Robustness by cutting planes and the Uncertain Set Covering Problem

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Abstract

In this paper we consider robust optimization as a tool to face uncertainty affecting some input parameter. Our starting point is the well-known Bertsimas-Sim (BS) approach to robustness, leading to a compact robust counterpart of a (pure or integer) linear program requiring the introduction of additional variables and constraints. Robustness can alternatively be enforced by working directly on the space of the original variables, without the need of artificial variables, at the expenses of the addition of an exponential number of "robustness cuts" that can be separated in a very efficient way. The practical performance of the corresponding cutting-plane approach is investigated by computational experiments, showing that a considerable speedup can be achieved with respect to the standard BS model.

We also point out that the cutting plane approach to robustness has some important features that can make it the method of choice (or even the only available option) in important applications. Indeed, the approach can be applied to handle problems whose nominal formulation is itself noncompact but requires an exponential number of constraints. In addition, the separation procedure for robustness cuts can take into account uncertainty domains that are much more complex (and realistic) than in the BS approach. A notable case arises when the uncertainty domain involves yes-no decisions that cannot be modeled by continuous variables (thus making the BS approach unapplicable), but can be described by a knapsack constraint. We describe a practically relevant application of this approach, namely the Uncertain Set Covering Problem where each column has a certain probability of "disappearing" (a yes-no event), and each row has to be covered with a high probability. An uncertain graph connectivity problem is also investigated. Computational experiments on both instances from the literature and on randomly generated instances are provided.

Key words: Robust Optimization, Cutting Planes, Uncertain Set Covering and Connectivity Problems.

1 Introduction

One of the main issues when facing with real-world optimization problems is the determination of *robust* solutions, i.e., solutions that are stable with respect to certain variations of the input parameters. An increasing amount of research has been devoted to this subject in the last years, since the exact value of the input data is generally unknown in real-world applications. Two main approaches have been proposed so far for dealing with uncertain data: stochastic programming and robust optimization.

Stochastic programming introduces additional variables and penalizes feasible solutions that are most likely to become infeasible due to uncertainty. Hence, it requires some knowledge of how uncertainty will act (which is not always the case in some practical applications) and often leads to very huge models, that can be extremely hard to be solved.

Robust optimization associates uncertainty with hard constraints restricting the solution space, i.e., one is required to find a solution that is still feasible for worst-case parameters chosen within a certain uncertainty domain. This is a simple way to model uncertainty, but it can lead to overconservative solutions that are quite bad in terms of cost (actually, a feasible solution may not exist at all). A breakthrough in robust optimization is the recent work by Bertsimas and Sim (BS) [7], where a compact way to model the robust counterpart of a given "nominal" model is proposed. The approach requires the introduction of a polynomial number of new variables and constraints into the nominal model, hence it does not increase the *theoretical* complexity of the problem to be solved with respect to the nominal one—though the practical solution time can be affected heavily. A simplified version of the BS approach named "light robustness" has been proposed recently by the authors in [14].

The starting point of the research reported in the present paper is the observation that BS robustness can alternatively be enforced by working directly on the space of the original (nominal) variables, without the need of artificial variables, at the expenses of the addition of an exponential number of linear constraints that reduce the feasibility region. As explained in more details in the sequel, the separation problem for these *robustness cuts* can be carried out very efficiently, as it amounts to the solution of the LP relaxation of a simple cardinality-constraint knapsack problem for each uncertain row of the nominal problem. As a result, an effective cutting-plane approach for robustness is conceivable, that generates robustness cuts on the fly. As far as we know, no computational experiments have been performed to compare the *practical* performance of the BS compact formulation and of its cutting plane counterpart. In this paper we investigate this issue, and show through extensive computation tests that the cutting plane approach can be significantly faster—up to 2 orders of magnitude, for certain LP instances.

