_1, i_2, i_3\} = \{i_j\}$ and $|W_j| \geq 2$ for $j = 1, 2$. Then the following* C3 inequality *is valid for both P and $\tilde{P}$ (see Figure 1.7):*

$$x(W_1, W_1) + x(W_2, W_2) + x(\{i_1\}, W_2) + x_{i_2 i_1} + x_{i_3 i_1} + x_{i_3 i_2} \leq |W_1| + |W_2| - 1.$$

Theorem 1.9. *[12] All D_k^+, D_k^-, C2, and C3 inequalities define facets of both P and $\tilde{P}$.*

Other classes of relevant ATS inequalities have been described in [15, 16], including the lifted cycle inequalities discussed in greater detail in the forthcoming Section 7. Finally, Grötschel and Wakabayashi [20, 21] described "bad" facets of $\tilde{P}$ related to hypo-Hamiltonian and hypo-semi-Hamiltonian subgraphs of G.

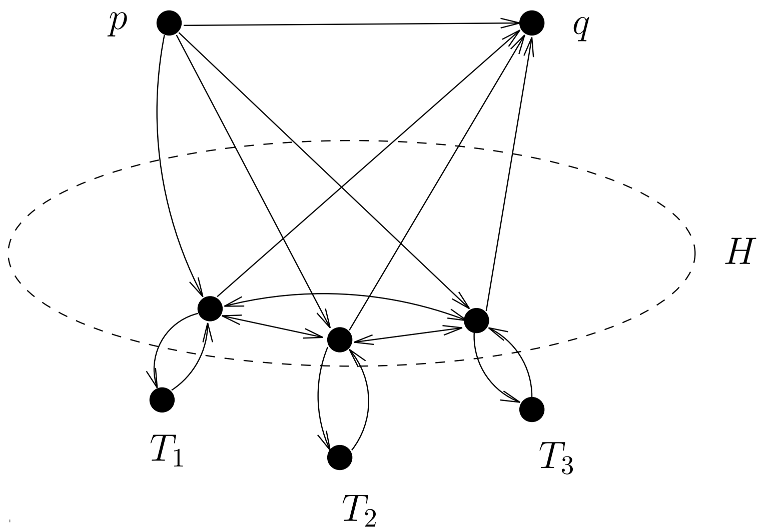


Figure 1.6: The support graph of a C2 inequality.

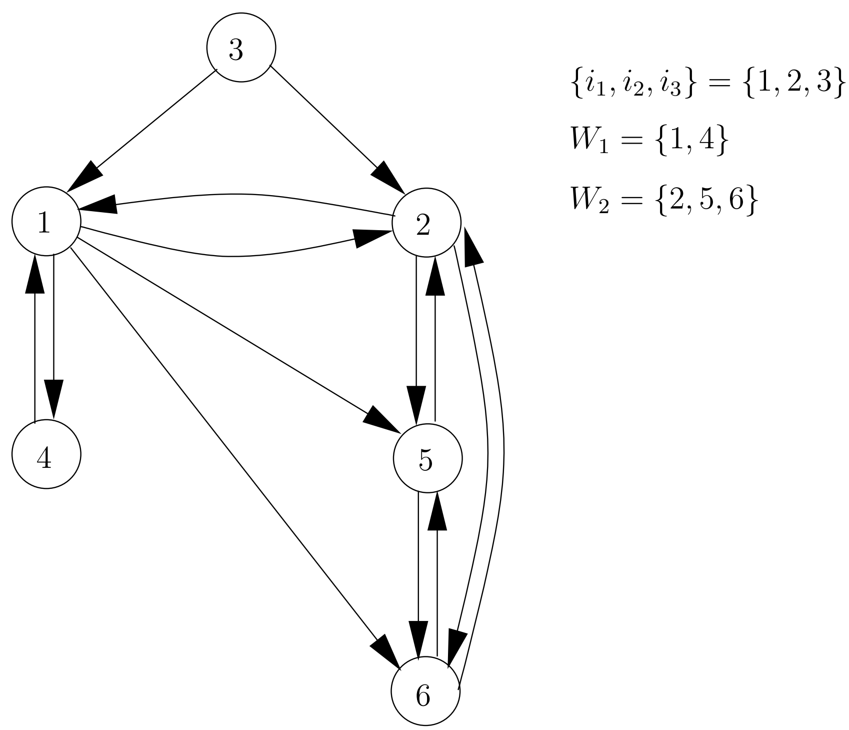


Figure 1.7: The support graph of a C3 inequality.

2 The monotone ATS polytope

Much of polyhedral theory deals with less than full dimensional polyhedra. The hallmark of such a polyhedron, say P, is the existence of certain equations satisfied by every point of P. Because of this, every valid inequality for P can be expressed in several forms, the equivalence of which is not always easy to recognize. This may cause technical difficulties when analyzing the facial structure of P. Moreover, polyhedral proof techniques often rely on interchange arguments, involving extreme points that differ only in a few components: the smaller the number of such components, the simpler the argument. Often the presence of equations in the defining system of P implies that this number cannot be very small. In particular, in the case of the ATS polytope any two distinct extreme points differ by at least six components, since at least three arcs have to be deleted from a tour, and three others added to it, in order to obtain a different tour.

A widely used technique to overcome these difficulties is to enlarge $P \subset \mathbb{R}^A$ so as to make it full dimensional, i.e. to obtain a polyhedron $\tilde{P}$ of dimension $|A|$. When P lies in the positive orthant, a standard way of doing this is to "enlarge P below" to obtain the *(downward) monotonization*, or *submissive*, of P, defined as

$$\tilde{P} := \{y \in \mathbb{R}^A : 0 \leq y \leq x \text{ for some } x \in P\} = (P - \mathbb{R}^A_+) \cap \mathbb{R}^A_+.$$

Under appropriate conditions, optimizing a linear function over P is equivalent to optimizing a related linear function over its monotonization $\tilde{P}$, so $\tilde{P}$ can replace P insofar as optimization is concerned. Hence the interest in the study of the polyhedral structure of $\tilde{P}$.

It is easy to see that $\tilde{P}$ is full dimensional if and only if for each $a \in A$, there exists $x \in P$ with $x_a \neq 0$. When $P = \text{conv}\{x \in \mathbb{Z}^A_+ : A_1 x = b_1,\ A_2 x \leq b_2\}$, $\tilde{P}$ can in some cases be obtained by replacing $A_1 x = b_1$ by $A_1 x \leq b_1$. This is true, for instance, in the case of the symmetric and asymmetric TS polytopes defined on a complete graph or digraph, when $A_1 x = b_1$ consists of degree equations. In this situation the monotone polytope is the convex hull of incidence vectors of path systems, i.e. families of simple node-disjoint paths (including single nodes). But it is not true in general in the case of the same polytopes defined on graphs other than complete: while the monotone polytope contains the incidence vectors of all subsets of tours, the polytope obtained by replacing $=$ with $\leq$ may contain incidence vectors of arc sets that are not part of any tour.

Other useful relaxations (not covered in the present chapter) are the *Fixed-Outdegree 1-Arborescence polytope* studied in [3] (where the outdegree equations (0.1) are only imposed for a given subset F of nodes), the *Asymmetric Assignment polytope* addressed in [1] (where all SEC's (0.3) with $|S| \geq 3$ are relaxed), and the *Graphical Asymmetric Traveling Salesman polyhedron* of Chopra and Rinaldi [8] (where the degree conditions (0.1)-(0.4) are relaxed into $x(\delta^+(v)) = x(\delta^-(v))$ for all $v \in V$, and the SEC's (0.2) are replaced by their cut form $x(\delta^+(S)) \geq 1$ for all $\emptyset \subset S \subset V$).

We analyze next the main properties of general monotone polyhedra and their implications for the study of the ATS polytope. The reader is referred to [5] for more details and for the proof of the main propositions, as well as for a parallel treatment of the monotone STS polytope.

2.1 Monotonizations of polyhedra

Consider next an arbitrary polyhedron P contained in $\mathbb{R}^N$, where $N = \{1, \dots, n\}$, and assume that P is given in the form $P := \{x \in \mathbb{R}^N : Ax \leq b\}$. Although we make no assumption on the dimensionality of P, the construction to follow becomes meaningful only when P is less than full dimensional. Let $A^= x = b^=$ be a full row rank *equality system* for P (i.e., every equation satisfied by all points of P is a linear combination of $A^= x = b^=$), having $r := n - \dim(P)$ rows.

We introduce a generalized concept of monotonization of a polyhedron. For any partition $[N^L, N^U]$ of N, let $b_j \in \mathbb{R} \cup \{-\infty\}$, $j \in N^L$, and $b_j \in \mathbb{R} \cup \{+\infty\}$, $j \in N^U$, be such that $x_j \geq b_j$ for $j \in N^L$ and $x_j \leq b_j$ for $j \in N^U$ for every $x \in P$. Then the *g-monotonization* (g for generalized) of P is defined as

$$\text{g-mon}(P) := \left\{ y \in \mathbb{R}^N \;\middle|\; \begin{array}{c} b_j \leq y_j \leq x_j,\ j \in N^L \text{ and } x_j \leq y_j \leq b_j,\ j \in N^U \\ \text{for some } x \in P \end{array} \right\}.$$

Notice that when $P \subseteq \mathbb{R}^N_+$, if $N^U = \emptyset$ and $b_j = 0$, $j \in N^L = N$, then g-mon$(P) = \{y \in \mathbb{R}^N : 0 \leq y_j \leq x_j,\ j \in N$, for some $x \in P\}$ coincides with the submissive of P. Moreover, if $N^L = \emptyset$ and $b_j = +\infty$, $j \in N^U = N$, then g-mon$(P) = \{y \in \mathbb{R}^N : y_j \geq x_j,\ j \in N^U$, for some $x \in P\}$ defines the so-called *dominant* of P.

Next we address the dimension of g-mon(P).

Proposition 2.1. *Let* $W := \{j \in N : |b_j| < \infty,\ x_j = b_j$ *for all* $x \in$ g-mon$(P)\}$. *Then* $\dim(\text{g-mon}(P)) = n - |W|$.

For the rest of this subsection we will assume that $W = \emptyset$, i.e. g-mon(P) is full dimensional.

Proposition 2.2. *The inequalities (i)* $x_j \geq b_j$ *for* $j \in N^L$ *such that* $|b_j| \neq \infty$, *and (ii)* $x_j \leq b_j$ *for* $j \in N^U$ *such that* $|b_j| \neq \infty$ *define facets of* g-mon(P).

We will call *trivial* the facets of g-mon(P) induced by $x_j \geq b_j$ or $x_j \leq b_j$. By extension, we will also call trivial every nonempty face of g-mon(P), of whatever dimension, contained in a trivial facet.

Proposition 2.3. *Let* $\alpha x \leq \alpha_0$ *define a nontrivial face of g-mon*(P). *Then* $\alpha_j \geq 0$ *for all* $j \in N^L$ *and* $\alpha_j \leq 0$ *for all* $j \in N^U$.

Definition 2.4 ([19]). *For a given valid inequality* $\alpha x \leq \alpha_0$ *defining a nontrivial face of g-mon*(P), *let* $N^0 := \{j \in N : \alpha_j = 0\}$ *and let* $A^=_0$ *denote the* $r \times |N^0|$ *submatrix of* $A^=$ *whose columns are indexed by* N^0. *The inequality* $\alpha x \leq \alpha_0$ *is said to be* support reduced *(with respect to* P*) if* $A^=_0$ *has full row rank.*

It can be shown that, if $\alpha x \leq \alpha_0$ defines a nontrivial facet of g-mon(P), then either $\alpha x \leq \alpha_0$ is support reduced, or else it is a linear combination of equations satisfied by all points in P (i.e., there exists $\mu \in \mathbb{R}^r \setminus \{0\}$ such that $(\alpha, \alpha_0) = \mu(A^=, b^=)$). This property leads to the following result.

Theorem 2.5. *Let $\alpha x \leq \alpha_0$ be a valid inequality for* g-mon(P)*, that defines a nontrivial facet of P. Then $\alpha x \leq \alpha_0$ defines a facet of* g-mon(P) *if and only if it is support reduced.*

We next address the important question of whether a facet defining inequality for g-mon(P) is also facet defining for P.

Definition 2.6 ([2, 5]). *Let $\alpha x \leq \alpha_0$ be a support reduced inequality defining a nontrivial facet of g-mon(P), and let $F := \{x \in P : \alpha x = \alpha_0\}$ denote the face of P induced by $\alpha x \leq \alpha_0$. Suppose w.l.o.g. that $N^0 := \{j \in N : \alpha_j = 0\} = \{j_1, \ldots, j_q\}$, where $q \geq r$, and $j_1, \ldots, j_r$ index independent columns (i.e., a basis) of $A_0^=$. Inequality $\alpha x \leq \alpha_0$ is said to be* strongly support reduced *with respect to P if for every $k \in \{r+1, \ldots, q\}$, if any, there exists a pair $x^1, x^2 \in F$ such that $\{j_k\} \subseteq \Delta(x^1, x^2) \subseteq \{j_1, \ldots, j_k\}$, where $\Delta(x^1, x^2) := \{j \in N : x_j^1 \neq x_j^2\}$.*

Theorem 2.7. *Let $\alpha x \leq \alpha_0$, where $\alpha_0 \neq 0$, define a nontrivial facet of* g-mon(P)*. If $\alpha x \leq \alpha_0$ is strongly support reduced with respect to P, then it also defines a facet of P.*

2.2 Properties of the monotone ATS polytope

We now turn to the ATS polytope, P, and examine its relation to its submissive $\tilde{P}$.

Given the complete digraph $G = (V, A)$, we define a bipartite graph $B[G] := (V^+ \cup V^-, E)$ having two nodes, $v^+ \in V^+$ and $v^- \in V^-$, for every node v of G, and an edge $[i^+, j^-] \in E$ for every arc (i, j) of G. It is easy to see that the coefficient matrix of the degree equations coincides with the node-edge incidence matrix of $B[G]$, hence a subset $\tilde{A}$ of the arcs of G indexes a basis of the equality system of P if and only if $\tilde{A}$ induces a spanning tree of $B[G]$.

Note that two edges of $B[G]$ are adjacent if and only if the corresponding arcs of G have either the same tail or the same head. Thus two nodes of $B[G]$ are connected by a path if and only if the corresponding nodes of G are connected by an *alternating path*, i.e., by a path of alternating arc directions.

Now let $\alpha x \leq \alpha_0$ define a nontrivial facet of $\tilde{P}$, and denote $A^0 := \{(i,j) \in A : \alpha_{ij} = 0\}$, $A^+ := A \setminus A^0$, $G^0 := (V, A^0)$, and $G^+ := (V, A^+)$.

Since connectedness is the necessary and sufficient condition for $B[G^0]$ to contain a spanning tree and thus for A^0 to contain a basis of $A^=$, it follows that an inequality $\alpha x \leq \alpha_0$ is support reduced with respect to P if and only if the bipartite graph $B[G^0]$ is connected. Thus when P is the ATS polytope, Theorem 2.5 specializes to the following:

Theorem 2.8. *Let $\alpha x \leq \alpha_0$ be a nontrivial facet-defining inequality for P, that is valid for $\tilde{P}$. Then $\alpha x \leq \alpha_0$ defines a facet of $\tilde{P}$ if and only if the bipartite graph $B[G^0]$ is connected.*

A relevant special case in which $B[G^0]$ is obviously connected arises when

$$G^+ \text{ has an isolated node } h, \text{ and } G^0 \text{ has an arc } a^* \notin \delta(h) \tag{2.1}$$

as in this case the $2n-1$ edges of $B[G^0]$ corresponding to the arcs in $\delta(h) \cup \{a^*\}$ induce a spanning tree of $B[G^0]$. Condition (2.1) is satisfied by many nontrivial facet inducing inequalities for the submissive of P, and is related to the concept of h-canonical form discussed in Section 4.

Suppose now that condition (2.1) is satisfied, guaranteeing that $\alpha x \le \alpha_0$ is support reduced. We next give a sufficient condition for $\alpha x \le \alpha_0$ to be strongly support reduced.

Let $G_h := G - \{h\}$, where h is the isolated node in (2.1). Two arcs of G_h are called *α-adjacent* (in G_h) if they are contained in a tour T of G_h such that $\alpha(T) = \alpha_0$. We define the (undirected) *α-adjacency graph* $G_h^* := (V^*, E^*)$ of G_h, as having a node for every arc in $A^0 \setminus \delta(h)$, and an edge for every pair $a, b \in V^*$ such that arcs a and b are α-adjacent in G_h. It can be shown that, under the assumption (2.1), a sufficient condition for $\alpha x \le \alpha_0$ to be strongly support reduced is that graph G_h^* be connected. This leads to the following characterization that is trivial to verify without constructing G_h^* explicitly and holds in most of the relevant cases.

Theorem 2.9. *Let $\alpha x \le \alpha_0$ define a nontrivial facet of $\tilde{P}$ and a proper face of P. Assume G^+ has two isolated nodes, say h and k. If the bipartite graph $B[G^0 - \{h, k\}]$ is connected, then $\alpha x \le \alpha_0$ is strongly support reduced, hence it defines a facet of P.*

Corollary 2.10. *Let $\alpha x \le \alpha_0$ define a nontrivial facet of $\tilde{P}$ and a proper face of P. If G^+ has three isolated nodes, then $\alpha x \le \alpha_0$ defines a facet of P.*

Similar results apply to the STS polytope as well [5].

