MIPping closures: an instant survey

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Abstract

A relevant amount of work has been devoted very recently to modeling and (heuristically) solving the NP-hard separation problem of famous classes of valid inequalities for mixed integer linear programs (MIPs). This task has been accomplished by using, in turn, mixed-integer linear models for the separation problem, and a general-purpose solver to actually find violated cuts—the so-called *MIPping* approach. We *instantly* survey these attempts by discussing their computational outcome and pointing out their practical interest for future integration in the MIP solvers.

Key words: mixed integer programs, separation problems, cutting plane methods.

1 Introduction

Mixed-integer linear programming (MIP) plays a central role in modeling difficult-to-solve (NP-hard) combinatorial problems. Exact MIP solvers are very sophisticated tools designed to deliver, within acceptable computing time, a provable optimal solution of the input MIP model, or at least a heuristic solution with a practically-acceptable error.

Modern MIP solvers exploit a rich arsenal of tools to attack hard problems, some of which include the solution of LP models to control the branching strategy (strong branching), the cut generation (lift-and-project), the heuristics (reduced costs), etc. As a matter of fact, it is well known by the OR community that the solution of very hard MIPs can take advantage of the solution of a series of auxiliary LPs intended to guide the main steps of the MIP solver.

Also well known is the fact that finding good-quality MIP solutions often requires a computing time that is just comparable to that needed to solve the LP relaxation of the problem at hand. This leads to the idea of "translating into a MIP model" (*MIPping*) some crucial decisions to be taken within a MIP algorithm (in particular: How to improve the incumbent solution? How to cut?), with the aim of bringing the MIP technology well within the MIP solver.

An example of the benefits deriving from the use of a black-box MIP solver to produce heuristic primal solutions for a generic MIP is the recently-proposed *local branching* paradigm that uses a general-purpose MIP solver to explore large solution neighborhoods defined through the introduction in the MIP model of invalid linear inequalities called *local branching cuts* [14]. An application to the Vehicle Routing Problem of the MIPping idea is instead reported by De Franceschi, Fischetti and Toth in [12], where the critical step of client reallocation and resequencing within a metaheuristic framework is just MIPped and solved through a general-purpose solver.

Very recently, the MIPping approach has been extensively applied to modeling and solving (possibly in a heuristic way) the NP-hard separation problems of famous classes of valid inequalities for mixed integer linear programs. Besides the theoretical interest in evaluating the strength of these classes of cuts computationally, the approach proved successful also in practice, and allowed for the solution of very hard MIPLIB instances [4] that could not be solved before.

The present paper *instantly* surveys these attempts by discussing their computational outcome and pointing out their practical interest for future integration in the solvers. The paper is organized as follows. In Section 2 we introduce our basic notation and definitions. In Section 3 we discuss the separation of Chvátal-Gomory cuts for pure integer programs through a natural MIP model. In Section 4 we address the more general (and powerful) family of split cuts by first discussing the separation of a generalization of Chvátal-Gomory cuts (Section 4.1) and then addressing their separation problem in two different ways, namely a *parametric* mixed integer programming approach (Section 4.2) and a *nonlinear* programming approach (Section 4.3). Finally, in Section 5 we discuss computational aspects of these models and we report results on the strength of the addressed closures.

2 Basic and Preliminaries

Consider first the pure integer linear programming problem $\min\{c^T x : Ax \leq b, x \geq 0, x \text{ integral}\}$ where A is an $m \times n$ rational matrix, $b \in \mathbb{Q}^m$, and $c \in \mathbb{Q}^n$, along with the two associated polyhedra $P := \{x \in \mathbb{R}^n_+ : Ax \leq b\}$ and $P_I := conv\{x \in \mathbb{Z}^n_+ : Ax \leq b\} = conv(P \cap \mathbb{Z}^n)$.