Besides practical considerations, the cutting plane approach has some important features that can make it the method of choice (or even the only available option) in important applications. First of all, it can be applied to handle problems whose nominal formulation is itself noncompact but requires an exponential number of constraints; e.g., routing problems such as the Traveling Salesman Problem (TSP) that involve connectivity constraints. More importantly, the separation procedure for robustness cuts can take into account uncertainty domains that are much more complex (and realistic) than in the BS approach. A notable case arises when the uncertainty domain involves yes-no decisions that cannot be modeled by continuous variables (thus making the BS approach—that relies LP duality—unapplicable), but can be described by a knapsack constraint. In this situation, separation can be still be performed in an effective way, though its theoretical complexity becomes exponential. We will address explicitly a practically relevant application of this approach, namely the *Uncertain Set Covering Problem* (USCP) where each column has a certain probability of "disappearing" (a yes-no event), and each row has to be covered with a high probability. Our USCP model finds applications also in problems that can be reformulated through set covering constraints, e.g., in the uncertain counterpart of connectivity problems in graphs modeled through cut conditions.

The present paper is organized as follows. In Section 2 we present the main methods proposed in the literature to handle uncertainty, focusing on the compact formulation by Berstimas and Sim [7]. In the same section we introduce our cutting plane approach and the associated separation procedure that implements the projection of the BS model onto the space of the original variables. Section 3 introduces the Uncertain Set Covering Problem, where cutting planes are the most natural way to face uncertainty. For this specific problem, we also propose an ad-hoc USCP formulation and analyze its computational behavior. Section 3.2 addresses a connectivity problem that has a set covering substructure that allows for the application of our USCP formulation. Finally, Section 4 summarizes our findings and draws some conclusions.

2 Modeling robustness

In this paper we concentrate on the robust counterpart of a generic "nominal" Linear Program (LP) of the form

$$\min \sum_{j \in N} c_j \, x_j \tag{1}$$

$$\sum_{j \in N} a_{ij} \, x_j \leq b_i, \qquad i \in M, \tag{2}$$

$$x_j \geq 0, \qquad j \in N, \tag{3}$$

where $N = \{1, \cdots, n\}$ and $M = \{1, \cdots, m\}$ denote the variable and constraint index sets, respectively. Our results can be applied in a straightforward way to Mixed-Integer (or Pure) Linear Programs as well—as a matter of fact, enumerative methods reduce the solution of these problems to the solution of a sequence of LPs.

The first attempt to handle data uncertainty through mathematical models was performed by Soyster [21], who considered uncertain problems of the form

$$\min \; \{\sum_{j \in N} c_j \; x_j \mid \sum_{j \in N} A_j \; x_j \leq b, \; \forall A_j \in \mathcal{K}_j, j \in N\}$$

where $\mathcal{K}_j$ are convex sets associated with "column-wise" uncertainty. This approach tends to lead to overconservative models, thus to poor solutions in term of optimality. Later, Ben-Tal and Nemirovski [2, 3, 4] defined less conservative models by considering ellipsoidal uncertainties. Moreover, [2] shows that the robust counterpart of an uncertain LP is equivalent to an explicit computationally tractable problem, provided that the uncertainty is itself "tractable". On the contrary, when the problem to be considered is an ILP, these nonlinear (convex) models become computationally hard problems.

2.1 The Bertsimas and Sim approach

Recently, Bertsimas and Sim [7, 6] considered a different concept of robustness. Taking into account the fact that we require $x \geq 0$ in our setting, the BS definition of robustness can be outlined as follows. It is assumed that each coefficient in the constraint matrix A can take any value $\widetilde{a}_{ij} \in [a_{ij} - \hat{a}_{ij}, a_{ij} + \hat{a}_{ij}]$ and the deviation of such value with respect to the nominal one is independent of the changes of the remaining coefficients. As it is unlikely that all coefficients take their worst value, a solution is considered *robust* if it remains feasible when at most Γ_i coefficients (chosen in any way) in row i ($i \in M$) take their worst value. The underlying assumption here is that, when more than Γ_i coefficients change, their deviation will tend to compensate one each other, the net effect being comparable with the worst-case deviation of no more than Γ_i coefficients.