This gives an unexpectedly simple answer to an open problem posed by Grötschel and Padberg:

"**Research problem.** Find (reasonable) sufficient conditions which imply that an inequality defining a facet for $\tilde{Q}_T^n$ (the monotone STS polytope) also defines a facet for Q_T^n (the STS polytope)."[18, p. 271]

3 Facet-lifting procedures

In this section we discuss general facet lifting procedures for the ATS polytope, which can be used to obtain new classes of inequalities from known ones. Unlike the traditional lifting procedure which calculates the coefficients of the lifted variables sequentially, often in non-polynomial time, and with outcomes that depend on the lifting sequence, these procedures calculate sequence-independent coefficient values, often obtained by closed form expressions. We concentrate on ATS-specific liftings, and refer the reader to Queyranne and Wang [31] and Chopra and Rinaldi [8] for facet composition and lifting operations that apply to symmetric ATS inequalities only.

The subsection on cloning and clique lifting is based on [4], whereas those on T-lifting, merge lifting, and 2-cycle cloning are based on [13].

3.1 Cloning and clique lifting

Let $\mathcal{F}$ be a given family of valid ATS inequalities, all defining proper faces, and denote by $\mathcal{F}_n$ the restriction of $\mathcal{F}$ to P_n.

Definition 3.1. *Let $\alpha x \le \alpha_0$ be a member of $\mathcal{F}_n$, and h, k any two distinct nodes. Then h and k are called* clones *(with respect to $\alpha x \le \alpha_0$ and $\mathcal{F}$) if:*

(a) *$\alpha_{ih} = \alpha_{ik}$ and $\alpha_{hi} = \alpha_{ki}$ for all $i \in V \setminus \{h, k\}$;*

(b) *$\alpha_{hk} = \alpha_{kh} = \max\{\alpha_{ik} + \alpha_{kj} - \alpha_{ij} : i, j \in V \setminus \{h, k\},\ i \neq j\}$;*

(c) *the inequality $\sum_{(i,j)\in A\setminus\delta(h)} \alpha_{ij}x_{ij} \le \alpha_0 - \alpha_{kh}$ belongs to $\mathcal{F}_{n-1}$.*

If there exists no pair of clones with respect to $\alpha x \le \alpha_0$, the inequality $\alpha x \le \alpha_0$ is said to be primitive.

We call a facet *regular* if it is nontrivial and its defining inequality is not the 2-cycle inequality $x_{ij} + x_{ji} \le 1$ for some $i, j \in V$. The following lemma is used in the proof of Theorem 3.3, the main result of this subsection.

Lemma 3.2 ([1, 4]). *Let $\alpha x \le \alpha_0$ be any inequality defining a regular facet of P, and let $k \in V$ be any node. Then there exists a sequence of $2n-3$ tours $x^{(t)}$ with $\alpha x^{(t)} = \alpha_0$, where each tour $x^{(t)}$ $(t = 1, \ldots, 2n-3)$ is associated with an arc $(i_t, j_t) \in \delta^-(k) \cup \delta^+(k)$ such that $x^{(t)}_{i_tj_t} = 1$ and $x^{(1)}_{i_tj_t} = x^{(2)}_{i_tj_t} = \cdots = x^{(t-1)}_{i_tj_t} = 0$.*

Theorem 3.3. *If all the primitive inequalities of $\mathcal{F}$ define regular facets of the corresponding polytopes P, then so do all the inequalities of $\mathcal{F}$.*

Proof. The proof is by induction on the number q of nodes that are clones. For $q = 0$ the statement is true by assumption. Suppose the theorem holds when the family $\mathcal{F}$ is restricted to inequalities with respect to which the corresponding digraph has no more than $q_0 \ge 0$ clones, and consider any inequality of the original family with $q = q_0+1$ clones. Let $h, k \in V$ be any pair of clones relative to each other, with h as the "new" (q-th) clone. We will now construct a set $X \subset P$ containing $\dim(P)$ affinely independent tours x satisfying $\alpha x = \alpha_0$.

Let $\tilde{\alpha} y \le \tilde{\alpha}_0$ be the member of $\mathcal{F}_{n-1}$ defined at point (c) of Definition 3.1, where $\tilde{\alpha}_0 = \alpha_0 - \alpha_{hk}$. From the induction hypothesis, $\tilde{\alpha} y \le \tilde{\alpha}_0$ defines a regular facet of the ATS polytope $P(\tilde{G})$ associated with the complete digraph $\tilde{G} = (\tilde{V}, \tilde{A})$ induced by $\tilde{V} := V \setminus \{h\}$. Thus there exists a set Y of $(n-1)^2 - 3(n-1) + 1$ affinely independent tours $y \in P(\tilde{G})$ satisfying $\tilde{\alpha} y = \tilde{\alpha}_0$. We initialize the set X by taking for each $y \in Y$, the tour x obtained from y by inserting node h right after k, i.e. by replacing the arc leaving k, say (k, j), with the arcs (k, h) and (h, j). All these tours satisfy $\alpha x = \tilde{\alpha} y + \alpha_{kh} = \tilde{\alpha}_0 + \alpha_{kh} = \alpha_0$, and are easily seen to be affinely independent. Indeed, the above transformation induces a 1-1 correspondence, φ, between $\tilde{A}$ and a subset of A, such that for all $(i, j) \in \tilde{A}$, $x_{\varphi(i,j)} = 1$ if and only if $y_{ij} = 1$. It follows that the affine dependence of X would imply that of Y. In addition, all $x \in X$ satisfy $x_{kh} = 1$, and hence $x_{ij} = 0$ for each $(i, j) \in Q := (\delta^+(k) \cup \delta^-(h) \cup \{(h, k)\}) \setminus \{(k, h)\}$.

Now let $\{y^{(t)} : t = 1, \ldots, 2(n-1) - 3\}$ be a sequence of tours of $\tilde{G}$ with $\tilde{\alpha} y^{(t)} = \tilde{\alpha}_0$, each tour being associated with an arc (i_t, j_t) in $\tilde{Q} := (\delta^-(k) \cup \delta^+(k)) \cap \tilde{A}$ such that $y^{(t)}_{i_tj_t} = 1$ and $y^{(1)}_{i_tj_t} = \cdots = y^{(t-1)}_{i_tj_t} = 0$. The existence of such a sequence follows from Lemma 3.2. For $t = 1, \ldots, 2n-5$, put into X the tour $x^{(t)}$ obtained from $y^{(t)}$ by inserting node h just before node k, i.e. by replacing the arc of $y^{(t)}$ entering k, say (i, k), with arcs (i, h) and (h, k). Clearly $\alpha x^{(t)} = \tilde{\alpha} y^{(t)} + \alpha_{hk} = \tilde{\alpha}_0 + \alpha_{hk} = \alpha_0$. As to the affine independence of the set X, it suffices to note that each $x^{(t)}$ contains an arc $(r_t, s_t) \in Q$ not contained in any previous

$x \in X$ (with $(r_t, s_t) = (i_t, j_t)$ if $i_t = k$, and $(r_t, s_t) = (h, j_t)$ if $j_t = k$). At this point, the set X has $|Y| + 2n - 5 = n^2 - 3n$ members. The last tour put into X is then any tour x^* such that $\alpha x^* = \alpha_0$ and $x^*_{hk} = x^*_{kh} = 0$. To show that such x^* exists, let $(r, s) \in \tilde{A} \setminus \delta(k)$ be an arc for which the maximum in the definition of α_{hk} is attained (see point (b) of Definition 3.1), i.e. such that $\alpha_{hk} = \alpha_{rk} + \alpha_{ks} - \alpha_{rs}$ $(= \max_{i,j}\{\alpha_{ik} + \alpha_{kj} - \alpha_{ij}\})$, and let $y^* \in Y$ be a tour in $\tilde{G}$ such that $\tilde{\alpha} y^* = \tilde{\alpha}_0$ and $y^*_{rs} = 1$ (such a tour exists, since $\tilde{\alpha} y \leq \tilde{\alpha}_0$ defines a nontrivial facet of $P(\tilde{G})$.). Then x^* can be obtained from y^* by inserting h into the arc (r, s), i.e. by replacing (r, s) with the pair $(r, h), (h, s)$. The affine independence of X then immediately follows from the fact that $x^*_{kh} + x^*_{hk} = 0$, while $x_{kh} + x_{hk} = 1$ for all the other members x of X.

Thus $\alpha x \leq \alpha_0$ defines a facet of P. To see that this facet is regular, it suffices to notice that for every $(i, j) \in A$, X contains a tour x such that $x_{ij} = 1$ as well as a tour x' such that $x'_{ij} + x'_{ji} = 0$.□

We now describe a constructive procedure, based on Theorem 3.3, to lift an inequality by cloning some nodes of G. Unlike the well known lifting procedure for general 0-1 polytopes [26], which calculates the lifted coefficients by solving a sequence of integer programs, one for each new coefficient, the procedure lifts simultaneously all the variables corresponding to the arcs incident with a node (or group of nodes), and calculates their coefficients by a closed form expression. Let $G' = (V', A')$ be the complete digraph induced by the node set $V' := V \cup \{n + 1\}$ (where $n = |V|$).

Theorem 3.4. *Suppose the inequality $\alpha x \leq \alpha_0$ defines a regular facet of $P(G)$, and let k be an arbitrary but fixed seed node. Let $\delta_k := \max\{\alpha_{ik} + \alpha_{ki} - \alpha_{ij} : i, j \in A \setminus \delta(k)\}$, and $\alpha'_0 := \alpha_0 + \delta_k$. Moreover, for each $(i, j) \in A'$ let $\alpha'_{ij} := \delta_k$ if $(i, j) \in \{(k, n + 1), (n + 1, k)\}$; $\alpha'_{ij} := \alpha_{ik}$ if $j = n + 1,\ i \neq k$; $\alpha'_{ij} := \alpha_{kj}$ if $i = n + 1,\ j \neq k$; and $\alpha'_{ij} := \alpha_{ij}$ otherwise.*

Then the lifted inequality $\alpha' x' \leq \alpha'_0$ defines a regular facet of $P(G')$.

Consider now the situation arising when the above described lifting procedure is applied to a sequence of nodes according to the following scheme. An initial inequality $\alpha x \leq \alpha_0$ is given, defining a nontrivial facet of the ATS polytope associated with the complete digraph $G^1 = (V^1, A^1)$. For $\ell = 1, \ldots, m$, choose any "seed" node $k^\ell \in V^\ell$, define the enlarged digraph $G^{\ell+1} = (V^{\ell+1}, A^{\ell+1})$ induced by $V^{\ell+1} := V^\ell \cup \{|V^\ell| + 1\}$, compute the coefficients α_{ij} of the new arcs $(i, h) \in A^{\ell+1} \setminus A^\ell$ according to Theorem 3.4, and repeat. W.l.o.g., one can assume that the seed nodes k^ℓ are always chosen in $V^1 \subseteq V^\ell$ (since each node in $V^\ell \setminus V^1$ is a clone of a node in V^1). It is not difficult to show that, for any $k \in V^1$, δ_k does not depend on the iteration in which k is selected as the seed node of the lifting. As a result, a more general lifting procedure based on Theorem 3.3 can be outlined as follows.

Procedure CLIQUE-LIFTING:

1. Let $\alpha x \leq \alpha_0$ be any valid inequality defining a regular facet of $P(G)$.

2. For all $k \in V$, compute $\delta_k := \max_{i,j}\{\alpha_{ik} + \alpha_{kj} - \alpha_{ij}\}$.

3. Define an enlarged complete digraph $G^* = (V^*, A^*)$ obtained from G by replacing each node k with a clique S_k containing $|S_k| \geq 1$ nodes.

4. For all $(i,j) \in A^*$, let $i \in S_{k_i}$ and $j \in S_{k_j}$, and define $\alpha^*_{ij} := \alpha_{k_i,k_j}$ if $k_i \neq k_j$, $\alpha^*_{ij} := \delta_{k_i}$ otherwise.

5. The inequality $\alpha^* x^* \leq \alpha^*_0 := \alpha_0 + \sum_{k \in V} \delta_k(|S_k| - 1)$, where $x^* \in \mathbb{R}^{A^*}$, is then valid and defines a regular facet of $P(G^*)$.

Thus, given a complete digraph $G^* = (V^*, A^*)$, and an induced subgraph $G = (V, A)$ of G^* along with a primitive facet defining inequality $\alpha x \leq \alpha_0$ for $P(G)$, any node of $V^* \setminus V$ can be made a clone of any node of V. Applying this procedure recursively in all possible cloning combinations thus gives rise to $|V|^{|V^*|-|V|}$ distinct lifted inequalities, all facet defining for $P(G^*)$.

Clique lifting carries over to the monotone ATS polytope $\tilde{P}$. Indeed, it can easily be shown that, if the inequality $\alpha x \leq \alpha_0$ of Theorem 3.4 is support-reduced, then so is the lifted inequality $\alpha^* x^* \leq \alpha^*_0$. Therefore, as discussed in Section 2, one has the following

Theorem 3.5. *If $\alpha x \leq \alpha_0$ defines a nontrivial facet of both $P(G)$ and $\tilde{P}(G)$, then $\alpha^* x^* \leq \alpha^*_0$ defines a nontrivial facet of both $P(G')$ and $\tilde{P}(G')$.*

Two earlier lifting procedures for facets of P, introduced in [15], require the presence, in the support of the inequality to be lifted, of a clique satisfying certain conditions on the coefficient values. When these conditions apply, the resulting inequalities are a subset of our family, obtained by cloning a vertex of the above mentioned clique.

More recently, Padberg and Rinaldi [28] have discussed clique-liftability for the symmetric TS polytope Q^V defined on the complete undirected graph with node set V. In their terminology, a facet inducing inequality $cx \leq c_0$ for Q^V is *clique-liftable* if for every $v \in V$ there exists a real number $\alpha(c, v) \geq 0$ depending on c and v, such that for any set W with $W \cap V = \emptyset$, there exists an inequality $c^* y \leq c^*_0$, facet inducing for $Q^{V \cup W}$, whose coefficients are given by $c^*_{uw} := c_{uw}$ if $u, w \in V$, $c^*_{uw} := c_{uv}$ if $u \in V \setminus \{v\}$, $w \in W$, and $c^*_{uv} := \alpha(c, v)$ if $u, w \in W \cup \{v\}$, whereas $c^*_0 := c_0 + |W| \cdot \alpha(c, v)$. From this perspective, the above results mean that in the case of the ATS polytope *all* regular facet-inducing inequalities are clique-liftable (with a definition duly modified to take into account directedness), since for each $v \in V$, $\alpha(c, v)$ exists and is equal to δ_v.

Also related to clique lifting is the *0-node lifting* introduced by Naddef and Rinaldi [24, 25] in their study of the graphical relaxation of the STS polytope (as defined in Chapter 4) and extended to the graphical ATS polytope (as defined in Section 2) by Chopra and Rinaldi [8]. An inequality $\pi x \geq \pi_0$, $\pi \geq 0$, is said to satisfy the *shortest path condition* if, for each $(i,j) \in A$, the π-length of every directed path from i to j is greater or equal to π_{ij} (a necessary condition for an inequality to be nontrivial facet-defining for the graphical ATS polytope [8]). On a complete digraph, this is equivalent to requiring that $\pi_{ij} \leq \pi_{ik} + \pi_{kj}$ for all triples $i, j, k \in V$ of distinct nodes. Moreover, the inequality $\pi x \geq \pi_0$ is said to be *tight triangular* whenever it satisfies the shortest path condition and, for each $k \in V$, there exists $(i,j) \in A \setminus \delta(k)$ such that $\pi_{ij} = \pi_{ik} + \pi_{kj}$. 0-node lifting then consists of replacing a given node v by a clique of clones connected one to each other by arcs with $\pi_{v'v''} = 0$ (hence the name 0-node lifting). This operation was first studied in the context of the graphical STS polytope by Naddef and Rinaldi [24, 25], who also provided necessary and sufficient conditions under which the 0-node lifting of a tight triangular inequality preserves the facet-defining property with respect to the standard (sometimes called *flat*) STS polytope. At a later time, Chopra and Rinaldi [8] proved that 0-node lifting always preserves the facet-defining property for the graphical ATS polytope, while they did not address the question of whether this is also true with respect to the flat ATS polytope. Observe that an inequality $\pi x \geq \pi_0$ can trivially be put in the form $\alpha x \leq \alpha_0$ by simply setting $(\alpha, \alpha_0) := -(\pi, \pi_0)$, hence $\pi x \geq \pi_0$ is tight triangular if and only if $\alpha \leq 0$ and $\alpha_{vv} = 0$ for all $v \in V$. From this perspective, 0-node lifting reduces to clique lifting, thus it has the nice property (not known to be shared by its symmetric counterpart) of always preserving the facet-defining quality with respect to the flat) ATS polytope.