A *Chvátal-Gomory* (*CG*) *cut* (also known as *Gomory fractional* cut) [16, 7] is an inequality of the form $\lfloor u^T A \rfloor x \leq \lfloor u^T b \rfloor$ where $u \in \mathbb{R}^m_+$ is a vector of multipliers, and $\lfloor \cdot \rfloor$ denotes the lower integer part. Chvátal-Gomory cuts are valid inequalities for P_I. The *Chvátal closure* of P is defined as

$$P^1 := \{x \geq 0 : Ax \leq b, \lfloor u^T A \rfloor x \leq \lfloor u^T b \rfloor \text{ for all } u \in \mathbb{R}^m_+\}. \tag{1}$$

Thus $P_I \subseteq P^1 \subseteq P$. By the well-known equivalence between optimization and separation [18], optimizing over the first Chvátal closure is equivalent to solving the *CG separation problem* where we are given a point $x^* \in \mathbb{R}^n$ and are asked to find a hyperplane separating x^* from P^1 (if any). Without loss of generality we can assume that $x^* \in P$, since all other points can be cut by simply enumerating the members of the original inequality system $Ax \leq b$, $x \geq 0$. Therefore, the separation problem we are actually interested in reads:

CG-SEP: Given any point $x^* \in P$ find (if any) a CG cut that is violated by x^*, i.e., find $u \in \mathbb{R}^m_+$ such that $\lfloor u^T A \rfloor x^* > \lfloor u^T b \rfloor$, or prove that no such u exists.

It was proved by Eisenbrand [13] that CG-SEP is NP-hard, so optimizing over P^1 also is.

Analogously, Gomory [17] proposed a stronger family of cuts, the so-called *Gomory Mixed Integer (GMI) cuts*, that apply to the mixed integer case. Such a family of inequalities has been proved to be equivalent to two other families, the so-called *split cuts* defined by Cook, Kannan and Schrijver [8], and the *Mixed Integer Rounding (MIR) cuts* introduced by Nemhauser and Wolsey [21].

For the purpose of this survey, we skip the formal definition of the well-known GMI inequalities to concentrate on those of split cuts and MIR cuts. The reader is referred to Cornuéjols and Li [10] for formal proofs of the correspondence among those families, and to Cornuéjols [9] for a very recent survey on valid inequalities for mixed integer linear programs. Let us consider a generic MIP of the form

$$\min\{c^T x + f^T y : Ax + Cy \leq b, x \geq 0, x \text{ integral},\ y \geq 0\} \tag{2}$$

where A and C are $m \times n$ and $m \times r$ rational matrices respectively, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$, and $f \in \mathbb{Q}^r$. We also consider the two following polyhedra in the (x, y)-space:

$$P(x,y) := \{(x,y) \in \mathbb{R}^n_+ \times \mathbb{R}^r_+ : Ax + Cy \leq b\} \tag{3}$$

$$P_I(x,y) := conv(\{(x,y) \in P(x,y) : x \text{ integral}\}). \tag{4}$$

Split cuts were introduced by Cook, Kannan and Schrijver [8]. They are obtained as follows. For any $\pi \in \mathbb{Z}^n$ and $\pi_0 \in \mathbb{Z}$, the disjunction $\pi^T x \leq \pi_0$ or $\pi^T x \geq \pi_0 + 1$ is of course valid for $P_I(x,y)$, i.e., $P_I(x,y) \subseteq conv(\Pi_0 \cup \Pi_1)$ where

$$\Pi_0 := P(x,y) \cap \{(x,y) : \pi^T x \leq \pi_0\} \tag{5}$$

$$\Pi_1 := P(x,y) \cap \{(x,y) : \pi^T x \geq \pi_0 + 1\}. \tag{6}$$

A valid inequality for $conv(\Pi_0 \cup \Pi_1)$ is called a *split cut*. The convex set obtained by intersecting $P(x,y)$ with all the split cuts is called the *split closure* of $P(x,y)$. Cook, Kannan and Schrijver proved that the split closure of $P(x,y)$ is a polyhedron.

Nemhauser and Wolsey [21] introduced the family of *MIR cuts*, whose basic (2-dimensional) version can be obtained in the following way. Let $1 < \hat{b} < 0$ and $\bar{b} \in Z$, and consider the two-variable mixed integer program $T = \{(x,y) : x + y \geq \hat{b} + \bar{b}, y \geq 0\}$. Then, it is easily seen that the points in T with $x \in Z$ satisfy the *basic MIR* inequality

$$\hat{b}x + y \geq \hat{b}(\bar{b} + 1), \tag{7}$$

that turns out to be a split cut derived from the disjunction $x \leq \bar{b}$ and $x \geq \bar{b}+1$. The hardness of separation of split cuts (and hence of MIR inequalities) has been settled by Caprara and Letchford [6].