For the sake of simplicity, we will assume that all Γ_i's are integer, although this is not required in original BS approach. Note that $\Gamma_i = 0$ means that robustness is not taken into account and the nominal constraint is considered, whereas $\Gamma_i = |N|$ means that each coefficient in row i can take its worst value, and corresponds to the conservative method by Soyster [21].

According to BS, given the nominal LP (1)–(3) one defines another LP (in an extended space) whose optimal solution remains feasible for every change of, at most, Γ_i coefficients (up to their worst value) in each row $i \in M$. To this end, the i-th constraint of the nominal problem is first replaced by

$$\sum_{j \in N} a_{ij}\ x_j + \beta(x, \Gamma_i) \leq b_i \tag{4}$$

where term

$$\beta(x, \Gamma_i) = \max_{S \subseteq N : |S| \leq \Gamma_i} \sum_{j \in S} \hat{a}_{ij}\ x_j \tag{5}$$

indicates the level of protection of the solution found with respect to the uncertainty associated with row i (recall that we assume $x \geq 0$). By using LP duality, the robust counterpart of the nominal problem (1)–(3) then becomes:

$$\min \sum_{j \in N} c_j\ x_j \tag{6}$$

$$\sum_{j \in N} a_{ij}\ x_j + \Gamma_i\ z_i + \sum_{j \in N} p_{ij} \leq b_i, \qquad i \in M, \tag{7}$$

$$-\hat{a}_{ij}\ x_{ij} + z_i + p_{ij} \geq 0, \qquad i \in M, j \in N, \tag{8}$$

$$z_i \geq 0, \qquad i \in M, \tag{9}$$

$$p_{ij} \geq 0, \qquad i \in M, j \in N, \tag{10}$$

$$x_j \geq 0, \qquad j \in N. \tag{11}$$

2.2 Robust optimization through cutting planes

The robust BS formulation (6)-(11) given in the previous section has the very nice property of being *compact*, in the sense that it involves a number of variables and constraints that is polynomial in the input size. Its projection onto the space of the original x variables requires however an exponential number of cuts, and can be obtained along the following lines.

Given the i-th constraint of the nominal problem, the robust constraint (4) can be expressed through the following *robustness cuts*:

$$\sum_{j \in N} a_{ij}\ x_j + \sum_{j \in S} \hat{a}_{ij}\ x_j \leq b_i, \qquad \forall S \subseteq N : |S| \leq \Gamma_i. \tag{12}$$

The separation problem for robustness cuts can be stated as follows: given the current solution x^*, find a set $S \subseteq N$ such that: (i) $|S| \leq \Gamma_i$, and (ii) $\sum_{j \in S} \hat{a}_{ij}\ x_j^*$ is a maximum. The former condition imposes that at most Γ_i coefficients change with respect to their nominal values, while the latter looks for the constraint (12) associated with row i that is most violated by the current solution x^*.

For each row i, the above separation problem can be solved easily by associating each variable x_j with a *profit* $\hat{a}_{ij}\ x_j^*$ and by finding a maximum-profit subset S with at most Γ_i elements. If the sum of the profits of the selected items exceeds $b_i - \sum_{j \in N} a_{ij}\ x_j^*$, a violated constraint is found; otherwise all constraints (12) associated with row i are satisfied by x^*. The separation problem can therefore be solved by just selecting the (at most) Γ_i variables with largest positive profit, and requires $O(n)$ time by using a partial-sorting technique for determining the Γ_i-th largest entry in the profit array [13].

It then follows that the robust BS model can be solved in polynomial time also by cutting planes. Obviously, the approach leads to a branch-and-cut algorithm when (M)ILPs (as opposed to LPs) have to be faced.

2.3 Computational experiments

In this section we compare computationally the performance of the original BS approach and of our cutting planes algorithm.