3.2 T-lifting

T-lifting aims at extending the following construction [16]: given the SEC $x(S, S) \leq |S| - 1$ associated with a subset S with cardinality k, $2 \leq k \leq n - 2$, choose three distinct nodes, say $p, q \in V \setminus S$ and $w \in S$, and add the term $x_{pw} + x_{wq} + x_{pq}$ to the left-hand side of the SEC, and 1 to its right-hand side, thus obtaining a T_k inequality which is known to be facet-defining for P_n for $k \neq n - 3$.

Now let $\alpha x \leq \alpha_0$ be any valid inequality for P_n. To simplify notation, in the sequel we will often use notation α_{kk} ($= \max\{\alpha_{ik} + \alpha_{kj} - \alpha_{ij} : (i, j) \in A \setminus \delta(k)\}$) to represent the clique-lkifting coefficient δ_k defined in Theorem 3.4. We assume:

(a) $\alpha_{ij} \geq 0$ and integer for all $(i, j) \in A$;

(b) there exist two distinct *isolated* nodes p, q, i.e., $\alpha_{ih} = \alpha_{hi} = 0$ for all $i \in V$ $(h = p, q)$;

(c) a node $w \in V \setminus \{p, q\}$ with $\alpha_{ww} = 1$ exists, such that $\alpha^w y \leq \alpha_0 - 1$ (where α^w is the restriction of α onto $G(V \setminus \{w\})$) is valid and defines a nonempty face of P_{n-1}.

Note that the above conditions imply $\alpha_{ij} \in \{0, 1\}$ for all $(i, j) \in \delta(w)$. Starting with $\alpha x \leq \alpha_0$, we define the *T-lifted inequality* $\beta x := \alpha x + x_{pw} + x_{wq} + x_{pq} \leq \beta_0 := \alpha_0 + 1$, obtained by adding (and then rounding) the following 6 valid inequalities, all weighed by 1/2: $x(\delta^+(i)) \leq 1$ for $i \in \{p, w\}$; $x(\delta^-(j)) \leq 1$ for $j \in \{w, q\}$, $\alpha x \leq \alpha_0$, and the inequality obtained from $\alpha^w y \leq \alpha_0 - 1$, see condition (c) above, by re-introducing node w as a clone of p.

We now address the important question of whether $\beta x \leq \beta_0$ is facet inducing, assuming that this is the case for the starting inequality. An example of an application that does not produce a facet corresponds to T_k inequalities on $n = k + 3$ nodes. Actually, T-lifting can in some cases even produce an inequality with *non-maximal* coefficients, i.e., some β_{ij} can singularly be increased without affecting the validity of $\beta x \leq \beta_0$.

Let $\overline{V} := V \setminus \{p, w, q\}$, $J^+ := \{j \in \overline{V} : \alpha_{wj} > 0\}$, $J^0 := \overline{V} \setminus J^+$, $I^+ := \{i \in \overline{V} : \alpha_{iw} > 0\}$, and $I^0 := \overline{V} \setminus I^+$. Moreover, let $F := \{x \in P_n : \beta x = \beta_0\}$ denote the face induced by $\beta x \leq \beta_0$.

Theorem 3.6. *Let $\alpha x \leq \alpha_0$ satisfy conditions (a) to (c) above, and let the inequality $\alpha^q y \leq \alpha_0 - 1$ induce a nontrivial facet of P_{n-1}. Assume that all the coefficients β_{pj}, $j \in J^+$, and β_{iq}, $i \in I^+$, are maximal with respect to the T-lifted inequality $\beta x \leq \beta_0$. Then F is* *either a facet of P_n, or else a face of dimension* $\dim(P_n) - 2$ *contained in the hyperplane* $H := \{x \in \mathbb{R}^A : x_{qp} = x(p, J^*) + x(I^*, q)\}$, *where* $J^*, I^* \subseteq \overline{V}$ *are such that* $J^0 \subseteq J^*$ *and* $I^0 \subseteq I^*$.

In the case of a T_k inequality on $n = k + 3$ nodes, the non-maximal face induced by $\beta x \leq \beta_0$ is contained in the hyperplane H defined by the equation $x_{qp} = x_{pz} + x_{zq}$, where z is the unique node in $V \setminus (S \cup \{p, q\})$. The existence of H is however a pathological occurrence for T-lifting, as shown in the following.

Corollary 3.7. *Under the assumptions of Theorem 3.6, F is a facet of P_n if there exists $x^* \in F$ such that $x^*_{qp} = 0$ and $x^*_{ij} = 1$ for a certain $(i, j) \in (p, J^0) \cup (I^0, q)$.*

It is not hard to see that the above corollary applies, e.g., when there exists $r \in \overline{V}$ such that $\alpha_{ir} = \alpha_{ri} = 0$ for all i, and $(J^0 \cup I^0) \setminus \{r\} \neq \emptyset$. Therefore the most restrictive requirement in Theorem 3.6 is the coefficient maximality. The reader is referred to [13] for a discussion of necessary and sufficient conditions for such a maximality.

A large class of inequalities can be obtained by iterating the application of T-lifting on clique tree inequalities. Let $C = (W_1, \ldots, W_r)$ be a clique tree on $n \geq 7$ nodes, and let $\alpha x \leq \alpha_0$ be the associated facet-inducing inequality. Now let $T := \{p_j, w_j, q_j : j = 1, \ldots, m\}$ contain $3m$ nodes such that, for $j = 1, \ldots, m$, nodes p_j and q_j are not covered by C, whereas w_j belongs to a tooth and to no handles. Moreover, assume that each tooth contains at least one node not belonging to any handle nor to $\{w_1, \ldots, w_m\}$. Then for $n \geq 7$ and $m \geq 0$, the *T-clique tree inequality* $\sum_{i=1}^{r} x(W_i, W_i) + \sum_{j=1}^{m} (x_{p_j w_j} + x_{w_j q_j} + x_{p_j q_j}) \leq \sigma(C) + m$ is valid and facet-inducing for P_n (except when either $r = 1$, $m = 1$ and $n = |W_1| + 3$, or $r = 1$, $m = 2$ and $n = |W_1| + 4$).

Different classes of facet-inducing inequalities can be obtained similarly, by applying T-lifting to different facet-defining $\alpha x \leq \alpha_0$ such as, e.g., D_k^+ and D_k^--inequalities.

3.3 Merge lifting

Merge lifting is an operation that produces (under appropriate conditions) a facet inducing inequality for P_{n+1} starting from two 'almost identical' inequalities for P_n. Roughly speaking, the new inequality is obtained by swapping the coefficients of arcs in $\delta^+(h)$ and $\delta^-(h)$ for a certain node h (where $h = n$ is assumed for the sake of easy notation).

More specifically, we are given two valid inequalities for P_n, say $\alpha x \leq \alpha_0$ and $\beta x \leq \beta_0$, which are 'almost identical' in the sense that $\alpha_{ij} = \beta_{ij}$ for all $(i, j) \in A \setminus \delta(n)$. For a real parameter ω, we define an inequality $\gamma x' \leq \omega$ for P_{n+1} as follows: $\gamma_{ij} := \alpha_{ij} (= \beta_{ij})$, for all $i, j \in A \setminus \delta(n)$, $\gamma_{n,n+1} := \omega - \alpha_0, \gamma_{n+1,n} := \omega - \beta_0$, whereas for all $i \in V \setminus \{n\}$ we set $\gamma_{in} := \alpha_{in}, \gamma_{n+1,i} := \alpha_{ni}, \gamma_{ni} := \beta_{ni}, \gamma_{i,n+1} := \beta_{in}$ (see Figure 3.1 for an illustration). By construction, the above inequality is satisfied by all $x' \in P_{n+1}$ such that $x'_{n,n+1} + x'_{n+1,n} = 1$, independently of ω. It then follows that $\gamma x' \leq \omega$ is valid for P_{n+1} if and only if ω is chosen greater or equal to $\gamma_0 := \max\{\gamma x' : x' \in P_{n+1},\ x'_{n,n+1} = x'_{n+1,n} = 0\}$. When $\omega = \gamma_0$, we say that $\gamma x' \leq \omega$ has been obtained from $\alpha x \leq \alpha_0$ and $\beta x \leq \beta_0$ through *merge lifting*.

It can be proved that $\gamma x' \leq \gamma_0$ defines a facet of P_{n+1} if $\alpha x \leq \alpha_0$ is facet defining for P_n, and $\beta x \leq \beta_0$ satisfies a certain regularity condition [13]. Observe that node cloning can be

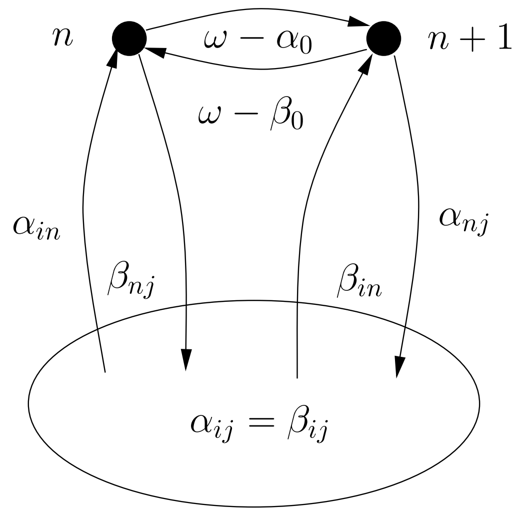


Figure 3.1: Illustration of the merge lifting operation.

viewed as a special case of merge lifting, arising when $(\alpha, \alpha_0) = (\beta, \beta_0)$; in this case, γ_0 can easily be computed as $\alpha_0 + \alpha_{nn}$.

As an example of merge lifting application, let $\alpha x \leq \alpha_0$ be the inequality associated with any clique tree $C = (W_1, \ldots, W_r)$ that leaves node n uncovered, and let $\beta x \leq \beta_0$ be the one associated with the clique tree C' obtained from C by including node n into, say, $W_1 \setminus (W_2 \cup \ldots \cup W_r)$. Therefore $\alpha x := \sum_{i=1}^{r} x(W_i, W_i) \leq \alpha_0 := \sigma(C)$, and $\beta x := x(W_1 \cup \{n\}, W_1 \cup \{n\}) + \sum_{i=2}^{r} x(W_i, W_i) \leq \beta_0 := \sigma(C) + 1$. Both inequalities are known to define facets of P_n. By construction, the two inequalities are suitable for merge lifting. The inequality $\gamma x' \leq \omega$ is in this case $\sum_{i=1}^{r} x'(W_i, W_i) + x'(n, W_1) + x'(W_1, n+1) + (1+\theta)x'_{n,n+1} + \theta x'_{n+1,n} \leq \sigma(C) + 1 + \theta$, where we have defined $\theta := \omega - \sigma(C) - 1$ to simplify notation. For sufficiently large δ, this inequality is valid for P_{n+1}. In order to have a facet, however, one has to choose not too large a value for θ, namely $\theta = 1$ if W_1 is a tooth of C, and $\theta = 0$ if W_1 is a handle of C (in this latter case, if C is a comb we obtain a C2 inequality). Notice that, unlike the starting inequalities $\alpha x \leq \alpha_0$ and $\beta x \leq \beta_0$, the merge-lifted inequality $\gamma' x \leq \gamma_0$ is asymmetric.

The iterative application of merge lifting on the T-clique tree inequality defined in the previous subsection, leads to large classes of facets. We describe one these classes, containing clique tree, C2, and T_k inequalities as special cases. Let C be a given clique tree, $T = \{p_j, q_j, w_j : j = 1, \ldots, m\}$ be as defined in the previous subsection, and let $Z = \{a_j, b_j : j = 1, \ldots, k\}$ contain $2k$ nodes not covered by C nor by T. Each pair a_j, b_j is associated with a distinct clique of C, say $W_{i(j)}$, and with a value θ_j (with $\theta_j = 1$ if $W_{i(j)}$ is a tooth of C,

$\theta_j = 0$ otherwise). Then for $n \geq 7$, $m \geq 0$ and $k \geq 0$ the following *ZT-clique tree inequality*

$$\sum_{i=1}^{r} x(W_i, W_i) + \sum_{j=1}^{m} (x_{p_j w_j} + x_{w_j q_j} + x_{p_j q_j}) +$$
$$\sum_{j=1}^{k} \Big(\, x(a_j, W_{i(j)}) + x(W_{i(j)}, b_j) + (1+\theta_j)\, x_{a_j b_j} + \theta_j \, x_{b_j a_j} \, \Big) \leq \sigma(C) + m + \sum_{j=1}^{k} (1+\theta_j)$$

is valid and facet-inducing (except in a few pathological cases) for P_n.

3.4 Two-cycle cloning

We now describe 2-cycle cloning, an operation that under appropriate conditions constructs a facet-inducing inequality for P_{n+2} starting from a facet of P_n. This operation is related to the *edge cloning* procedure proposed for the symmetric TSP by Naddef and Rinaldi [25], and analyzed by Chopra and Rinaldi [8] in the context of symmetric ATS inequalities. However, 2-cycle cloning is ATS-specific, in that it allows one to lift both symmetric and asymmetric inequalities.

A 2-cycle is simply a cycle of length 2. Let $\alpha x \leq \alpha_0$ be a valid inequality for P_n. Given two distinct nodes h and k such that $\Delta(h,k) := (\alpha_{hh} + \alpha_{kk}) - (\alpha_{hk} + \alpha_{kh}) > 0$, let $\alpha^* x' \leq \alpha^*_0 := \alpha_0 + \alpha_{hh} + \alpha_{kk}$ be the inequality for P_{n+2} obtained from $\alpha x \leq \alpha_0$ by adding nodes $h' := n+1$ and $k' := n+2$ as clones of h and k, respectively. We define the following (not necessarily valid) inequality for P_{n+2}:

$$\beta x' := \alpha^* x' - \Delta(h,k)\, x'_{hh'} + x'_{h'h} + x'_{kk'} + x'_{k'k} \leq \beta_0 := \alpha^*_0 - \Delta(h,k) = \alpha_0 + \alpha_{hk} + \alpha_{kh}$$

and say that $\beta x' \leq \beta_0$ is obtained from $\alpha x \leq \alpha_0$ by *cloning the 2-cycle induced by h and k*. In other words, $\beta x' \leq \beta_0$ is obtained from $\alpha^* x' \leq \alpha^*_0$ by adding the invalid constraint $-(x'_{hh'} + x'_{h'h} + x'_{kk'} + x'_{k'k}) \leq -1$ weighted by $\Delta(h,k)$. Note that, although similar, nodes h and h', as well as k and k', are no longer clones with respect to $\beta x' \leq \beta_0$ because of the coefficient reduction.

Theorem 3.8. *Assume that $\alpha x \leq \alpha_0$ defines a regular facet of P_n. If $\beta x' \leq \beta_0$ is valid, then it defines a regular facet of P_{n+2}.*

By construction, $\beta x' \leq \beta_0$ is valid for P_{n+2} if and only $\alpha^* x' \leq \alpha^*_0 - \Delta(h,k)$ holds for all $x' \in P_{n+2}$ such that $x'_{hh'} = x'_{h'h} = x'_{kk'} = x'_{k'k} = 0$. In several cases, α is integer and $\Delta(h,k) = 1$, hence the above condition simply requires that no $x' \in P_{n+2}$ with $\alpha^* x' = \alpha^*_0$ exists, such that $x'_{hh'} = x'_{h'h} = x'_{kk'} = x'_{k'k} = 0$. Easy-to-check conditions for the validity of $\beta x' \leq \beta_0$ were given recently by Caprara, Fischetti and Letchford [7] in case a rank-1 Chvátal truncation proof of validity is available for $\alpha x \leq \alpha_0$.

A relevant application of 2-cycle cloning arises when $\alpha x \leq \alpha_0$ is the ZT-clique tree inequality of the previous subsection, and the two nodes h, k belong to a 2-node *pendant* tooth (i.e., to a tooth intersecting only one handle). This enlarges the ZT-clique tree class so as to cover the Padberg-Hong [27] *chain inequalities*, a special case arising when $\alpha x \leq \alpha_0$ is associated with a comb and 2-cycle cloning is iterated on a single pendant tooth.

From Theorem 3.8, the members of this new class define facets of P_n (this extends the corresponding result for chain inequalities, due to Queyranne and Wang [31]). Moreover, one can show that the 2-cycle induced by the pair p, q chosen for T-lifting can always be cloned, in the sense that the resulting inequality is valid (and hence facet-inducing). The same holds for the node pairs a_j, b_j of ZT-clique tree inequalities. Therefore, one can make the class of ZT-clique tree inequalities even larger by iteratively applying 2-cycle cloning, thus obtaining inequalities that generalize properly clique tree, C2, T_k, chain inequalities, as well as the CAT inequalities [1] described in Section 5

Other examples of application of 2-cycle cloning are given in [7], where extended versions of C3 inequalities and of SD inequalities (the latter described in Section 6) are analyzed.

4 Equivalence of inequalities and canonical forms

We will consider only valid inequalities with rational coefficients defining proper, nonempty faces of P. Two given inequalities $\alpha x \leq \alpha_0$ and $\alpha' x' \leq \alpha'_0$ are said to be *equivalent* if one of them is a linear combination of the other and the degree constraints, i.e., if there exist $\rho \in \mathbb{R}_+$ and $u_i, v_i \in \mathbb{R}$ $(i \in V)$ such that $\alpha'_0 = \rho\alpha_0 + \sum_{i \in V}(u_i + v_i)$ and, for all $(i,j) \in A$, $\alpha'_{ij} = \rho\alpha_{ij} + u_i + v_j$. It is well known that $\alpha x \leq \alpha_0$ and $\alpha' x \leq \alpha'_0$ define the same facet of P (assuming that they are both facet inducing) if and only if they are equivalent.