3 Chvátal-Gomory cuts

In [15] we addressed the issue of evaluating the practical strength of P^1 in approximating P_I. Our approach was to model the CG separation problem as a MIP, which is then solved through a general-purpose MIP solver. To be more specific, given an input point $x^* \in P$ to be separated, CG-SEP calls for a CG cut $\alpha^T x \leq \alpha_0$ which is (maximally) violated by x^*, where $\alpha = \lfloor u^T A \rfloor$ and $\alpha_0 = \lfloor u^T b \rfloor$ for some $u \in \mathbb{R}^m_+$. Hence, if A_j denotes the jth column of A, CG-SEP can be modeled as:

$$\max \quad \alpha^T x^* - \alpha_0 \tag{8}$$
$$\alpha_j \leq u^T A_j, \quad \forall j = 1, \ldots, n \tag{9}$$
$$\alpha_0 + 1 - \epsilon \geq u^T b, \tag{10}$$
$$u_i \geq 0, \qquad \forall i = 1, \ldots, m \tag{11}$$
$$\alpha_j \quad \text{integer}, \quad \forall j = 0, \ldots, n \tag{12}$$

where ϵ is a small positive value. In the model above, the integer variables α_j ($j = 1, \ldots, n$) and α_0 play the role of coefficients $\lfloor u^T A_j \rfloor$ and $\lfloor u^T b \rfloor$ in the CG cut, respectively. Hence the objective function (8) gives the amount of violation of the CG cut evaluated for $x = x^*$, that we want to maximize. Because of the sign of the objective function coefficients, the rounding conditions $\alpha_j = \lfloor u^T A_j \rfloor$ can be imposed through upper bound conditions on variables α_j ($j = 1, \ldots, n$), as in (9), and with a lower bound condition on α_0, as in (10). Note that this latter constraint requires the introduction of a small value ϵ so as to avoid an integer $u^T b$ be rounded to $u^T b - 1$.

Model (8)-(12) can also be explained by observing that $\alpha^T x \leq \alpha_0$ is a CG cut if and only if (α, α_0) is an integral vector, as stated in (12), and $\alpha^T x \leq \alpha_0 + 1 - \epsilon$ is a valid inequality for P, as stated in (9)-(11) by using the well-known characterization of valid inequalities for a polyhedron due to Farkas.

4 Split closure by steps

The computational results reported in [15] show that P^1 often gives a surprisingly tight approximation of P, so a natural question is whether the same result generalizes to mixed integer linear programming problems.

Unfortunately, model (8)-(12) does not extend immediately to the mixed integer case, where one typically concentrates on the stronger split/MIR(/GMI) cuts[1].

Although, as in CG case, it is easy to find a split cut that separates a basic solution of the linear programming relaxation that is not integer feasible, separating over the split closure is NP-hard as pointed out in Section 2. However, such a separation is even more tricky since no natural MIP model like (8)-(12)

[1] Of course, the separation of split/MIR cuts turns out to be important in the pure integer case too.

is known, while a natural mixed integer nonlinear model has been suggested in [6].

Our first step towards the optimization over the split/MIR closure is to address in Section 4.1 a generalization of the Chvátal-Gomory cuts to the mixed integer case, called *projected Chvátal-Gomory (pro-CG) cuts* [5]. This generalization provides a first approximation of the split closure for mixed integer problems. Finally, we face the overall problem of separating split cuts either by solving a parametric mixed integer problem [3] (Section 4.2) or a nonlinear mixed integer problem [11] (Section 4.3).

4.1 Projected Chvátal-Gomory cuts

Bonami, Cornuéjols, Dash, Fischetti and Lodi [5] extended the concept of Chvátal-Gomory cuts to the mixed integer case. Such an extension of the classical definition of Chvátal-Gomory cuts to the mixed integer case is interesting in itself, and has the advantage of identifying a large class of cutting planes whose resulting separation problem retains the simple structure of model (8)-(12). We define the projection of $P(x, y)$ onto the space of the x variables as:

$$P(x) \quad := \{x \in \mathbb{R}^n_+ : \text{there exists } y \in \mathbb{R}^r_+ \text{ s.t. } Ax + Cy \le b\} \tag{13}$$

$$= \{x \in \mathbb{R}^n_+ : u^k A \le u^k b,\ k = 1, \ldots, K\} \tag{14}$$

$$=: \{x \in \mathbb{R}^n_+ : \bar{A}x \le \bar{b}\} \tag{15}$$

where $u^1, \ldots, u^K$ are the (finitely many) extreme rays of the projection cone $\{u \in \mathbb{R}^m_+ : u^T C \ge 0^T\}$. Note that the rows of the linear system $\bar{A}x \le \bar{b}$ are of Chvátal rank 0 with respect to $P(x, y)$, i.e, no rounding argument is needed to prove their validity.

