On the knapsack closure of 0-1 Integer Linear Programs

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Abstract

Many inequalities for Mixed-Integer Linear Programs (MILPs) or pure Integer Linear Programs (ILPs) are derived from the Gomory corner relaxation, where all the nonbinding constraints at an optimal LP vertex are relaxed. Computational results show that the corner relaxation gives a good approximation of the integer hull for problems with general-integer variables, but the approximation is less satisfactory for problems with 0-1 variables only. A possible explanation is that, for 0-1 ILPs, even the non-binding variable bound constraints $x_j \geq 0$ or $x_j \leq 1$ play an important role, hence their relaxation produces weaker bounds.

In this note we address a relaxation for 0-1 ILPs that explicitly takes all variable bound constraints into account. More specifically, we introduce the concept of *knapsack closure* as a tightening of the classical Chvátal-Gomory (CG) closure. The knapsack closure is obtained as follows: for *all* inequalities $w^T x \leq w_0$ valid for the LP relaxation, add to the original system *all* the valid inequalities for the knapsack polytope $conv\{x \in \{0,1\}^n : w^T x \leq w_0\}$. A MILP model for the corresponding separation problem is also introduced.

Keywords: Integer Linear Programs, Knapsack Problem, Cutting plane separation.

1 Motivation

Given a Mixed-Integer Linear Program (MILP) and an optimal vertex x^* of the associated Linear Programming (LP) relaxation, the *Gomory's corner relaxation* (called "Gomory integer program" in [9]) is obtained by dropping from the MILP all the constraints that are not binding at x^*. This definition of the corner relaxation depends on the choice of vertex x^* but not on the corresponding optimal LP basis. The relaxation is a variant of the well-known *group relaxation* introduced by Ralph Gomory [4] for pure Integer Linear Programs (ILPs), which is obtained by dropping nonnegativity constraints on the variables that are basic in a given optimal LP basis. The group relaxation was deeply investigated by Gomory and Johnson [5,6] who exploited an interpretation in terms of mod-1 equations to obtain its complete facial characterization through subadditive functions. Since then, the theoretical study of the group relaxation received a considerable attention in the literature as witnessed, e.g., by the recent papers [7,8,10], among others.

According to the computational analysis reported in [2], Gomory's corner relaxation gives a good approximation of the integer hull for MILPs with general-integer variables. The approximation is however less effective for problems with 0-1 variables only, as observed already by Balas [1]. A possible explanation is that, for 0-1 ILPs, even the non-binding variable bound constraints $x_j \geq 0$ or $x_j \leq 1$ play an important role, hence their relaxation produces weaker bounds.

A natural question is therefore: "How can we take the variable bound constraints $0 \leq x_j \leq 1$ into account when generating Gomory-like cuts?" In this note we introduce the concept of *knapsack closure* as a tightening of the classical Chvátal-Gomory (CG) concept.

Roughly speaking, the knapsack closure is defined as follows: For *all* inequalities $w^T x \leq w_0$ valid for the LP relaxation, add to the original system

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all the valid inequalities for the knapsack polytope

$$conv\{x \in \{0,1\}^n : w^T x \leq w_0\}.$$

More specifically, consider the 0-1 ILP

$$\min\{c^T x : x \in P \cap X\}, \tag{1}$$

where

$$P := \{x \in \Re^n : Ax \leq b, x \geq 0\} \tag{2}$$

is a given polyhedron and $X \subseteq Z^n$ is such that the optimization of every linear function over X is a "practically tractable" problem, e.g.,

$$X := \{x \in Z^n : 0 \leq x \leq 1\}. \tag{3}$$

Now let $w^T x \leq w_0$ be any valid inequality for P, called *source KP inequality* in the sequel, and let

$$KP(w, w_0) := \{x \in X : w^T x \leq w_0\} \tag{4}$$

define a corresponding *KP relaxation* of the original ILP problem.

Given a (fractional) point $x^* \in \Re^n$, we are interested in the following *Separation problem*: Find a linear inequality $\alpha^T x \leq \alpha_0$ that is valid for $KP(w, w_0)$ but violated by x^* (if any).

2 The "easy" case: the source KP inequality is given

If the source KP inequality is given, the separation problem amounts to the solution of a series of knapsack problems, i.e., of optimizations of a linear function over the KP relaxation $KP(w, w_0)$ [3]. Indeed, one can in principle enumerate all the members of $KP(w, w_0)$, say $x^1, \ldots, x^K$, and write the following LP model for the separation:

$$\max\ \alpha^T x^* - \alpha_0 \tag{5}$$

$$\alpha^T x^i \leq \alpha_0, \qquad \forall i = 1, \ldots, K \tag{6}$$

$$-1 \leq \alpha_j \leq 1, \quad \forall j = 0, \ldots, n \tag{7}$$

where (7) is an example of possible normalization conditions.