As in the previous section, for all $k \in V$ we define $\alpha_{kk} := \max\{\alpha_{ik} + \alpha_{kj} - \alpha_{ij} : (i,j) \in A \setminus \delta(k)\}$. Note that equivalence transformations affect the values α_{kk} (which are the same as the values δ_k defined in Theorem 3.4) the same way as the other coefficients α_{ij}, i.e., if $\rho > 0$ and $\alpha'_{ij} = \rho\alpha_{ij} + u_i + v_j$, for all $(i,j) \in A$, then also $\alpha'_{kk} = \rho\alpha_{kk} + u_k + v_k$ for each $k \in V$. Indeed, $\alpha'_{kk} := \max_{i,j}\{\alpha'_{ik} + \alpha'_{kj} - \alpha'_{ij}\} = \max_{i,j}\{\rho(\alpha_{ik} + \alpha_{kj} - \alpha_{ij}) + u_k + v_k\} = \rho \max_{i,j}\{\alpha_{ik} + \alpha_{kj} - \alpha_{ij}\} + u_k + v_k = \rho\alpha_{kk} + u_k + v_k$.

We will analyze next the notion of h-canonical form introduced in Balas and Fischetti [3] (see also Remark 9.2 in [17]).

Definition 4.1. *An inequality $\beta x \leq \beta_0$ is said to be in h-*canonical form *with respect to any given $h \in V$ if:*

(i) *$\beta_{ih} = \beta_{hi} = 0$ for all $i \in V \setminus \{h\}$.*

(ii) *$\beta \geq 0$ and there exists $(r,s) \in A$ with $r, s \neq h$ and $\beta_{rs} = 0$,*

(iii) *the coefficients β_{ij} and the right hand side β_0 are relatively prime integers.*

The h-canonical form of a given inequality $\alpha x \leq \alpha_0$ is an inequality $\beta x \leq \beta_0$ in h-canonical form, which is equivalent to $\alpha x \leq \alpha_0$. Any inequality $\alpha x \leq \alpha_0$ can be put in this form simply by defining $\beta_{ij} := \sigma[\alpha_{ij} + (\alpha_{hh} - \alpha_{ih}) + (-\alpha_{hj})] = \sigma[\alpha_{hh} - (\alpha_{ih} + \alpha_{hj} - \alpha_{ij})] \geq 0$ for all $i, j \in V$, and $\beta_0 := \sigma[\alpha_0 + \sum_{i \in V}(\alpha_{hh} - \alpha_{ih} - \alpha_{hi})]$, where $\sigma > 0$ is a scaling factor chosen so as to satisfy the normalization condition (iii).

Theorem 4.2. *Two inequalities $\alpha x \leq \alpha_0$ and $\alpha' x \leq \alpha'_0$ are equivalent if and only if their h-canonical forms $\beta x \leq \beta_0$ and $\beta' x \leq \beta'_0$ are the same.*

We now state three interesting properties of the h-canonical form.

Theorem 4.3. *Let the inequality $\beta x \leq \beta_0$ be in h-canonical form for some $h \in V$. If there exists a symmetric inequality $\alpha x \leq \alpha_0$ equivalent to $\beta x \leq \beta_0$, then $\beta x \leq \beta_0$ is symmetric.*

Theorem 4.4. *If two given nodes s and t are clones with respect to some inequality $\alpha x \leq \alpha_0$, then s and t are also clones with respect to the h-canonical form $\beta x \leq \beta_0$ of $\alpha x \leq \alpha_0$.*

Thus the h-canonical form conspicuously exhibits all clones, some of which could be hidden in other formulations. In addition, when checking equivalence among the members of a given family of inequalities through transformation to h-canonical form, we can restrict ourselves to considering only the primitive inequalities of the family.

Theorem 4.5. *Inequalities in h-canonical form are support reduced.*

As a consequence of the above theorem, if $\alpha x \leq \alpha_0$ defines a nontrivial facet of P, then its h-canonical form $\beta x \leq \beta_0$ defines a facet of both P and $\tilde{P}$ (see Theorem 2.5). Note that this is not necessarily true of other inequalities equivalent to $\alpha x \leq \alpha_0$, some of which can even be invalid for $\tilde{P}$.

Finally we note that if $\alpha x \leq \alpha_0$ defines the *improper* face of P, i.e., if for some multipliers u_i, v_j we have $\alpha_{ij} := u_i + v_j$, $(i,j) \in A$, and $\alpha_0 := \sum_i (u_i + v_i)$, then its h-canonical form $\beta x \leq \beta_0$ has $\beta = 0$ and $\beta_0 = 0$. Indeed, $\beta_{ij} = \sigma(\alpha_{hh} - \alpha_{ih} - \alpha_{hj} + \alpha_{ij}) = 0$, $\forall i, j \in V$ (note that $\alpha_{hh} = \max_{i,j}\{\alpha_{ih} + \alpha_{hj} - \alpha_{ij}\} = u_h + v_h$).

The use of h-canonical forms goes beyond determining whether two given ATS inequalities are equivalent. In particular, Queyranne and Wang [29] exploited extensively h-canonical forms to establish whether a given inequality $\alpha x \leq \alpha_0$ defines a facet of the ATS polytope. Indeed, according to the so-called *indirect method* [18] the proof of the latter consists of showing that any valid ATS inequality $\beta x \leq \beta_0$ such that $\{x \in P : \alpha x = \alpha_0\} \subseteq \{x \in P : \beta x = \beta_0\} \subset P$ is indeed equivalent to $\alpha x \leq \alpha_0$, a property which is easier to establish if one assumes without loss of generality that both $\alpha x \leq \alpha_0$ and $\beta x \leq \beta_0$ are in h-canonical form with respect to some (arbitrary) node h.

We conclude this subsection by observing that the h-canonical form has the disadvantage of depending on h, which can cause some problems when comparing two members of a family of (as opposed to single) inequalities. Thus, when choosing h as a particular vertex of the first inequality, one should consider all possible roles for h in the second inequality. A different canonical form, which does not depend on the node h, has been studied in [4]. Roughly speaking, the dependence on node h is eliminated by adding the h-canonical forms for all $h \in V$. More precisely, the *canonical form* $\beta x \leq \beta_0$ of a given inequality $\alpha x \leq \alpha_0$ is defined as follows: Let first $\Delta := \sum_{h \in V} \alpha_{hh}$, $r_i := \sum_{j \in V} \alpha_{ij}$ for $i \in V$, $c_j := \sum_{i \in V} \alpha_{ij}$ for $j \in V$, $\gamma_0 := n\Delta + n\alpha_0 - \sum_{i \in V}(r_i + c_i)$, and $\gamma_{ij} := \Delta + n\alpha_{ij} - r_i - c_j$ for $i, j \in V$, and then set $\beta_0 := \sigma(\gamma_0 - n\varepsilon)$ and $\beta_{ij} := \sigma(\gamma_{ij} - \varepsilon)$ for $i, j \in V$, where $\varepsilon := \min\{\gamma_{ij} : i, j \in V\}$ and $\sigma > 0$ is a scaling factor making the coefficients β_{ij} and β_0 relatively prime integers.

As an example, consider the following T_2 inequality, defined on a digraph with 4 nodes (see Figure 4.1): $\alpha x := x_{12} + x_{13} + x_{23} + x_{24} + x_{42} \leq 2$. Its canonical form is obtained by computing $(\alpha_{hh}) = (1, 2, 1, 1)$, $\Delta = 5$, $(r_i) = (3, 4, 1, 2)$, $(c_j) = (1, 4, 3, 2)$, $\varepsilon = 0$ and $\sigma = 1$, leading to the inequality $\beta x := 2(x_{12} + x_{23} + x_{34} + x_{41}) + 3(x_{13} + x_{31} + x_{24} + x_{42}) \leq 8$.

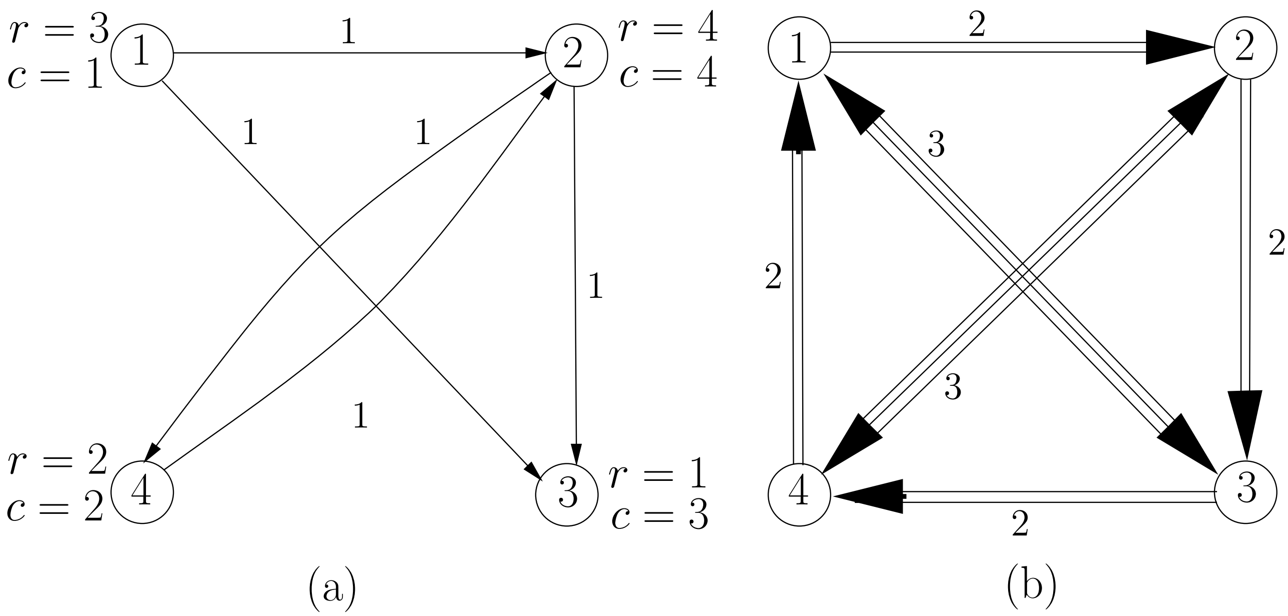


Figure 4.1: A T_2 inequality (a) and its canonical form (b).

Theorem 4.6. *Theorems 4.2 to 4.4 remain valid if the h-canonical form is replaced by the canonical form of the inequality involved.*

An example of application of canonical form is given in Section 6, where it is used to point out the existence of equivalent members in the class of the SD inequalities.

5 Odd closed alternating trail inequalities

Odd Closed Alternating Trail (CAT) inequalities were introduced in [1]. An *assignment* in the complete digraph G is a spanning subgraph that is the node-disjoint union of directed cycles. A frequently used relaxation of the ATS problem is the *Assignment Problem* (AP), whose constraint set is obtained from the ATS formulation (0.1)-(0.3) by removing all SEC's (0.3). An *asymmetric* assignment is one that contains no directed 2-cycles, i.e. that satisfies the 2-node SEC's:

$$x_{ij} + x_{ji} \leq 1 \qquad \text{for all } (i,j) \in A,\ i < j. \tag{5.1}$$

The *Asymmetric Assignment* (AA) *polytope* P_A is the convex hull of incidence vectors of asymmetric assignments, i.e.

$$P_A := \text{conv}\ \{x \in \{0,1\}^A : x \text{ satisfies } (0.1), (0.4), \text{ and } (5.1\}.$$

Unlike the standard AP, minimizing a linear function over P_A is an NP-hard problem [32].

An arc set that is the node disjoint union of directed cycles and/or paths will be called an (asymmetric) *partial assignment*. The *Asymmetric Partial Assignment* (APA) polytope on G, or monotonization of P_A, is

$$\tilde{P}_A := \text{conv}\{x \in \{0,1\}^A : (5.1),\ \text{and}\ x(\delta^+(v)) \leq 1,\ x(\delta^-(v)) \leq 1 \text{ for all } v \in V\}.$$

In this section we describe classes of facet inducing inequalities for the ATS polytope P that correspond to facets of the AA polytope P_A. They are associated with certain subgraphs of G (called closed alternating trails) that correspond to odd holes of the intersection graph of the coefficient matrix of the AA polytope. The reader is referred to [1] for proofs of the main results.

Let $G^* = (V, E)$ be the intersection graph of the coefficient matrix of the system (0.1)-(0.4) and (5.1). Then G^* has a vertex for every arc of G; and two vertices of G^* corresponding, say, to arcs (p, q) and (r, s) of G, are joined by an edge of G^* if and only if either $p = r$, or $q = s$, or $p = s$ and $q = r$. Two arcs of G will be called G^*-*adjacent* if the corresponding vertices of G^* are adjacent. Clearly, there is a 1-1 correspondence between APA's in G and vertex packings (independent vertex sets) in G^*, hence the APA polytope $\tilde{P}_A$ defined on G is identical to the vertex packing polytope defined on G^* .

We define an *alternating trail* in G as a sequence of distinct arcs $T = (a_1, \ldots, a_t)$, such that for $k = 1, \ldots, t-1$, a_k and a_{k+1} are G^*-adjacent, but a_k, a_ℓ, $\ell > k+1$, are not (with the possible exception of a_1 and a_t). If a_1 and a_t are G^*-adjacent, the alternating trail T is *closed*. An arc $a_k = (p, q)$ of T is called *forward* if p is the tail of a_{k-1}, or q is the head of a_{k+1}, or both; it is called *backward* if q is the head of a_{k-1}, or p is the tail of a_{k+1}, or both. The definition of an alternating trail T implies that the direction of the arcs of T alternates between forward and backward, except for pairs a_k, a_{k+1} that form a directed 2-cycle entered and exited by T through the same node, in which case a_k and a_{k+1} are both forward or both backward arcs. It also implies that all the 2-cycles of T are node-disjoint (since two 2-cycles of G that share a node define a 4-cycle in G^*). Notice that T traverses a node at most twice, and the number of arcs of T incident from (incident to) any node is at most 2.

Let $G[T]$ denote the subdigraph of G generated by T, i.e., $G[T]$ has T as its arc set, and the endpoints of the arcs of T as its node set. Further, for any $v \in N$, let $\deg_T^+(v)$ and $\deg_T^-(v)$ denote the outdegree and indegree of node v in $G[T]$, respectively.

The *length* of an alternating trail is the number of its arcs. An alternating trail will be called *even* if it is of even length, *odd* if it is of odd length. We will be interested in *closed alternating trails* (CATs for short) of odd length, the reason being the following.

Proposition 5.1. *There is a 1-1 correspondence between odd CATs in G and odd holes (chordless cycles) in G^*.*

It is well known [26] that the odd holes of an undirected graph give rise to facets of the vertex packing polytope defined on the subgraph generated by the odd hole, and that these facets in turn can be lifted into facets of the polytope defined on the entire graph. In order to make the lifting procedure conveniently applicable to the particular vertex packing polytope associated with G^*, we need some structural information concerning adjacency relations on G^*.

Let T be a CAT in G. A node of $G[T]$ will be called a *source* if it is the common tail of two arcs of T, and a *sink* if it is the common head of two arcs of T. A node of $G[T]$ can thus be a source, or a sink, or both, or none. A node of $G[T]$ that is neither a source nor a sink will be called *neutral.* A neutral node is incident only with the two arcs of a 2-cycle. A 2-cycle will be called *neutral* if it contains a neutral node. Several odd CATs are illustrated in Figure 5.1. The sources and sinks of $G[T_1]$ are nodes 1, 2 and 2, 4, respectively, while 3 is neutral. $G[T_2]$ has three neutral nodes, 1, 4 and 6, while nodes 2, 3 and 5 are both sources and sinks. $G[T_3]$ has sources 1 and 4, sinks 2, 3 and 4, while 5 is a neutral node. Further, $G[T_1]$, $G[T_2]$ and $G[T_3]$ have one, three and one neutral 2-cycles, respectively, and $G[T_3]$ also has a non-neutral 2-cycle.