We then define a *projected Chvátal-Gomory (pro-CG) cut* as a CG cut derived from the system $\bar{A}x \le \bar{b},\ x \ge 0$, i.e., an inequality of the form $\lfloor w^T \bar{A} \rfloor x \le \lfloor w^T \bar{b} \rfloor$ for some $w \ge 0$. Since any row of $\bar{A}x \le \bar{b}$ can be obtained as a linear combination of the rows of $Ax \le b$ with multipliers $\bar{u} \ge 0$ such that $\bar{u}^T C \ge 0^T$, it follows that a pro-CG cut can equivalently (and more directly) be defined as an inequality of the form

$$\lfloor u^T A \rfloor x \le \lfloor u^T b \rfloor \qquad \text{for any } u \ge 0 \text{ such that } u^T C \ge 0^T. \tag{16}$$

As such, its associated separation problem can be modeled as a simple extension of (8)-(12), through the following MIP:

$$\max \quad \alpha^T x^* - \alpha_0 \tag{17}$$

$$\alpha_j \le u^T A_j, \quad \forall j = 1, \ldots, n \tag{18}$$

$$0 \le u^T C_j, \quad \forall j = 1, \ldots, r \tag{19}$$

$$\alpha_0 + 1 - \epsilon \ge u^T b \tag{20}$$

$$u_i \ge 0, \quad \forall i = 1, \ldots, m \tag{21}$$

$$\alpha_j \quad \text{integer}, \quad \forall j = 0, \ldots, n. \tag{22}$$

Projected Chvátal-Gomory cuts are dominated by split cuts, and therefore $P^1(x, y)$ contains the split closure of $P(x, y)$. The following result gives the precise relation between the two classes of cuts.

Theorem 4.1 *[5] Let $S(x, y)$ denote the intersection of $P(x, y)$ with all the split cuts where one of the sets Π_0, Π_1 defined in (5) and (6) is empty. Then*

$$P^1(x, y) = S(x, y).$$

4.2 Split cuts solving a parametric MIP

Balas and Saxena [3] directly addressed the separation problem of the most violated split cut of the form $\alpha^T x + \gamma^T y \geq \beta$ by looking at the union of the two polyhedra (5) and (6) defined in Section 2. In particular, they addressed a generic MIP of the form

$$\min\{c^T x + f^T y : Ax + Cy \geq b, x \text{ integral}\} \tag{23}$$

where the variable bounds are included among the explicit constraints, and wrote a first nonlinear separation model for split cuts as follows:

$$\begin{aligned}
\min \quad \alpha^T x^* &+ \gamma^T y^* - \beta && (24)\\
\alpha_j &= u^T A_j - u_0 \pi_j \quad \forall j = 1, \ldots, n && (25)\\
\gamma_j &= u^T C_j \qquad\qquad \forall j = 1, \ldots, r && (26)\\
\alpha_j &= v^T A_j + v_0 \pi_j \quad \forall j = 1, \ldots, n && (27)\\
\gamma_j &= v^T C_j \qquad\qquad \forall j = 1, \ldots, r && (28)\\
\beta &= u^T b - u_0 \pi_0 && (29)\\
\beta &= v^T b + v_0(\pi_0 + 1) && (30)\\
1 &= u_0 + v_0 && (31)\\
u,\ v,\ u_0,\ v_0 &\geq 0 && (32)\\
\pi,\ \pi_0 &\quad \text{integer} && (33)
\end{aligned}$$