Our algorithm was coded in C language and embedded into the commercial solver `ILOG-Cplex` version 11.0, by using its default parameter setting. At each iteration, the most violated robustness cut associated with each nominal constraint is added to the current problem. All codes were run on a PC AMD Athlon 4200+ with 4 GB ram.

We considered a large testbed of LP instances from the NETLIB, as suggested by Ben-Tal and Nemirovski in [3]. In particular, we addressed all instances with at least 1000 variables and 1000 constraints, for which all variables are required to be nonnegative and that include at least one inequality constraint (as customary, the coefficients arising in equality constraints, if any, are assumed not to be affected by uncertainty).

Table 1 reports, for each instance in our test bed, the number of variables n and constraints m of the nominal problem and the number of additional variables n' and constraints m' required by the compact formulation (6)–(11).

Instance	n	m	n'	m'
BNL2	3489	2325	11555	9437
D2Q06C	5167	2172	17426	13766
DEGEN3	1818	1504	23692	22591
GANGES	1681	1310	2006	1609
PILOT	3652	1442	44330	40911
SCTAP2	1880	1091	7804	6394
SCTAP3	2480	1481	10354	8494
SHIP12L	5427	1152	11068	6686
SHIP12S	2763	1152	5740	4022
STOCFOR2	2031	2158	6459	5571
STOCFOR3	15695	16676	50013	43147
WOODW	8405	1099	9891	2571

Table 1: LP instances from the NETLIB in our test bed.

Uncertainty was modeled by allowing each coefficient appearing in an inequality constraint to differ by at most 1% from its nominal value, in a similar way to what suggested in [7]. Moreover, we considered $\Gamma_i = \Gamma$ for each row i, where $\Gamma \in \{1, 10, 50\}$.

Table 2 reports computational results on LP instances and gives, for each instance, the optimal solution value z and associated computing time T for the nominal problem. In addition, for each value of Γ we give the percentage increase of the robust solution value z_R

with respect to the nominal one (namely, $\%\Delta z = 100 * (z_R - z)/|z|$), the computing time t_{BS} required for solving model (6)–(11), the computing time t_{CP} required by our cutting plane method, along with the number $\#cuts$ of robustness cuts added. Note that we let `ILOG-Cplex` choose the most appropriate algorithm for solving both model (6)–(11) and the first LP within our cutting plane scheme (i.e., the nominal problem), whereas LPs after separation are reoptimized by using the dual simplex. Finally, the last line of the table gives the average computing time of each method over all the instances in the testbed.

Instance			$\Gamma = 1$				$\Gamma = 10$				$\Gamma = 50$			
	z	T	$\%\Delta z$	t_{BS}	t_{CP}	$\#cuts$	$\%\Delta z$	t_{BS}	t_{CP}	$\#cuts$	$\%\Delta z$	t_{BS}	t_{CP}	$\#cuts$
BNL2	1811.237	0.07	0.790	1.26	0.50	647	1.840	1.69	0.97	839	1.847	1.96	0.30	411
D2Q06C	122784.211	2.71	`inf.`	18.76	3.84	285	`inf.`	41.77	5.26	306	`inf.`	25.90	5.84	308
DEGEN3	-987.294	0.43	`inf.`	58.34	1.45	492	`inf.`	88.69	1.84	492	`inf.`	339.47	2.07	496
GANGES	-109585.736	0.01	0.053	0.02	0.01	25	0.430	0.02	0.05	31	0.474	0.02	0.02	24
PILOT	-557.490	3.05	`inf.`	661.72	12.95	1091	`inf.`	58.95	4.85	1094	`inf.`	47.14	4.95	1094
SCTAP2	1724.807	0.01	1.533	0.12	0.24	403	2.814	0.18	0.62	631	2.844	0.15	0.05	126
SCTAP3	1424.000	0.02	1.602	0.17	0.39	466	2.995	0.24	1.46	823	3.040	0.23	0.08	150
SHIP12L	1470187.919	0.04	0.060	0.10	0.05	45	0.346	0.11	0.08	63	0.353	0.14	0.09	61
SHIP12S	1489236.134	0.02	0.062	0.06	0.03	57	0.386	0.09	0.05	77	0.390	0.08	0.05	63
STOCFOR2	-39024.409	0.06	0.759	0.56	0.18	484	1.522	0.55	0.17	459	1.522	0.72	0.18	459
STOCFOR3	-39976.784	1.58	0.733	11.68	12.95	4135	1.482	10.41	11.61	3958	1.482	16.86	11.59	3958
WOODW	1.304	0.09	0.447	0.12	0.12	5	1.280	0.15	0.11	3	1.280	0.15	0.10	3
avg.		0.67		62.74	2.73			16.90	2.26			36.07	2.11	