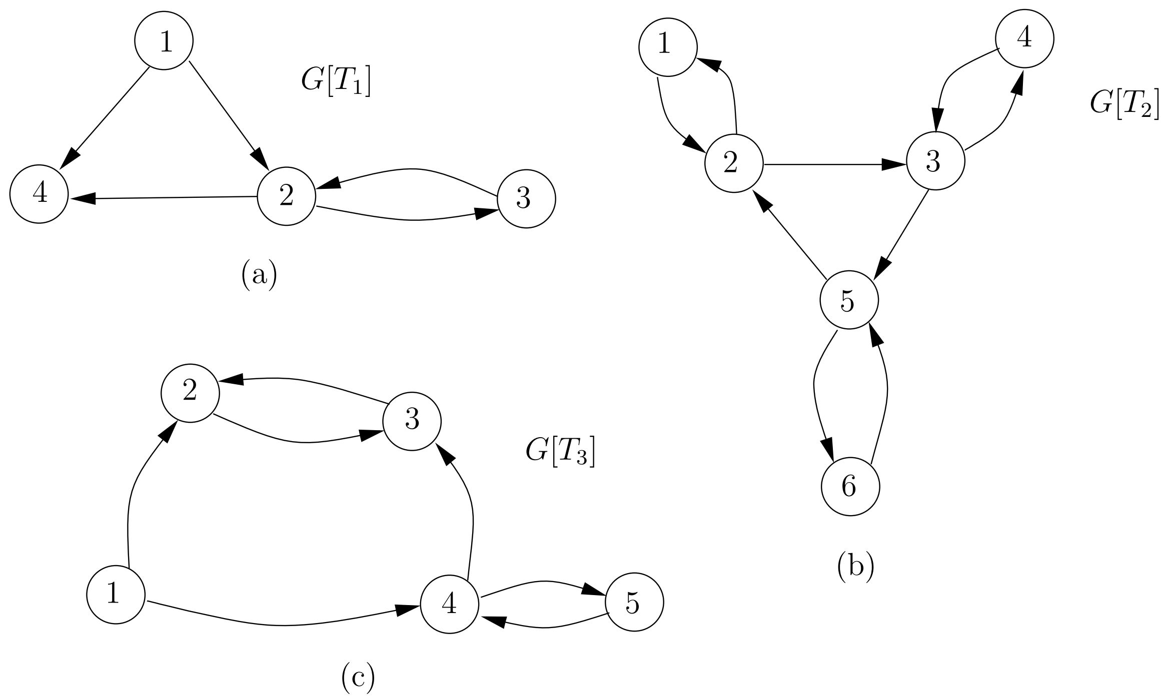


Figure 5.1: Some odd CAT's.

A *chord* of a CAT T is an arc $a \in A \setminus T$ joining two nodes of $G[T]$. One can distinguish between several types of chords, of which the most important ones are those of *type 1.* A chord (u, v) is of type 1 if it joins a source to a sink (i.e. $deg_T^+(u) = deg_T^-(v) = 2$).

We are now ready to characterize the class of facet inducing inequalities of the APA polytope $\tilde{P}_A$ associated with odd CATs. We consider first the subgraph generated by an odd CAT.

Proposition 5.2. *Let T be an odd CAT of length t, and let $\tilde{P}_A(G[T])$ be the APA polytope defined on $G[T]$. Then the inequality*

$$x(T) \leq (t-1)/2 \tag{5.2}$$

defines a facet of both $\tilde{P}_A(G[T])$ and $\tilde{P}(G[T])$.

Next we apply sequential lifting to (5.2) and identify inequalities of the form

$$x(T) + \sum(\alpha_{ij}x_{ij} : (i,j) \in A \setminus T) \leq (t-1)/2 \tag{5.3}$$

that define facets of $\tilde{P}_A$. We take the arcs of $A \setminus T$ in any order such that the chords of Type 1 precede all other arcs. It is then not hard to show that, for any such lifting sequence, the lifting coefficient of each arc $a \in A \setminus T$ is $\alpha_a = 1$ if α is a chord of Type 1, and $\alpha_a = 0$ otherwise. This leads to the following result.

Theorem 5.3. *Let T be an odd CAT of length t, and let C_1 be the set of its chords of type 1. Then the* odd CAT inequality

$$x(T \cup C_1) \leq (t-1)/2 \tag{5.4}$$

defines a facet of both $\tilde{P}_A$ and $\tilde{P}$.

Proof outline. One can show that a chord of type 1, if lifted first, gets a coefficient of 1. It then follows by induction that all remaining arcs of C_1, if lifted before those in $A \setminus (T \cup C_1)$, get a coefficient of 1. Finally, all arcs lifted after those in C_1 get a coefficient of 0.□

If a different lifting sequence is used, one can get facet inducing inequalities with a coefficient of 1 for some chords not in C_1 and a coefficient of 0 for some chords in C_1; but these inequalities do not share the simplicity and regularity of the family (5.4) (see [1] for details).

The arc sets corresponding to the support of each odd CAT inequality in the digraph with 6 nodes are shown (up to isomorphism) in Figure 5.2, with the arcs of T and C_1 shown in solid and shaded lines, respectively.

Remark 5.4. *The odd CAT inequalities (5.4) have Chvátal rank 1.*

(For a definition and discussion of Chvátal rank see e.g. [9].)

Proof. Each inequality (5.4) associated with an odd CAT T can be obtained by adding the outdegree equations (0.1) associated with each source of $G[T]$, the indegree equations (0.4) associated with each sink of $G[T]$, and the inequalities (5.1) associated with each two-cycle of $G[T]$; then dividing by two the resulting inequality and rounding down all coefficients to the nearest integer.□

We finally address the polytopes P_A and P, and show that, with the exception of one special case for $n = 5$ and one for $n = 6$, the odd CAT inequality defines a facet of the ATS polytope P. It then follows that it also defines a facet of the AA polytope P_A.

We consider first three cases with small n and $|T|$. The shortest odd CAT has length 5 and it uses 4 nodes, i.e. (5.4) is defined only for $|T| \geq 5$ and $n \geq |V[T]| \geq 4$.

Proposition 5.5. *Let $|T| = 5$ and $|V[T]| = 4$. Then the odd CAT inequality (5.4) defines a facet of P_A and P if $n = 4$ and $n = 6$, but not if $n = 5$.*

Proposition 5.6. *Let $|T| = 7$ and $|V[T]| = 5$. Then the odd CAT inequality (5.4) defines a facet of P_A and P for $n = 5$ and $n = 6$.*

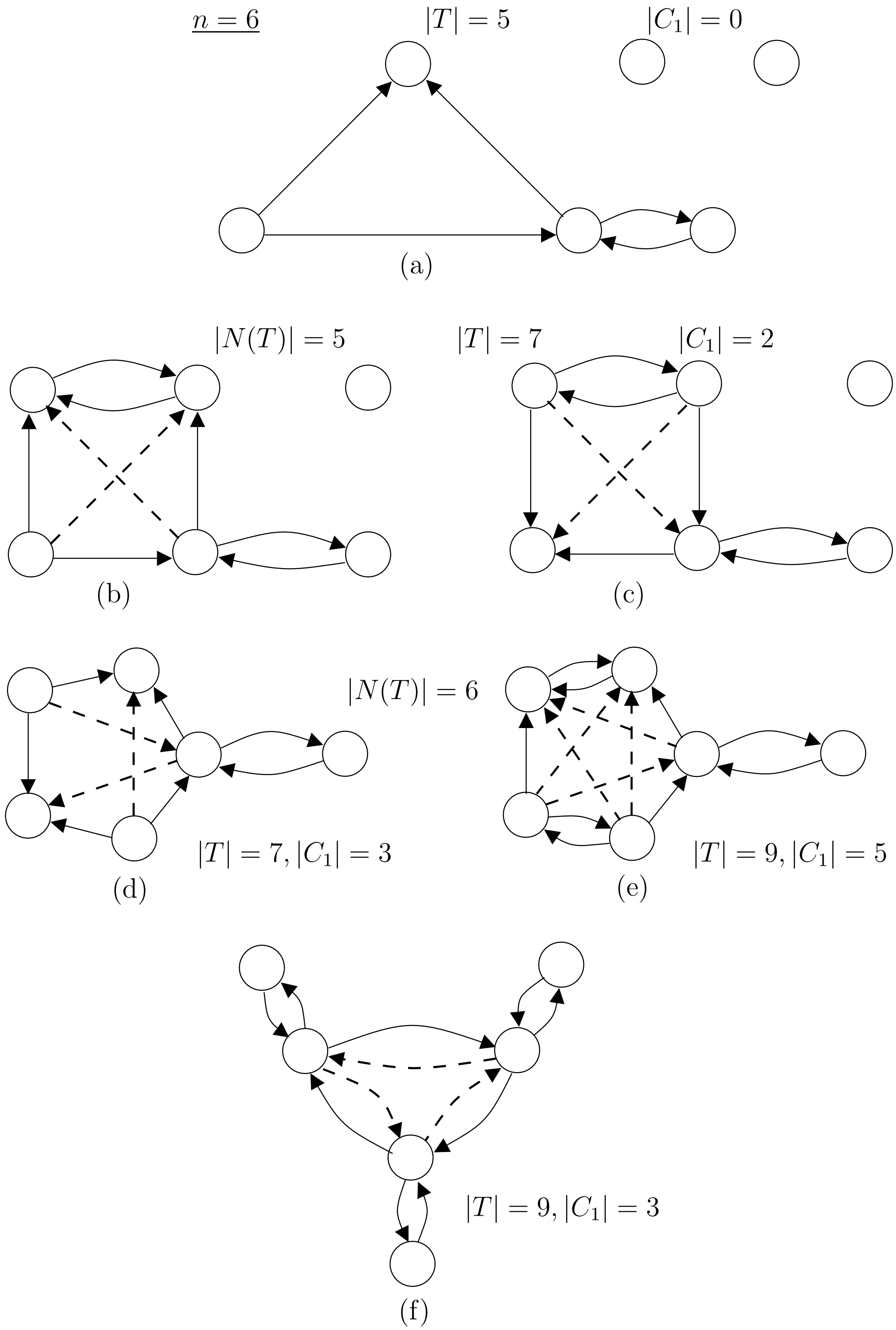


Figure 5.2:

Finally, for $|V[T]| = 6$ we will denote by T^* the odd CAT of Figure 5.2(f), i.e., T^* consists of three neutral 2-cycles whose non-neutral nodes are joined in a 3-cycle.

Proposition 5.7. *If $n = 6$ and $T = T^*$, then the odd CAT inequality (5.4) does not define a facet of P_A or P.*

Here is the main result of this section.

Theorem 5.8. *For all $n \geq 6$ the odd CAT inequality (5.4) defines a facet of P_A and P (except in case $n = 6$ and $T = T^*$).*

The proof of this theorem relies heavily on the following property of *all* nontrivial facet defining inequalities for P [1].

Theorem 5.9. *Suppose $\alpha x \leq \alpha_0$ defines a facet of P. Let X be the matrix whose rows are the incidence vectors of tours in G satisfying $\alpha x = \alpha_0$, and for any $p, q \in V$ let $A(p,q)$ be the set of columns (i,p), $i \in V \setminus \{p,q\}$, and (q,j), $j \in V \setminus \{p,q\}$, of X. Then the submatrix of X consisting of the columns in $A(p,q)$ has rank $|A(p,q)| - 1$; i.e., $2n-3$ if $p = q$ and $2n-5$ if $p \neq q$.*

6 Source-destination inequalities

We now describe a class of facet inducing inequalities introduced in [4] that properly generalizes several known classes of facets of the ATS polytope; the reader is referred to [4] for further details. We first introduce the primitive (i.e., clone free) members of the family.

Let (S, D, W, I, E, Q) be a partition of V with the following properties: S is the (possibly empty) set of *sources*; D is the (possibly empty) set of *destinations*; $H := W \cup I$ is the (nonempty) *handle*, with $0 \leq |W| \leq 1$ and $|I| = s \geq 1$; $I := \{i_1, \ldots, i_s\}$; $E := \{e_1, \ldots, e_s\}$; $T_j := \{i_j, e_j\}$ for $j = 1, \ldots, s$ are the *teeth*; $0 \leq |Q| \leq 1$; and $|S| + |D| + s$ is odd.

Theorem 6.1. *The (primitive) source-destination (SD) inequality*

$$x(S \cup H,\ D \cup H) + \sum_{j=1}^{s} x(T_j, T_j) \leq \frac{1}{2}(|S| + |D| + 2|H| + s - 1) \tag{6.1}$$

is valid for P and $\tilde{P}$ and has Chvátal rank 1.

Proof. Multiplying the following inequalities by $\frac{1}{2}$, adding them up and rounding down all coefficients produces (6.1): $x(i, V) \leq 1$ for $i \in S \cup H$, $x(V, j) \leq 1$ for $j \in D \cup H$, and $x(T_j, T_j) \leq 1$ for $j = 1, \ldots, s$.□

Figure 6.1 illustrates the support graph of an SD inequality on 9 nodes. Note that the nodes in S, although interchangeable, are not clones, because the arcs joining them to each other have coefficients equal to 0, instead of 1.

A primitive SD inequality becomes a comb inequality when $|S| = |D| = 0$ and $s \geq 3$, a C2 inequality when $|S| = |D| = 1$ and $s \geq 3$, a T_2 inequality when $|S| = |D| = 1$, $W = \emptyset$ and $s = 1$, and an odd CAT inequality when $|S| = |D|$ (with $W \neq \emptyset$ only if $s \geq 5$, or $|S| \geq 2$, or $|S| = 1$ and $s \geq 3$).

We now address the cases in which SD inequalities are *not* facet inducing.

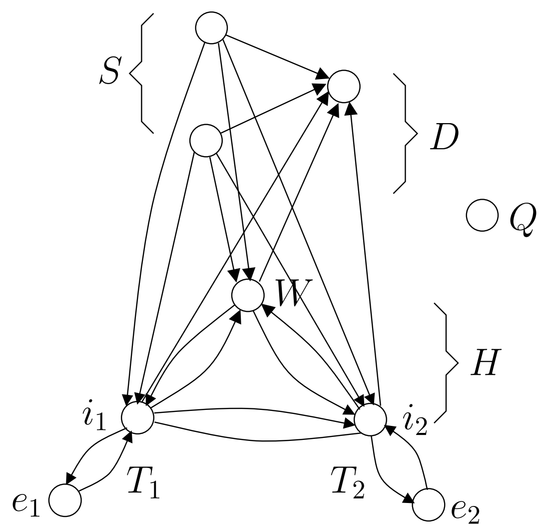


Figure 6.1: The support graph of a primitive SD inequality.

Theorem 6.2. *The primitive SD inequalities do not define facets of P in the following 3 pathological cases, all arising when $|S| = |D|$ and $n \leq 6$:*

(a) $s = 3$, $S = D = W = Q = \emptyset$ *(hence $n = 6$),*

(b) $n \geq 5$, $s = 1$, $|S| = |D| = 1$ *(hence $n \in \{5, 6\}$),*

(c) $s = 1$, $S = D = \emptyset$, $|W| = |Q| = 1$ *(hence $n = 4$).*

Moreover, when $|S| \neq |D|$ we have the following result.

Theorem 6.3. *Let $\alpha x \leq \alpha_0$ be any primitive SD inequality with $||S| - |D|| > \max\{s - 3, 0\}$. Then $\alpha x \leq \alpha_0$ does not define a facet of P.*

On the other hand, in all cases not covered by the previous theorems the SD-inequalities do define facets of P:

Theorem 6.4. *Let $\alpha x \leq \alpha_0$ be any primitive SD inequality, different from those of Theorem 6.2 and satisfying $||S| - |D|| \leq \max\{s - 3, 0\}$. Then $\alpha x \leq \alpha_0$ defines a facet of both P and $\tilde{P}$.*

We now consider the application to the primitive SD inequality $\alpha x \leq \alpha_0$ of the clique lifting procedure described in Section 3.1. To this end it is sufficient to compute the values δ_k defined in Theorem 3.4, thus obtaining $\delta_k = 0$ if $k \in Q$, $\delta_k = 2$ if $k \in I$, and $\delta_k = 1$

otherwise. This leads to the following (clique lifted) SD inequality:

$$
\begin{aligned}
x(S \cup H, H \cup D) &+ \sum_{i=1}^{s} x(T_i, T_i) + \sum_{i=1}^{\sigma} x(S_i, S_i) + \sum_{i=1}^{\delta} x(D_i, D_i) \\
&\leq |H| + \sum_{i=1}^{s}(|T_i| - 1) + \sum_{i=1}^{\sigma}(|S_i| - 1) + \sum_{i=1}^{\delta}(|D_i| - 1) \\
&\quad + \frac{\sigma + \delta - s - 1}{2}
\end{aligned}
$$

where $(H, T_1, \ldots, T_s)$ defines a comb with a possibly even number s of teeth, S and D are disjoint subsets of $V \setminus (H \cup T_1 \cup \cdots \cup T_s)$ which are partitioned properly into $S = \cup_{i=1}^{\sigma} S_i$ and $D = \cup_{i=1}^{\delta} D_i$, and $\sigma + \delta + s$ is an odd number.