Normalization constraint (31) allows one to simplify the model to the form below:

$$\begin{aligned}
\min u^T(Ax^* + Cy^* - b) &- u_0(\pi^T x^* - \pi_0) && (34)\\
u^T A_j - v^T A_j - \pi_j &= 0 \quad \forall j = 1, \ldots, n && (35)\\
u^T C_j - v^T C_j &= 0 \quad \forall j = 1, \ldots, r && (36)\\
-u^T b + v^T b + \pi_0 &= u_0 - 1 && (37)\\
0 < u_0 < 1 &\quad , \quad u, v \geq 0 && (38)\\
\pi,\ \pi_0 &\quad \text{integer} && (39)
\end{aligned}$$

where v_0 has been removed by using constraint (31), and one explicitly uses the fact that any nontrivial cut has $u_0 < 1$ and $v_0 < 1$ (see, Balas and Perregaard

[2]). Note that the nonlinearity only arises in the objective function; moreover, for any fixed value of parameter u_0 the model becomes a regular MIP.

The continuous relaxation of the above model yields a parametric linear program which can be solved by a variant of the simplex algorithm (see, e.g., Nazareth [20]). Balas and Saxena [3] however avoided solving the parametric mixed integer program through a specialized algorithm, and considered a grid of possible values for parameter u_0, say $u_0^1 < u_0^2 < \ldots < u_0^k$. The grid initialized by means of the set $\{0.05, 0.1, 0.2, 0.3, 0.4, 0.5\}$ and then is enriched, on the fly, by bisecting a certain interval $[u_0^t, u_0^{t+1}]$ through the ineserion of the new grid point $u_0' := (u_0^t + u_0^{t+1})/2$.

4.3 Split cuts solving a nonlinear MIP

Dash, Günlük and Lodi [11] addressed the optimization over the split closure by looking at the corresponding MIR inequalities and, more precisely, developed a mixed integer nonlinear model and linearized it in an effective way.

For the ease of writing the model, we slightly change the definition of polyhedron $P(x, y)$ by putting the constraints in equality form as:

$$P(x, y) = \{(x, y) \in R^n_+ \times R^r_+ \ : \ Ax + Cy + Is = b, \ s \geq 0\} \tag{40}$$

through the addition of nonnegative slack variables s.

We are looking for an MIR inequality in the form

$$u^+ s + \hat{\beta}\bar{\alpha}x \geq \hat{\beta}(\bar{\beta} + 1) \tag{41}$$

where $\bar{\alpha}, \bar{\beta}$ are constrained to be integer, u^+ is nonnegative, and $0 < \hat{\beta} < 1$.

In particular, inequality (41) is derived by solving the system

$$\begin{aligned}
\bar{\alpha} &= u^T A && (42)\\
u^T C &= 0^T && (43)\\
\hat{\beta} + \bar{\beta} &\leq u^T b && (44)\\
0 < \hat{\beta} &< 1 && (45)\\
\bar{\alpha}, \quad \bar{\beta} & \quad \text{integer.} && (46)
\end{aligned}$$

Of course, equality $u^T s + \bar{\alpha}^T x = u^T b$ is valid as well as inequality $u^+ s + \bar{\alpha}^T x \geq \hat{\beta} + \bar{\beta}$ where $u_i^+ = \max\{u_i, 0\}$. Using the basic MIR inequality (7) we obtain the validity of inequality (41) above.

Let $\sum_{k \in K} \epsilon_k < 1$ (e.g., $\epsilon_k = 2^{-k}$). We approximate $\hat{\beta}$ with $\sum_{k \in \bar{K}} \epsilon_k$ for some $\bar{K} \subset K$ and write the RHS of the MIR inequality as $\sum_{k \in \bar{K}} \epsilon_k \Delta$ where $\Delta = (\lceil \beta \rceil - \bar{\alpha}x^*)$. Using the fact that there is a violated MIR inequality if and only if there is one with $\Delta < 1$, we have the following formulation for the separation of the most violated MIR inequality, where for each $k \in K$ we set $\pi_k = 1$ if $k \in \bar{K}$, $= 0$ otherwise.