Table 2: Results on robust LP instances from the NETLIB (`inf.` indicates that no feasible robust solution exists)

Results of Table 2 show that, for most instances, the computing time required by our cutting plane approach is considerably smaller than that required to solve the compact BS formulation. In particular, a speedup of 1-2 orders of magnitude is achieved in the cases where a feasible robust solution does not exist.

3 Cutting planes as the only option

The computational experiments reported in Section 2.3 show that the cutting planes approach is typically more efficient in handling uncertainty with respect to the compact formulation.

In this section we address situations where a cutting plane approach is actually the *only* possible way to face uncertainty. These situations arise when the uncertainty domain cannot be fully described by a linear system, hence making it impossible to use the compact BS formulation that heavily relies on LP duality.

We first consider an uncertain version of the Set Covering Problem, and propose alternative ILP models for its solution. Then, our results are extended to a class of problems that admit a set covering formulation, focusing in particular on the connectivity problem on a "uncertain" graph whose edges can disappear with a certain probability.

3.1 The Uncertain Set Covering Problem

Given a 0-1 $m \times n$ matrix $A = (a_{ij})$ and an n-dimensional integer vector $c = (c_j)$ representing the cost of each column, the *Set Covering Problem* (SCP) requires to select a subset S of columns such that

- the sum of the costs of the selected columns is minimized;
- for each row i ($i = 1, \dots, m$) there exists at least one column $j \in S$ such that $a_{ij} = 1$.

Set Covering is a a useful model for several important practical problems, and arises as a subproblem in many applications; see Caprara et al. [10] and Gamache et al. [15] for applications to railways and airline crew scheduling, or Balas [1] and Ceria et al. [11] for surveys on applications of SCP to location, routing and other problems. The huge amount of literature on SCP includes both exact and heuristic algorithms; a computational comparison of the main solution methods is reported in Caprara et al. [9].

Let $N = \{1, \dots, n\}$ and $M = \{1, \dots, m\}$ be the column set and the row set, respectively. Moreover, for each row $i \in M$, let $J_i = \{j \in N : a_{ij} = 1\}$ denote the set of columns covering row i. A straightforward ILP model for SCP is as follows:

$$\min \sum_{j \in N} c_j \, x_j \tag{13}$$

$$\sum_{j \in J_i} x_j \geq 1, \qquad i \in M, \tag{14}$$

$$x_j \in \{0, 1\}, \qquad j \in N, \tag{15}$$

where each variable x_j takes value 1 if column j is selected, and 0 otherwise.

We next introduce a variant of SCP called the *Uncertain* Set Covering Problem (USCP), arising when each column $j \in N$ has an associated positive value p_j, denoting the probability that the column disappears (i.e., that all coefficients in column j become zero), and each row $i \in M$ has associated a positive value $\overline{P_i}$ representing the minimum probability for row i to be covered by at least one selected column. We assume that probabilities associated with columns are independent. The problem requires to determine a set S of columns that minimizes objective function (13) and satisfies the i-th constraint (14) with probability greater than $\overline{P_i}$. USCP is strongly NP-hard, since it generalizes SCP, and arises in many practical problems, including crew scheduling applications where columns are associated with feasible pairings, and a column disappearing from the model corresponds to the nonshow of a crew.