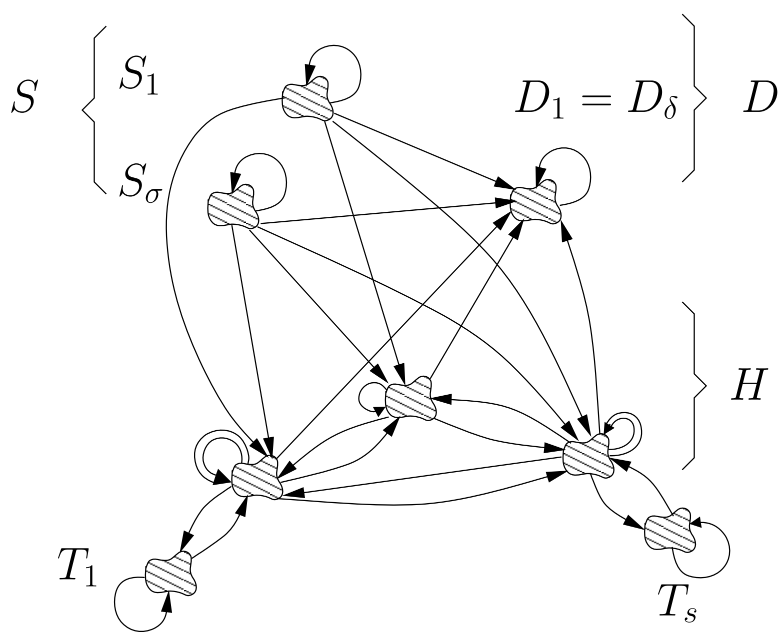


Figure 6.2: A general SD inequality

A general inequality of this type is illustrated in Figure 6.2, where the shaded regions represent cliques, while single and double lines represent coefficients of 1 and 2, respectively.

It can readily be shown that SD inequalities properly generalize the T_k, comb, C2, and odd CAT inequalities. Indeed, the T_k inequalities can be obtained starting from a primitive SD inequality with $s = 1$, $|S| = |D| = 1$, $W = \emptyset$ and introducing clones of nodes e_1 and $q \in Q$. Comb inequalities arise trivially from the case $S = D = \emptyset$ through the cloning of any node. C2 inequalities can be derived from the case $|S| = |D| = 1$ and $s \geq 3$, by adding clones of the nodes in $V \setminus (S \cup D)$. Finally, the odd CAT inequalities are obtainable from primitive SD inequalities with $|S| = |D|$ by introducing clones of $w \in W$ and at most one clone for each node in $S \cup D$.

We now address the possibility that two given (facet-defining) SD inequalities define the same facet of P, i.e., that they are equivalent. To this end we derive and compare their canonical form, as described at the end of Section 4. In view of Theorems 4.4 and 4.6 we will assume that the two SD inequalities to be compared are primitive. Then let $\alpha x \leq \alpha_0$ be any primitive SD inequality associated with the node partition $(S, D, W, Q, I := \{i_1, \ldots, i_s\},\ E := \{e_1, \ldots, e_s\})$ and with tooth set $T := \{\{i_j, e_j\} : j = 1, \ldots, s\}$, and consider any primitive SD inequality $\alpha' x \leq \alpha'_0$, different from $\alpha x \leq \alpha_0$ and associated with the node partition $(S', D', W', Q', I' := \{i'_1, \ldots, i'_s\},\ E' := \{e'_1, \ldots, e'_s\})$ and with tooth set $T' := \{\{i'_j, e'_j\} : j = 1, \ldots, s'\}$.

Theorem 6.5. *Two SD inequalities $\alpha x \leq \alpha_0$ and $\alpha' x \leq \alpha'_0$ are equivalent if $S' = D$, $D' = S$, $Q' = W$, $W' = Q$, $I' = E$, $E' = I$, and $T' = T$.*

The equivalence established in Theorem 6.5 was known earlier for comb inequalities, i.e., when $|S| = |D| = 0$; note that it also holds for the C2 and odd CAT inequalities.

A known case of equivalence not completely covered by Theorem 6.5 arises when $n = 4$ and $|S| = |D| = s = 1$, i.e. when considering T_2 inequalities on 4 nodes (see Figure 4.1). We give the corresponding result for the sake of completeness.

Theorem 6.6. *Let $n = 4$, and assume $|S| = |D| = s = 1$ and $|S'| = |D'| = s' = 1$. W.l.o.g. let $S = \{1\}$, $i_1 = 2$, $D = \{3\}$, and $e_1 = 4$. Then $\alpha' x \leq \alpha'_0$ is equivalent to $\alpha x \leq \alpha_0$ if and only if one of the following 4 cases occurs: (1) $S' = \{1\}$, $i'_1 = 2$, $D' = \{3\}$, and $e'_1 = 4$; (2) $S' = \{4\}$, $i'_1 = 1$, $D' = \{2\}$, and $e'_1 = 3$; (3) $S' = \{3\}$, $i'_1 = 4$, $D' = \{1\}$, and $e'_1 = 2$; (4) $S' = \{2\}$, $i'_1 = 3$, $D' = \{4\}$, and $e'_1 = 1$.*

We now show that, with the one exception of the pathology pointed out in Theorem 6.6, all the equivalences among facet-defining primitive SD inequalities are covered by Theorem 6.5. We will do this by transforming the generic primitive facet-inducing SD inequality $\alpha x \leq \alpha_0$ to its canonical form $\beta x \leq \beta_0$. Following the procedure stated at the end of Section 4, for all $h \in V$ we compute:

$$\alpha_{hh} := \begin{cases} 0 & \text{if } h \in Q, \\ 2 & \text{if } h \in I, \\ 1 & \text{otherwise,} \end{cases} \qquad \Delta := \sum_{h=1}^{n} \alpha_{hh} = 3s + |W| + |S| + |D|,$$

$$r_h := \begin{cases} 1 + |D| + |W| + s & \text{if } h \in S, \\ 1 & \text{if } h \in D, \\ |W| + s + |D| & \text{if } h \in W, \\ 0 & \text{if } h \in Q, \\ |W| + |D| + s + 2 & \text{if } h \in I, \\ 2 & \text{if } h \in E, \end{cases} \qquad c_h := \begin{cases} 1 & \text{if } h \in S, \\ 1 + |S| + |W| + s & \text{if } h \in D, \\ |W| + s + |S| & \text{if } h \in W, \\ 0 & \text{if } h \in Q, \\ |W| + |S| + s + 2 & \text{if } h \in I, \\ 2 & \text{if } h \in E. \end{cases}$$

Recall that $|S| + |D| + |W| + |Q| + 2s = n$. The canonical form $\beta x \leq \beta_0$ is then computed as $\beta_{ij} := \sigma(\Delta + n\alpha_{ij} - r_i - c_j - \varepsilon)$, $i, j \in V$, where $\sigma > 0$ and ε are defined in the normalization step. It is therefore not difficult to prove the following:

Theorem 6.7. *Let $n \geq 5$. Two distinct facet-inducing primitive SD inequalities are equivalent only in the case covered by Theorem 6.5.*

7 Lifted cycle inequalities

In this section we investigate the family of lifted cycle inequalities for the ATS polytope, and establish several properties that earmark it as one of the most important among the families of asymmetric facet inducing inequalities known to date.

Unless otherwise stated, all directed cycles considered in this section are simple. Let $S \subset N$, $S = \{i_1, i_2, \ldots, i_s\}$, and let $C := \{(i_1, i_2), (i_2, i_3), \ldots, (i_{s-1}, i_s), (i_s, i_1)\}$ be a directed cycle visiting all the nodes in S. For the sake of simplicity, we will use i_{j+1} and i_{j-1} to denote the successor and the predecessor, respectively, of node i_j in the cycle (hence $i_{s+1} \equiv i_1$ and $i_0 \equiv i_s$). A *chord* of C is an arc $(i_h, i_k) \in A$ such that $i_k \neq i_{h+1}$. Let R denote the set of chords of C. For every subset $F \subseteq A$, let $\tilde{P}(F) := \{x \in \tilde{P} : x_a = 0 \ \forall a \in F\}$. It is well known [15] that $x(C) \leq |C| - 1$ defines a facet of the polytope $\tilde{P}(R)$. Moreover, let $R = \{a_1, a_2, \ldots, a_m\}$; then the *lifted cycle inequality*

$$\alpha x := x(C) + \sum_{i=1}^{m} \alpha_{a_i} x_{a_i} \leq \alpha_0 := |C| - 1$$

defines a facet of $\tilde{P}$ [15, 18] where the lifting coefficients α_{a_i} $(i = 1, \ldots, m)$ are sequentially computed as the maximum value such that inequality $\alpha x \leq \alpha_0$ is valid for $\tilde{P}(\{a_{i+1}, \ldots, a_m\})$. It is well known that (a) different sequences $\{a_i;\ i = 1, \ldots, m\}$ may lead to different inequalities $\alpha x \leq \alpha_0$, and (b) the value of a given coefficient is largest if lifted first, and is a monotone nonincreasing function of its position in the lifting sequence (with the position of the other coefficients kept fixed). In the case of lifted cycle inequalities $\alpha x \leq \alpha_0$, it is also easily seen that $\alpha_{a_i} \in \{0, 1, 2\}$ for $i = 1, \ldots, m$.

We will study the lifted cycle inequalities on $\tilde{P}$ rather than P, since $\tilde{P}$ is full dimensional, and the following result holds by virtue of Corollary 2.10.

Theorem 7.1. *Any lifted cycle inequality for $\tilde{P}$ whose defining cycle has at most $n-3$ arcs, induces a facet of P.*

Unless otherwise stated, all results in the present section are from [6].

7.1 Two-liftable chord sets

In this section we characterize those sets of chords that can get a coefficient 2 in the lifting process. We first note that a given lifted cycle inequality $\alpha x \leq \alpha_0$ obtained, say, via the chord sequence $a_1, \ldots, a_m$, can always be obtained via an equivalent chord sequence in which all the chords with coefficient 2 appear (in any order) at the beginning of the sequence, while all the chords with coefficient 0 appear (in any order) at the end of the sequence. We call any such chord sequence *canonical*. Indeed, consider swapping the positions in the lifting sequence of two consecutive chords. Clearly, either (a) their coefficients remain unchanged, or (b) the coefficient of the chord moved to the left increases and the other one decreases. Case (b) cannot occur when a chord with coefficient 2 is moved to the left, or a chord with coefficient 0 is moved to the right. Therefore a sequence of swaps of the above type leads to the desired canonical form.

For any arc set F, we denote by $V(F)$ the set of nodes spanned by F. Given a cycle C and a chord (i,j), we denote by C_{ij} the shorter of the two cycles contained in $C \cup \{(i,j)\}$. A given set $H \subset R$ is called *2-liftable* if there exists a chord sequence producing a lifted cycle inequality $\alpha x \leq \alpha_0$ such that $\alpha_a = 2$ for all $a \in H$. Because of the above property, such a sequence can w.l.o.g. be assumed to be canonical. It then follows that H is 2-liftable if and only if $\alpha x := x(C) + 2x(H) \leq |C| - 1$ is a valid inequality for $\tilde{P}(R \setminus H)$. We will show that in order to impose this condition it is necessary and sufficient to forbid the presence in H of certain patterns of chords.

Two distinct arcs (i,j) and (u,v) are called *compatible* when there exists a tour containing both, i.e., when $i \neq u$, $j \neq v$, and (i,j) and (u,v) do not form a 2-cycle. Two arcs that are not compatible are called *incompatible*. Given a chord (i_a, i_b) of C, we call *internal* w.r.t. (i_a, i_b) the nodes $i_b, i_{b+1}, \ldots, i_a$; *external* the nodes $i_{a+1}, \ldots, i_{b-1}$. Thus the internal nodes w.r.t. (i_a, i_b) are those of $V(C_{i_a i_b})$, and the external ones are those of $V(C) \setminus V(C_{i_a i_b})$. Given two chords (i_a, i_b) and (i_c, i_d), we say that (i_c, i_d) *crosses* (i_a, i_b) if they are compatible and nodes i_c and i_d are not both internal or both external w.r.t. (i_a, i_b); see Figure 7.1 for an illustration. Note that (i_c, i_d) crosses (i_a, i_b) if and only if (i_a, i_b) crosses (i_c, i_d), i.e., the property is symmetric. Two chords that do not cross each other are called *noncrossing.* Thus all incompatible pairs are noncrossing.

We define a *noose* in $C \cup R$ as a simple alternating (in direction) cycle $Q := \{a_1, b_1, a_2, b_2, \ldots, a_q, b_q\}$ of $2q \geq 4$ distinct arcs $a_i \in R$ and $b_i \in C$ $(i = 1, \ldots, q)$, in which all adjacent arcs in the sequence (including b_q and a_1) are incompatible and all chords are pairwise noncrossing (see Figure 7.2).

Theorem 7.2. *A chord set $H \subset R$ is 2-liftable if and only if $C \cup H$ contains no pair of crossing chords and no noose.*

An immediate consequence of Theorem 7.2 is that the set of 2-liftable chords is never empty for $|C| \geq 3$. It then follows that the subtour elimination inequalities, in which every chord has a coefficient 1, are not sequentially lifted inequalities (they can be obtained from the corresponding cycle inequality by simultaneous lifting). Any chord that is lifted first in a sequential lifting procedure must get a coefficient 2.

We might note at this point that we have touched upon an important point of difference between the symmetric and asymmetric TS polytopes. In the case of the STS polytope, the subtour elimination inequalities are the only kinds of lifted cycle inequalities: whichever chord is lifted first, it gets a coefficient of 1, and so does the chord that is lifted last.

Assigning a chord (i,j) the coefficient $\alpha_{ij} = 2$ forces to 0 the coefficients of several other chords.

Theorem 7.3. *Let (i_a, i_b) be a chord of C such that $\alpha_{i_a i_b} = 2$. Then the following chords must have coefficient 0 (see Figure 7.3):*

(i) (i_j, i_{a+1}) *for all* $j = b, b+1, \ldots, a-1$*;*

(ii) (i_{b-1}, i_ℓ) *for all* $\ell = b+1, b+2, \ldots, a$*.*

Corollary 7.4. *If the chord (i_a, i_{a+2}) has coefficient 2, then all chords incident with node i_{a+1} must have coefficient 0.*

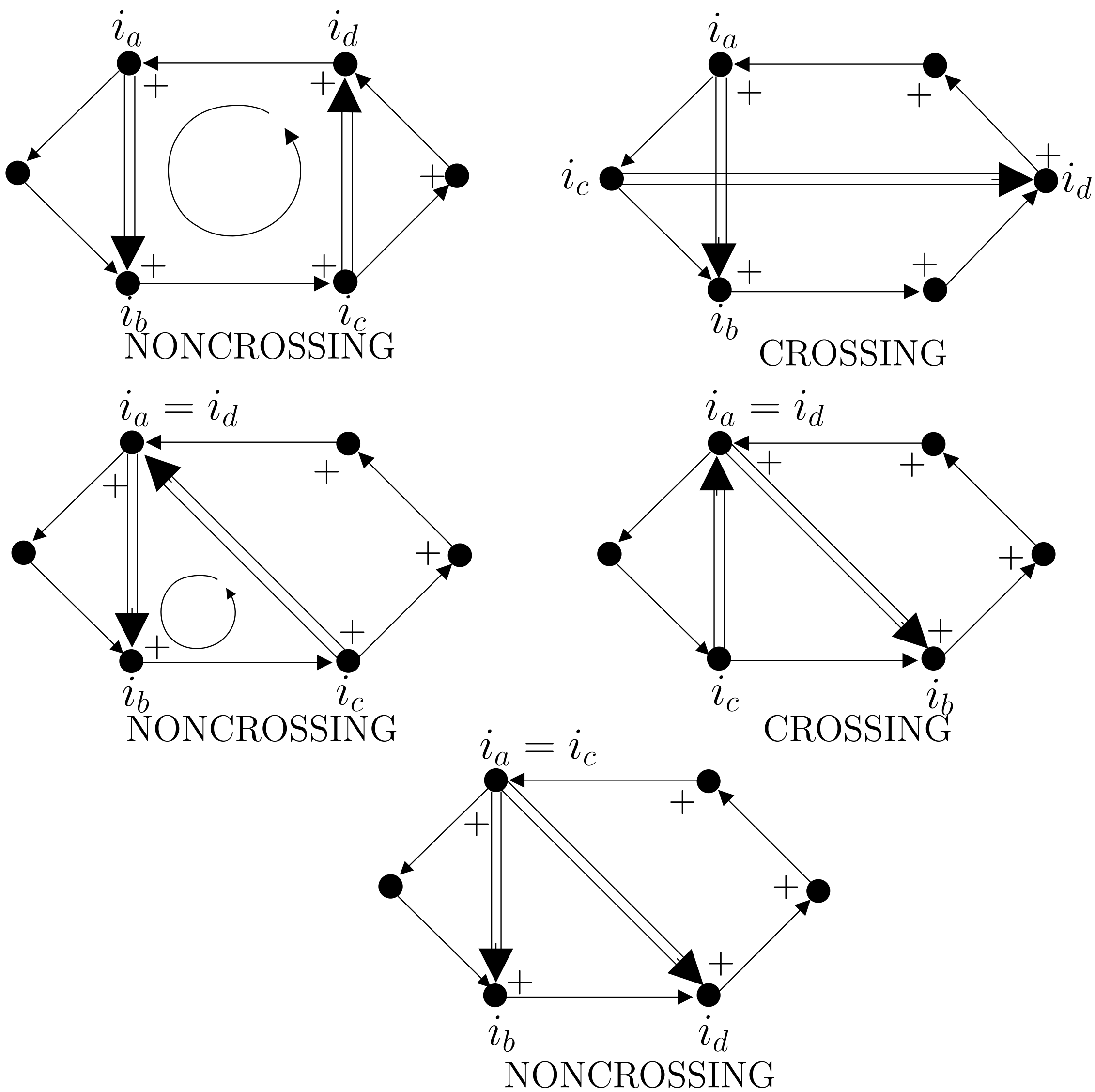


Figure 7.1: Crossing and noncrossing chords. The internal nodes w.r.t. (i_a, i_b) are marked with $+$, and the chords are drawn in double lines.