The above LP contains an exponential number of constraints, hence a standard run-time cut generation technique can be applied, where at each iteration the following steps are performed:

(i) Consider explicitly just a few solutions in $KP(w, w_0)$, say solutions $x^1, \cdots, x^h$ for some $h \ll K$ (initially, $h := 0$);

(ii) Compute an optimal solution $(\bar{\alpha}, \bar{\alpha}_0)$ of the corresponding restricted LP model

(8) $\max\ \alpha^T x^* - \alpha_0$

(9) $\alpha^T x^i \leq \alpha_0, \quad \forall i = 1, \ldots, h$

(10) $-1 \leq \alpha_j \leq 1, \quad \forall j = 0, \ldots, n$

(iii) If $\bar{\alpha} x^* - \bar{\alpha}_0 \leq 0$, then the method can be stopped as no violated inequality $\alpha^T x \leq \alpha_0$ exists;

(iv) Call an *oracle* to compute an optimal solution y^* of the KP problem

$$\max\{\bar{\alpha} y : y \in KP(w, w_0)\};$$

(v) If $\bar{\alpha} y^* \leq \bar{\alpha}_0$, then the inequality $\bar{\alpha} x \leq \bar{\alpha}_0$ is valid for $KP(w, w_0)$ and maximally violated, hence return it;

(vi) Include y^* in the separation model by setting $h := h + 1$ and $x^h := y^*$, and repeat.

3 The "hard" case: the source KP inequality is not given

We now address the intriguing case where the inequality $w^T x \leq w_0$ is *not* given a priori (nor read from the optimal LP tableau), but is completely general and is actually defined during the separation phase so as to maximize its effectiveness. Our approach produces a more powerful separation tool that goes beyond the separation over the first Chvátal closure, but requires the solution of a MILP (as opposed to the LP of the previous case).

Our MILP separation model is as follows:

(11) $\max\ \alpha^T x^* - \alpha_0$

(12) $w^T \leq u^T A,\ w_0 \geq u^T b,\ u \geq 0$

(13) $\alpha^T x^i \leq \alpha_0 + M\delta_i, \quad \forall i = 1, \ldots, Q$

(14) $w^T x^i \geq w_0 + \epsilon - M(1 - \delta_i), \quad \forall i = 1, \ldots, Q$

(15) $\delta_i \in \{0, 1\}, \quad \forall i = 1, \ldots, Q$

(16) $-1 \leq \alpha_j \leq 1, \quad \forall j = 0, \ldots, n$

where $X =: \{x^1, \ldots, x^Q\}$, and M and ϵ are a large and a small positive value, respectively. Notice that $u, w, w_0, \alpha, \alpha_0, \delta$ all play the role of variables in the above model.

The idea of the model above is to certify the validity of $w^T x \leq w_0$ for P (where w and w_0 are now variables) by using Farkas' characterization (12).

Because of (13), a point $x^i \in X$ can violate the inequality $\alpha^T x \le \alpha_0$ only by setting $\delta_i = 1$, in which case (14) imposes that the valid inequality $w^T x \le w_0$ cuts it off (hence this point cannot be feasible for the original ILP model).

The solution of the MILP separation model can be obtained along the same lines of its LP counterpart. Namely,

a. Find an optimal solution $(\bar{u}, \bar{w}, \bar{w}_0, \bar{\alpha}, \bar{\alpha}_0, \bar{\delta})$ of a *restricted* MILP separation problem taking into account only a subset of points $x^1, \dots, x^h$;

b. Invoke the KP oracle to solve

$$\max\{\bar{\alpha} y : y \in KP(\bar{w}, \bar{w}_0)\}$$

so as to certify the validity of $\bar{\alpha} x \le \bar{\alpha}_0$ for the current KP relaxation $KP(\bar{w}, \bar{w}_0)$, or else to produce a new point x^{h+1} to be inserted in the MILP separation model (along with the corresponding variable δ_{h+1}), and repeat.

4 Conclusions

We have introduced the new concept of *knapsack closure* for 0-1 Integer Linear Programs. This is a strengthening of the classical Chvátal-Gomory concept of closure, and is obtained by considering, for all inequalities $w^T x \le w_0$ valid for the LP relaxation, all the valid inequalities for the knapsack polytope $conv\{x \in \{0,1\}^n : w^T x \le w_0\}$.

A MILP model for the corresponding separation problem has been presented, whose practical viability is under investigation.

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