$$\min u^+ s^* - \epsilon^T \Phi \quad + \quad \hat{\gamma}^T y^* + \hat{\alpha}^T x^* \tag{47}$$

$$
\begin{aligned}
\hat{\gamma}_j &\geq u^T C_j \quad \forall j = 1, \ldots, r && (48)\\
\hat{\alpha}_j + \bar{\alpha}_j &\geq u^T A_j \quad \forall j = 1, \ldots, n && (49)\\
\hat{\beta} + \bar{\beta} &\leq u^T b && (50)\\
\hat{\beta} &= \sum_{k \in K} \epsilon_k \pi_k && (51)\\
\Delta &= (\bar{\beta} + 1) - \bar{\alpha}^T x^* && (52)\\
\Phi_k &\leq \Delta \qquad \forall k \in K && (53)\\
\Phi_k &\leq \pi_k \qquad \forall k \in K && (54)\\
u_i^+ &\geq u_i \qquad \forall i = 1, \ldots, M && (55)\\
u^+,\ \hat{\alpha},\ \hat{\beta}, \hat{\gamma} &\geq 0 && (56)\\
\bar{\alpha},\ \bar{\beta} \text{ integer}, \qquad & \pi \ \in \ \{0,1\}^{|K|} && (57)
\end{aligned}
$$

where $M := \{i : s_i^* > 0, i = 1, \ldots, m\}$, i.e., we define a variable u_i^+ only if the corresponding constraint i written in 'less or equal form' is not tight. The above approximate model turns out to be an exact model if K is chosen appropriately, as discussed in the following theorem.

Theorem 4.2 *[11] Let Γ be the least common multiple of all subdeterminants of $A|C$, $K = \{1, \ldots, log\Gamma\}$, and $\epsilon_k = 2^k/\Gamma, \forall k \in K$. Then, system (47)–(57) is an exact model.*

5 A Computational Overview

In this section we discuss some simple issues that turn out to be crucial to make the presented models solvable. Moreover, we show their strengthen by reporting computational results on MIPs included in the MIPlib 3.0 [4] and, finally, we discuss future directions that should be addressed to really make the models practical.

5.1 Making the models solvable

All papers discussed in the previous sections implement pure cutting plane approaches in which (as usual) the following steps are iteratively repeated:

1. the continuous relaxation of the mixed integer program at hand is solved;
2. the separation problem is (heuristically) solved and a set of violated constraints is eventually found;
3. the constraints are added to the original formulation.

Of course, the original formulation becomes larger and larger but in order to provide cuts of rank 1, the separation problem solved at step 2 above only uses the original constraints in the cut derivation. For what concerns the solution of those separation problems, it is important that state-of-the-art MIP solvers such as `ILOG-Cplex` or `Xpress Optimizer` are used, as they incorporate very powerful heuristics that are able to find (and then improve) feasible solutions in short computing time. Indeed, good heuristic solutions are enough for step 2 above, where the NP-hard separation problem does not need to be solved to optimality[2] since any feasible solution provides a valid inequality cutting off the current solution of step 1 above.

In order to make these MIPs solvable, a few issues have to be addressed.

All authors noted that only integer variables in the support of the fractional solution of step 1 above have to be considered, e.g., a constraint $\alpha_j \leq u^T A_j$ for j such that $x_j^* = 0$ is redundant because α_j (times x_j^*) does not contribute to the violation of the cut, while it can be computed a posteriori by an efficient post-processing procedure. It is easy to see that this is also the case of integer variables whose value is at the upper bound, as these variables can be complemented before separation.

The ultimate goal of the cutting plane sketched above is to find, for each fractional point (x^*, y^*) to be cut off, a "round" of cuts that are significantly violated and whose overall effect is as strong as possible in improving the current LP relaxation. A major practical issue for accomplishing such a goal is the strength of the returned cuts. As a matter of fact, several equivalent solutions of the separation problems typically exist, some of which produce very weak cuts for the MIP model. This is because the separation problem actually considers the face $F(x^*, y^*)$ of P_I where all the constraints that are tight at (x^*, y^*) (including the variable bounds) are imposed as equalities. Hence, for this face there exist several formulations of each cut, which are equivalent for $F(x^*, y^*)$ but not for P_I.

Fischetti and Lodi [15] experimented a practical relation between the strength of a cut and the sparsity of the vector of multipliers u generating it. In particular, they introduced a penalty term $-\sum_i w_i u_i$ (where i denotes the index of a constraint) in the objective function (8), whose side effect is also to make the cut itself sparser which has obvious advantages for the LP problems solved on step 1 of the cutting plane procedure[3].

The importance of making the cuts as sparse as possible has been also documented by Balas and Saxena [3], who noticed that split disjunctions with sparse support tend to give rise to sparse split cuts.