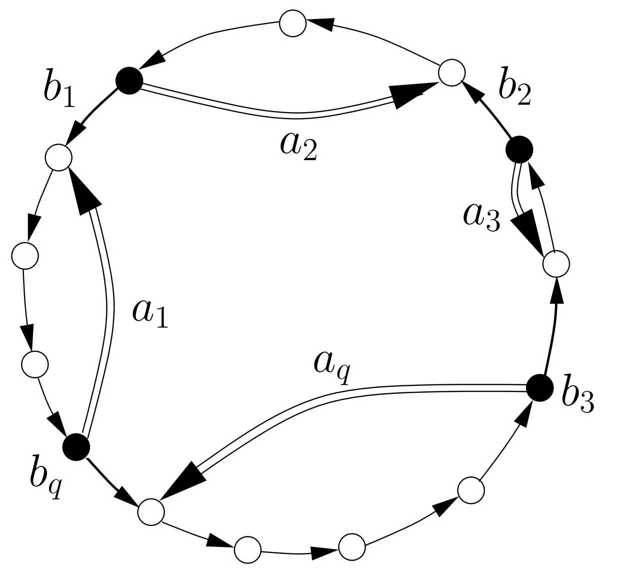


Figure 7.2: A noose Q with $q = 4$.

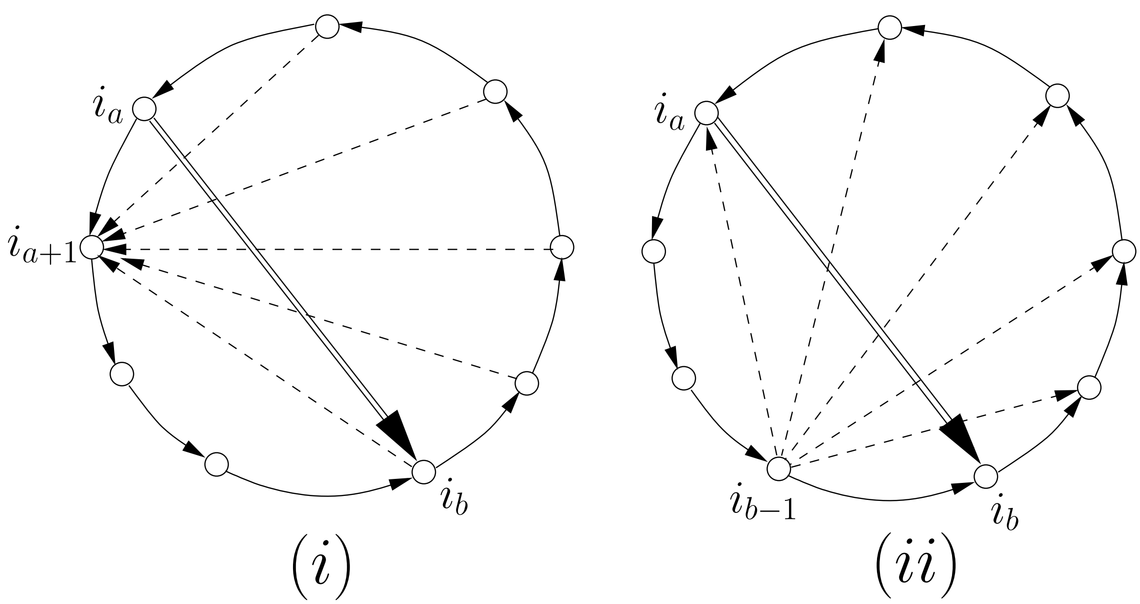


Figure 7.3: The dashed lines represent chords with coefficient 0.

7.2 Maximally 2-lifted cycle inequalities

Given a lifted cycle inequality with a given set of 2-liftable chords and a corresponding set of chords with coefficients forced to 0, the size of the remaining coefficients depends in general on the lifting sequence. Next we characterize a class of lifted cycle inequalities whose 0-1 coefficients are largely sequence-independent.

A set Q_2 of 2-liftable chords is termed *maximal* if no set of the form $Q_2 \cup \{(i,j)\}$, $(i,j) \in R \setminus Q_2$, is 2-liftable. A lifted cycle inequality whose set of chords with coefficient 2 is maximal 2-liftable, will be called a *maximally 2-lifted* cycle inequality.

Proposition 7.5. *The maximum cardinality of a 2-liftable chord set is* $|C| - 2$.

Notice that not all sequentially lifted cycle inequalities are maximally 2-lifted. Examples of (facet inducing) lifted cycle inequalities having a single chord with coefficient 2 are shown in Figure 7.4. (For the first graph of that figure, the appropriate lifting sequence is $(3,2)$, $(1,4)$, $(3,1)$, $(4,2)$ etc.).

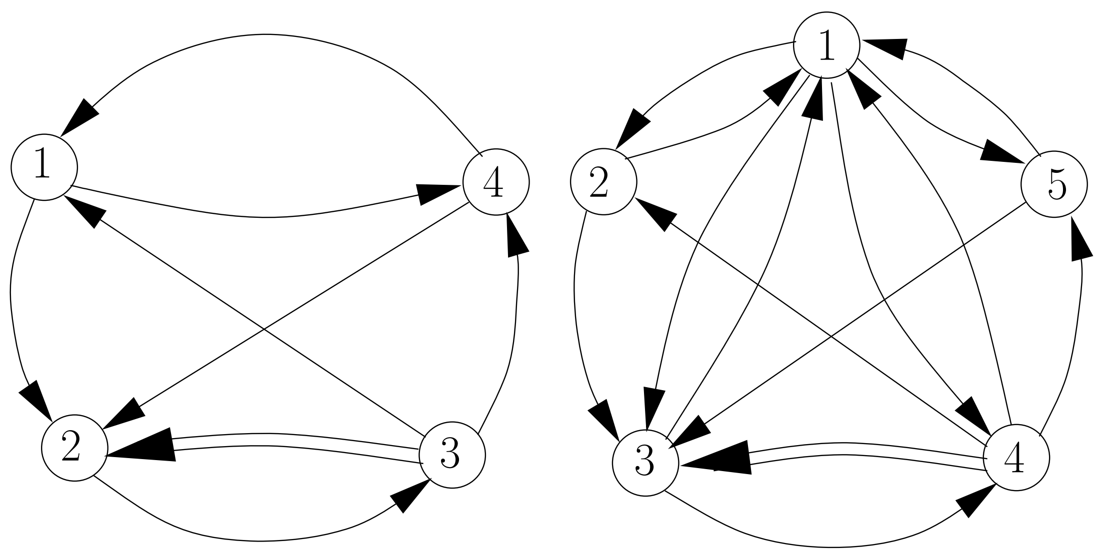


Figure 7.4: Support graphs of two lifted cycle inequalities having a single chord with coefficient 2.

On the other hand, we can identify some large classes of maximally 2-lifted cycle inequalities with useful properties. The next theorem gives a sufficient condition for a given inequality related to a cycle C to be a (facet defining) lifted cycle inequality for $\tilde{P}$.

Theorem 7.6. *Let $\alpha x \leq \alpha_0$ be an inequality with $\alpha_{ij} = 1$ for all arcs (i,j) of a given cycle C of length $|C| = \alpha_0 + 1 \leq n - 1$, where $\alpha_{ij} \in \{0,1,2\}$ for all chords (i,j) of C, and $\alpha_{ij} = 0$ for all other arcs. Let $Q_t := \{(i,j) \in R : \alpha_{ij} = t\}$ for $t = 0,1,2$, and assume the following conditions hold:*

(a) *Q_2 is a maximal 2-liftable set;*

(b) *all 0 coefficients are maximal, i.e., $\alpha_{ij} = 0$ implies the existence of $x^* \in \tilde{P}$ such that $\alpha x^* = \alpha_0$ and $x^*_{ij} = 1$;*

(c) *the inequality $\alpha x \leq \alpha_0$ is valid for $\tilde{P}$.*

Then $\alpha x \leq \alpha_0$ is a maximally 2-lifted cycle inequality, hence facet defining for $\tilde{P}$.

A particularly friendly (and, it turns out, rich) class of lifted cycle inequalities is that for which $Q_0 := \{(i,j) \in R : \alpha_{ij} = 0\}$ is just the set of chords whose coefficients are forced to 0 by the conditions of Theorem 7.3. For this class condition (b) holds automatically, and the only condition to be checked is the inequality validity (c).

Corollary 7.7. *Let $\alpha x \leq \alpha_0$ and let $Q_t := \{(i,j) \in R : \alpha_{ij} = t\}$ for $t = 0, 1, 2$. If Q_2 is a maximal 2-liftable set and Q_0 is the set whose coefficients are forced to 0 by the conditions of Theorem 7.3, then $\alpha x \leq \alpha_0$ is a (facet defining) lifted cycle inequality for $\tilde{P}$ if and only if it is valid for $\tilde{P}$.*

One familiar subclass of this class is that of the D_k^+ and D_k^- inequalities. Another subclass will be introduced in the sequel.

7.3 Maximally 2-lifted cycle inequalities of rank 1

In this section we introduce two new classes of lifted cycle inequalities of Chvátal-rank 1. We start by pointing out that the pattern of 2-liftable chords in a rank 1 inequality has to satisfy an additional condition (besides those required for 2-liftability): it has to define a nested family of node sets in the following sense. A family $\mathcal{F}$ of sets is *nested* if for every $S_1, S_2 \in \mathcal{F}$, either $S_1 \subseteq S_2$, or $S_2 \subseteq S_1$, or $S_1 \cap S_2 = \emptyset$. Given a cycle C and a chord (i_a, i_b) of C, recall that $C_{i_a i_b}$ denotes the shorter of the two cycles contained in $C \cup \{(i_a, i_b)\}$.

Theorem 7.8. *Let $\alpha x \leq \alpha_0$ be a lifted cycle inequality with cycle C, chord set R and $Q_2 := \{(i,j) \in R : \alpha_{ij} = 2\}$. If $\alpha x \leq \alpha_0$ is of Chvátal rank 1, then $V(C)$ and the node sets $V(C_{i_a i_b})$ for all $(i_a, i_b) \in Q_2$ form a nested family.*

An implication of the above theorem is that rank-1 lifted cycle inequalities form a very special subclass indeed: for every $(i_a, i_b) \in Q_2$, the path from i_b to i_a in C never meets the tail of any chord in Q_2 before meeting its head. This implies, among other things, that Q_2 cannot contain a 2-cycle.

Theorem 7.8 gives a necessary condition for $\alpha x \leq \alpha_0$ to be of Chvátal rank 1. Interestingly, this condition is not sufficient, as shown by the following example. Consider the cycle C and the 2-liftable set of chords of Figure 7.5. One can easily see that node sets $V(C_{i_a i_b})$ for $(i_a, i_b) \in Q_2$ form a nested family, as required in Theorem 7.8. However, no lifted cycle inequality $\alpha x \leq \alpha_0$ with the 2-chord pattern of Figure 10 can be of rank 1, as certified by the point x^* with $x^*_{ij} := \frac{2}{3}$ for all $(i,j) \in C \setminus \{(i_5, i_6)\}$, $x^*_{ij} := \frac{1}{3}$ for all $(i,j) \in Q_2 \setminus \{(i_9, i_2)\}$, $x^*_{ij} := 0$ for all other arcs. Indeed, x^* satisfies all degree and subtour elimination inequalities, but $\alpha x^* = \frac{28}{3} > \alpha_0 + 1 = 9$. This implies [30] that $\alpha x \leq \alpha_0$ cannot be of rank 1.

We now introduce two new large classes of maximally 2-lifted cycle inequalities of Chvátal rank 1.

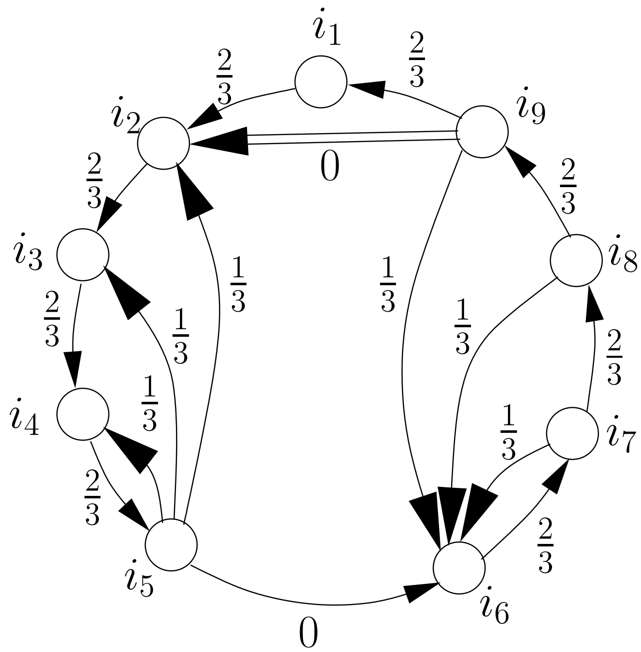


Figure 7.5: Counterexample for the converse of Theorem 7.8.

Theorem 7.9. *Let C be the cycle visiting in sequence the nodes $i_1, \ldots, i_k$ for some $4 \leq k \leq n-1$. Then the* shell inequality

$$\alpha x := x(C) + \sum_{\substack{3 \leq j \leq k \\ j \text{ odd}}} x(\{i_{j+1}, \ldots, i_k, i_1\}, i_j)$$
$$+ \sum_{j=3}^{k} x_{i_1 i_j} + \sum_{\substack{3 \leq j \leq k-1 \\ j \text{ odd}}} x_{i_{j+1} i_j} \leq k - 1 =: \alpha_0$$

is a rank-1 maximally 2-lifted cycle inequality, hence facet defining for $\tilde{P}$. (See Figure 7.6.)

Theorem 7.10. *Let $w, a_1, \ldots, a_{k_a}, b_1, \ldots, b_{k_b}$, be distinct nodes with $k_a, k_b \geq 1$, $k_b \in \{k_a, k_a+1\}$, and $k_a+k_b+1 \leq n-1$. Let C be the directed cycle visiting, in sequence, nodes $w, b_1, b_2, \ldots,$ $b_{k_b}, a_{k_a}, a_{k_a-1}, \ldots, a_1, w$, and define $F := \bigcup_{i=1}^{k_a} \{(a_i, b_j) : i \leq j \leq i+1,\ j \leq k_b\}$. Then the* fork inequality

$$\alpha x := x(C) + x(F) + \textstyle\sum_{i=1}^{k_a} \sum_{j=1}^{k_b} x_{a_i b_j} +$$
$$\textstyle\sum_{i=1}^{k_a-1} x(a_i, \{a_{i+1}, \ldots, a_{k_a}\}) + \sum_{j=1}^{k_b-1} x(\{b_{j+1}, \ldots, b_{k_b}\}, b_j) \leq k_a + k_b =: \alpha_0$$

is a rank-1 maximally 2-lifted cycle inequality, hence facet defining for $\tilde{P}$ (see Figure 7.7).

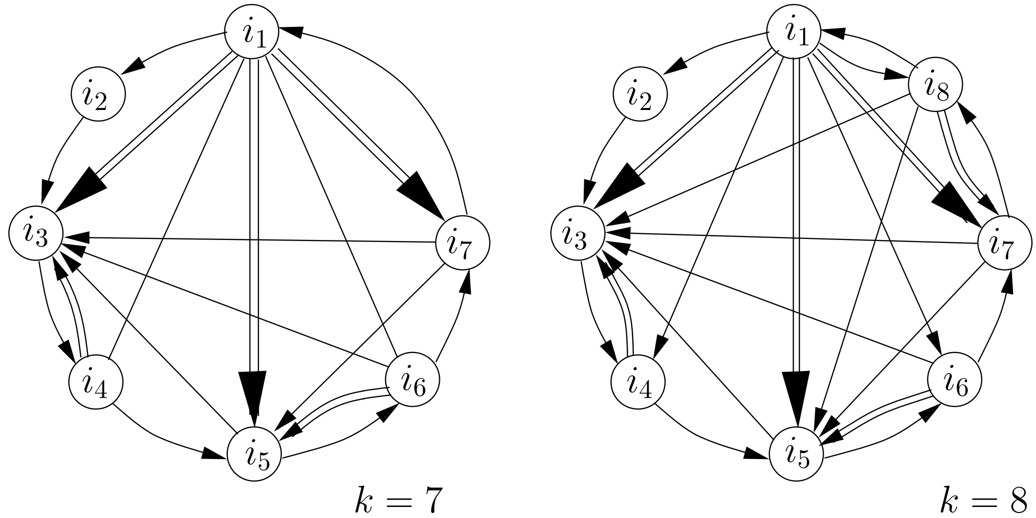


Figure 7.6: Support graphs of two shell inequalities.

7.4 Maximally 2-lifted cycle inequalities of unbounded rank

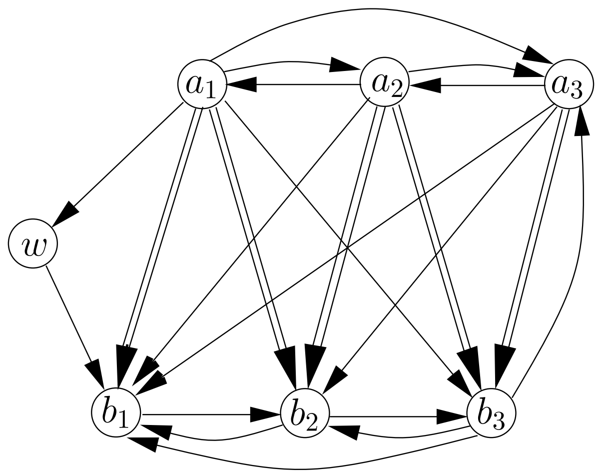
We next establish an important property of the family of lifted cycle inequalities.

Theorem 7.11. *The family of lifted cycle inequalities contains members of unbounded Chvátal rank.*

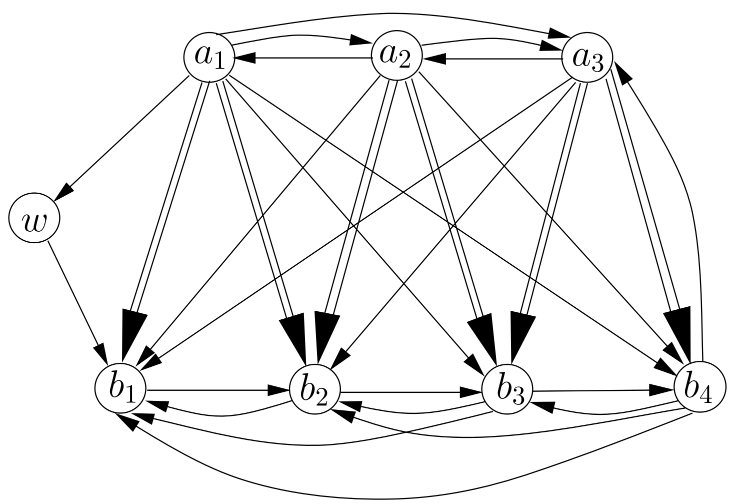
Proof. Let C be a cycle with node set $i_1, \dots, i_k$, $k \geq 8$ even, and consider the nonmaximal 2-liftable chord set $Q := \{(i_1, i_3), (i_3, i_5), \dots, (i_{k-1}, i_1)\}$. Let $\alpha x \leq \alpha_0$ be any lifted cycle inequality associated with C and such that $\alpha_{ij} = 2$ for all $(i, j) \in Q$. We claim that for any subset S of the set $V_{\text{even}} := \{i_2, i_4, \dots, i_k\}$ of even nodes, the subtour elimination inequality associated with $V(C) \setminus S$ must have a positive multiplier in any Chvátal derivation of $\alpha x \leq \alpha_0$. Since the number of subsets S is exponential in k (which in turn is bounded only by n), it then follows from Chvátal, Cook, and Hartmann [9] that for any $M > 0$ there exists a lifted cycle inequality in a sufficiently large digraph, whose Chvátal rank is at least M.

To prove the claim, we assume by contradiction that there exists a Chvátal derivation of $\alpha x \leq \alpha_0$ in which the SEC associated with some $S^* := V(C) \setminus S$ with $S \subseteq V_{\text{even}}$ has 0 multiplier. Then $\alpha x \leq \alpha_0$ must be valid for the polytope $\tilde{P}^*$ defined as the convex hull of points $x \in \{0, 1\}^A$ satisfying the degree inequalities and all the SEC's except for the one associated with S^*. But here is a point $x^* \in \tilde{P}^*$ with $\alpha x^* = |C| > \alpha_0$ that violates $\alpha x \leq \alpha_0$, whose support A^* is constructed as follows: start with $A^* := C$ and then, for each $i_j \in S$, remove from A^* the two arcs incident with i_j (having coefficient 1), and add the arc $(i_{j-1}, i_{j+1}) \in Q$ (which has coefficient 2).□

Next we introduce a large family of maximally 2-lifted cycle inequalities whose set Q_2 of 2-lifted chords is of the type used for the proof of Theorem 7.11, and which therefore



$k_a = k_b = 3$



$k_a = 3, k_b = 4$

Figure 7.7: Support graphs of two fork inequalities.

contains members with unbounded Chvátal rank. Notice that, in addition, these inequalities satisfy the condition of Corollary 7.7.

Theorem 7.12. *Let C be the cycle visiting in sequence the nodes $i_1, i_2, \ldots, i_{4k}$ for some integer $k \geq 2$ satisfying $4k \leq n-1$. Further, let $S_1 := \{i_j \in N(C) : j \text{ is odd}\}$, and $C_1 := \{(i_1, i_3), (i_3, i_5), \ldots, (i_{4k-1}, i_1)\}$. Then the* curtain inequality

$$\alpha x := x(C) + x(S_1, S_1) + x(C_1) + \sum_{\substack{j=3 \\ j \text{ odd}}}^{2k-1} (x_{i_j i_{4k-j+2}} + x_{i_{4k-j+2} i_j}) \leq 4k-1 =: \alpha_0$$

is a maximally 2-lifted, hence facet defining, cycle inequality for $\tilde{P}$.

Proof. Let $Q_t := \{(i,j) : \alpha_{ij} = t\}$ for $t \in \{0,1,2\}$. Clearly, Q_2 has no crossing chords and no nooses, so it is 2-liftable. Also, $|Q_2| = |C|/2 + (|C|-4)/2 = |C|-2$, hence Q_2 is maximal. Further, Q_0 consists precisely of those chords whose coefficient is forced to 0 by the conditions of Corollary 7.4. Hence by Corollary 7.7, we only have to show that the curtain inequality is valid for $\tilde{P}$. From the properties of the 2-liftable chord set Q_2, the inequality $x(C) + 2x(Q_2) \leq 4k-1$ is valid for $\tilde{P}(R \setminus Q_2)$. Thus the curtain inequality is satisfied by all $x \in \tilde{P}$ such that $x(Q_1) = 0$. Now let $x \in \tilde{P}$ be such that $x(Q_1) \geq 1$. Then subtracting this inequality from the sum of the $2k$ indegree inequalities and the $2k$ outdegree inequalities for the odd nodes $i_1, i_3, \ldots, i_{4k-1}$, produces an inequality $\beta x \leq 4k-1$, with $\beta \geq \alpha$. This proves the validity of the curtain inequality for $\tilde{P}$.□

The pattern of chords with coefficient 2 in a curtain inequality is illustrated in Figure 7.8, where odd and even nodes are marked by + and −, respectively. The chords with coefficient 1 are all those (not shown in the figure) joining pairs of odd nodes.

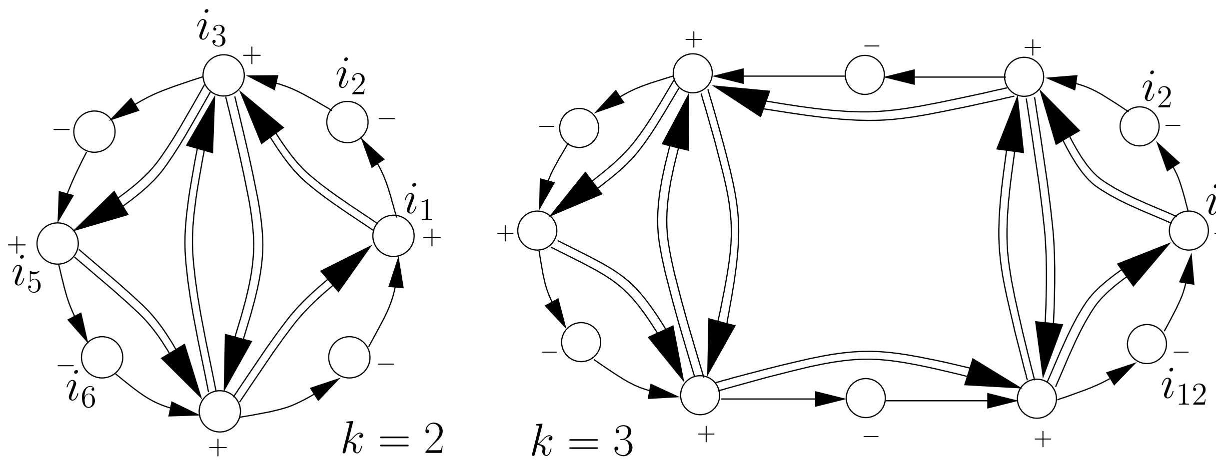


Figure 7.8: 2-liftable chord sets of curtain inequalities.

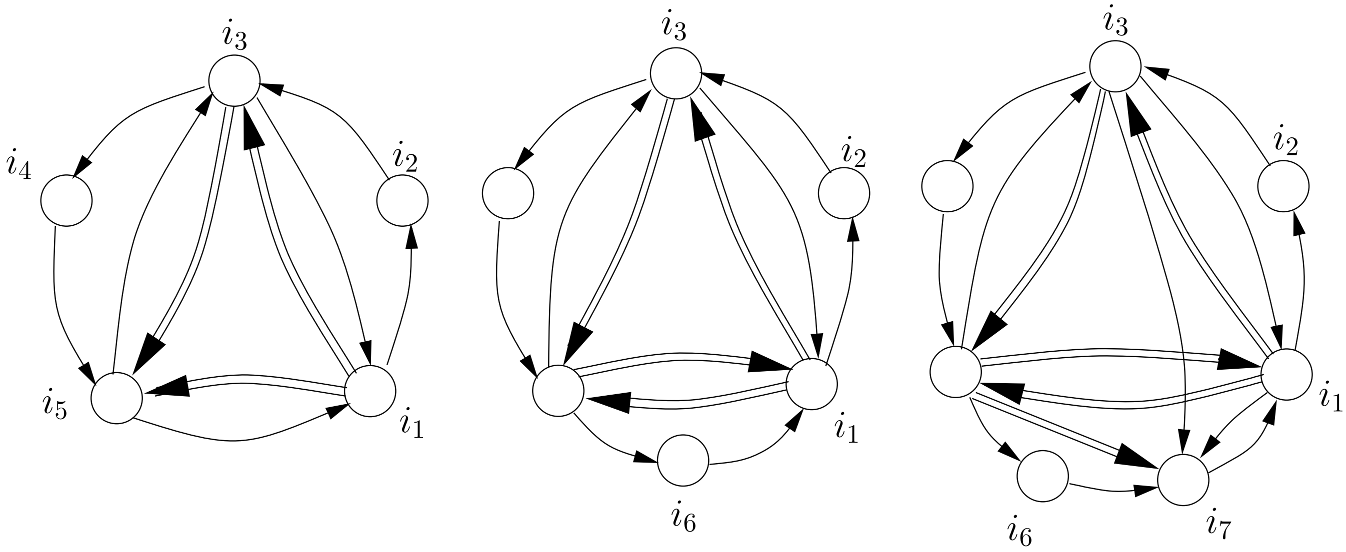


Figure 7.9: Support graphs of curtain inequalities for $|C| = 5, 6, 7$.

The class of curtain inequalities can be extended to cycles of length $|C| \neq 0 \pmod 4$. The corresponding graphs for $|C| = 5, 6, 7$ are shown in Figure 7.9.

For the cases $|C| = 1 \pmod 4$ and $|C| = 3 \pmod 4$ we have the following.

Theorem 7.13. *Let C be the cycle visiting in sequence the nodes $i_1, \ldots, i_{4k+1}$ for some integer k, $2 \leq k \leq (n-2)/4$. Further, let $S_1 := \{i_j \in V(C) : j \text{ is odd}\}$, and $P_1 := \{(i_1, i_3), (i_3, i_5), \ldots, (i_{4k-1}, i_{4k+1})\}$. Then the curtain inequality*

$$\alpha x := x(C) + x(S_1, S_1) + x(P_1) + \sum_{\substack{j=3 \\ j \text{ odd}}}^{2k-1} (x_{i_j i_{4k-j+2}} + x_{i_{4k-j+2} i_j}) + x_{i_1 i_{4k+1}} \leq 4k =: \alpha_0$$

is a maximally 2-lifted, hence facet defining, inequality for $\tilde{P}$.

Proof. Parallels the proof of Theorem 7.12. As in that case, we only have to show that the inequality is satisfied by all $x \in \tilde{P}$ such that $x(Q_1) \geq 1$. Let x have this property. Then adding

- the outdegree inequalities for nodes $i_1, i_3, \ldots, i_{4k-1}$
- the indegree inequalities for node $i_3, i_5, \ldots, i_{4k+1}$
- $1/2$ times the outdegree inequality for node i_{4k+1}
- $1/2$ times the indegree inequality for node i_1
- $1/2$ times the inequality $-x(Q_1) \leq -1$

we obtain an inequality $\beta x \leq 4k + 0.5$, with $\beta \geq \alpha$. Rounding down the coefficients on both sides then yields an inequality that implies $\alpha x \leq 4k$.□

Theorem 7.14. *Let C be the cycle visiting in sequence the nodes $i_1, \dots, i_{4k+3}$ for some integer k, $2 \leq k \leq (n-4)/4$. Further, let $S_1 := \{i_j \in N(C) : j \text{ is odd}\}$, and $P_1 := \{(i_1, i_3), (i_3, i_5), \dots, (i_{4k+1}, i_{4k+3})\}$. Then the curtain inequality*

$$\alpha x := x(C) + x(S_1, S_1) + x(P_1) + \sum_{\substack{j=1 \\ j \text{ odd}}}^{2k-1} (x_{i_j i_{4k-j+2}} + x_{i_{4k-j+2} i_j}) - \sum_{\substack{j=3 \\ j \text{ odd}}}^{4k+1} x_{i_{4k+3} i_j} \leq 4k + 2 := \alpha_0$$

is a maximally 2-lifted, hence facet defining, inequality for $\tilde{P}$.

Proof. As in the case of Theorem 7.12, Q_2 can easily be seen to be a maximal 2-liftable chord set. Also, from Theorem 7.3, all chords incident from or to even nodes have 0 coefficients. Further, from the same theorem as it applies to the 2-chord (i_{4k+1}, i_1), all chords incident from i_{4k+3} have 0 coefficients. Finally, to prove validity, the inequality can be shown to be satisfied by all $x \in \tilde{P}$ such that $x(Q_1) \geq 1$ by adding (1) the outdegree inequalities for nodes $i_1, i_3, \dots, i_{4k+1}$; (2) the indegree inequalities for nodes $i_1, i_3, \dots, i_{4k+3}$; and (3) the inequality $-x(Q_1) \leq -1$.□

Finally, for the case $|C| = 2 \pmod 4$ we have a stronger result, i.e., we can identify a larger class of facet defining inequalities that contains as a special case the curtain inequality with $|C| = 2 \pmod 4$.

Theorem 7.15. *Let C be an even length cycle visiting nodes $i_1, \dots, i_{|C|}$, with $|C| \leq n - 1$. Define the cycle $C_1 := \{(i_1, i_3), (i_3, i_5), \dots, (i_{|C|-1}, i_1)\}$ and let S_1 be the node set of C_1. Then for any maximally 2-liftable chord set Q_2 containing C_1, the inequality*

$$\alpha x := x(C) + x(S_1, S_1) + x(Q_2) \leq |C| - 1$$

is a maximally 2-lifted, hence facet defining, inequality for $\tilde{P}$.

Proof. Since Q_2 is maximally 2-liftable, every $x \in \tilde{P}(R \setminus Q_2)$ satisfies $x(C) + 2x(Q_2) \leq |C| - 1$. Moreover, Q_0 consists precisely of those chords whose coefficient is forced to 0 by the conditions of Theorem 7.3. Thus by Corollary 7.7, we only need to prove that the inequality of the theorem is valid for $\tilde{P}$. Clearly, all $x \in \tilde{P}$ such that $x(Q_1) = 0$, where Q_1 is the set of chords with coefficient 1, satisfies the inequality. Now let $x \in \tilde{P}$ be such that $x(Q_1) \geq 1$, and note that $|S_1| = |C|/2$. Then adding up (1) the outdegree inequalities for nodes $i \in S_1$; (2) the indegree inequalities for nodes $i \in S_1$; and (3) the inequality $-x(Q_1) \leq -1$, we obtain an inequality $\beta x \leq |C| - 1$, where $\beta \geq \alpha$.□

The curtain inequality for $|C| = 2 \pmod 4$ is then a special case of the inequality of Theorem 7.15.

Corollary 7.16. *Let C be the cycle visiting in sequence the nodes $i_1, i_2, \dots, i_{4k+2}$ for some integer $k \geq 2$ satisfying $4k + 2 \leq n - 1$. Further, let $S_1 := \{i_j \in V(C) : j \text{ is odd}\}$, and*

$C_1 := \{(i_1, i_3), (i_3, i_5), \ldots, (i_{4k+1}, i_1)\}$. *Then the* curtain inequality

$$\alpha x := x(C) + x(S_1, S_1) + x(C_1) + \sum_{\substack{j=3 \\ j \text{ odd}}}^{2k-1} (x_{i_j i_{4k-j+2}} + x_{i_{4k-j+2} i_j}) + x_{i_1 i_{4k+1}} \leq 4k + 1 =: \alpha_0$$

is a maximally 2-lifted, hence facet defining, cycle inequality for $\tilde{P}$.

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