Another interesting issue raising up to accelerate the cutting plane procedure is finding set of cuts whose overall behavior is as effective as possible, thus the overall cutting plane algorithm requires a relatively small number of iterations. This issue is by far the most crucial one in the attempt of making these methods computationally attractive, and is related to the need of finding a set of cuts which are "as diverse as possible" one each other. In this respect, Fischetti and Lodi [15] have observed that a positive diversification effect can be obtained by just allowing more freedom in the multiplier selection. Indeed, it is well know for integer programs that, in case the constraint matrix (A, b) is integral, one

[2]Except eventually in the last step, in which one needs a proof that no additional violated cut exists.

[3]The same sparsification trick is also used in Bonami et al. [5].

can constrain the Chvátal-Gomory multipliers by $u_i < 1$. Surprisingly enough, in [15] we discovered that removing those bounds speeds up the convergence of the overall cutting plane procedure, and we interpreted this phenomenon as a consequence of the enlarged range of multipliers allowed [4].

Finally, one can expect that diversification can be strongly improved by exploiting cuts obtained by heuristically solving two or more of the discussed separation models; promising results in this direction have been obtained by combining either CG or pro-CG cuts with MIR inequalities [19].

5.2 Strengthen of the closures

The strengthen of the closures, namely CG, pro-CG and split (or MIR), have been evaluated by running cutting plane algorithms for large (sometimes huge) computing times. Indeed, the goal of the investigation was in all cases to show the tightness of the closures, rather than investigating the practical relevance of the separation MIPping idea when used within a MIP solver. On the other hand, as discussed in the previous section, several techniques can be implemented to speed up the computation and, even in the current status, the MIPping separation approach is not totally impractical. Indeed, one can easily implement a hybrid approach in which the MIP-based separation procedures are applied (for a fixed amount of time) in a preprocessing phase, resulting in a tighter MIP formulation to be solved at a later time by a standard MIP solver. Using this idea, two unsolved MIPlib-2003 [1] instances, namely `nsrand-ipx` and `arki001`, have been solved to proven optimality for the first time by Fischetti and Lodi [15] and by Balas and Saxena [3], respectively. In other words, for very difficult and challenging problems it does pay to improve the formulation by adding cuts in these closures before switching to either general- or special-purpose solution algorithms.

In Tables 1 and 2 we report, in an aggregated fashion, the tightness of the closures for MIPlib 3.0 [4] instances, in terms of percentage of gap closed[5] for pure integer and mixed integer linear programs, respectively.

		Split closure	CG closure
% Gap closed	Average	71.71	62.59
% Gap closed	98-100	9 instances	9 instances
% Gap closed	75-98	4 instances	2 instances
% Gap closed	25-75	6 instances	7 instances
% Gap closed	< 25	6 instances	7 instances

Table 1: Results for 25 *pure* integer linear programs in the MIPlib 3.0.

Most of the results reported in the previous tables give a lower approximation of the exact value of the closures[6], due to the time limits imposed on the

[4] In addition, leaving more freedom tothe multipliers seems to improve the effectiveness of the heuristics used by `ILOG-Cplex` 9.

[5] Computed as $100 - 100(opt\_value(P_I) - opt\_value(P^1))/(opt\_value(P_I) - opt\_value(P))$.

[6] In particular, the time limit in [5] to compute a bound of the pro-CG closure is rather

		Split closure	pro-CG closure
% Gap closed	Average	84.34	36.38
% Gap closed	98-100	16 instances	3 instances
% Gap closed	75-98	10 instances	3 instances
% Gap closed	25-75	2 instances	11 instances
% Gap closed	< 25	5 instances	17 instances

Table 2: Results for 33 *mixed* integer linear programs in the MIPlib 3.0.

cutting plane algorithms. Nevertheless, the picture is pretty clear and shows that, although one can construct examples in which the rank of the facets for a polyhedron is very large, in most practical cases the inequalities of rank 1 already give a very tight approximation of the convex hull of integer and mixed integer programs.

Future directions of work should therefore concentrate on the possibility of speeding-up the separation phase by avoiding the explicit definition of the MIP separation models to be solved by a black-box solver, and should address the designing of very fast ad-hoc heuristics that use the underlying MIP separation model only in an implicit way.

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short, 20 CPU minutes, and there are pathological instances for which such a closures is ineffective, see [5] for details.

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