USCP is akin to a probabilistic problem from the literature known as the *Probabilistic* Set Covering Problem (PSCP) [5], in which one wants to optimize over the set of binary vectors x such that $\mathbb{P}\{A^T \, x \geq \xi\} \geq p$, where ξ is a random binary right-hand side and p is a threshold input value. Clearly, requiring that constraints $A^T \; X \geq \xi$ are satisfied for all possible realizations of ξ would lead to deterministic SCP; on the contrary, PSCP requires to satisfy constraints with a probability no smaller than p. PSP was first studied in the literature by Beraldi and Ruszczyński [5], who proposed an exact algorithm based on the iterative solution of deterministic SCPs. Recently, Saxena et al. [20] reformulated PSCP as a mixed integer nonlinear program, linearized the corresponding model, and solved the resulting MILP with a general purpose commercial code.

Using again variables x_j ($j \in N$), USCP can be formulated as follows:

$$\min \sum_{j \in N} c_j \, x_j \tag{16}$$

$$\mathbb{P}\{a_i^T \, x \geq 1\} > \overline{P_i}, \qquad i \in M, \tag{17}$$

$$x_j \in \{0, 1\}, \qquad j \in N. \tag{18}$$

Recall that probabilities associated with columns are assumed to be independent.

Given a solution x^* we will denote by $J_i^* = \{j \in J_i : x_j^* = 1\}$ the set of columns covering row $i \in M$. If solution x^* satisfies the i-th nominal constraint (14), the probability that the associated uncertain constraint is violated is given by $\mathbb{P}\{a_i^T\ x^* < 1\} = \prod_{j \in J_i^*} p_j$. We define as feasible only those solutions for which this probability is not larger than $1 - \overline{P_i}$, i.e., the solutions such that, for each row $i \in M$

$$\mathbb{P}\{a_i^T\ x < 1\} = \prod_{j \in J_i} p_j \leq 1 - \overline{P_i} \tag{19}$$

Defining the nonnegative quantities $w_j = -\ln p_j$ $(j \in N)$ and $\overline{W_i} = -\ln(1 - \overline{P_i})$, the feasibility condition with respect to row i reads:

$$\sum_{j \in J_i} w_j \geq \overline{W_i} \tag{20}$$

Without loss of generality we assume that, for each row i, condition (19) is satisfied by selecting $J_i^* = J_i$, since otherwise it would be impossible to cover row i with the required probability $\overline{P_i}$, and USCP would be infeasible—despite the feasibility of the deterministic Set Covering Problem (13)–(15).

3.1.1 A basic ILP model for USCP

Following the approach described in Section 2.2, a model for USCP can be derived by (16)–(18) by replacing the i-th covering constraint with its "uncertain" counterpart

$$\sum_{j \in J_i} x_j - \beta(x, \overline{W_i}) \geq 1, \tag{21}$$

where term $\beta(x, \overline{W_i})$ represents the maximum decrease of the left-hand side associated with uncertain situations we want to take care of. In our problem, these situations are those that can arise with a probability not smaller than $1 - \overline{P_i}$.

Our first way to model constraints (21) in a linear way is through the following (noncompact) ILP:

$$\text{(M1)} \qquad \min \sum_{j \in N} c_j\ x_j \tag{22}$$

$$\sum_{j \in J_i} x_j - \sum_{j \in S} x_j \geq 1, \qquad S \subseteq J_i : \sum_{j \in S} w_j < \overline{W_i},\ i \in M, \tag{23}$$

$$x_j \in \{0, 1\}, \qquad j \in N. \tag{24}$$

Constraints (23) impose that each row $i \in M$ must be covered by a subset of columns having a small probability to disappear all together. This means that a nonempty set S of columns corresponding to a solution which is not feasible according to (20), is not enough for covering row i: some additional column must be selected in order to provide a feasible solution.

The exponential number of constraints in the model immediately suggests to adopt a cutting planes algorithm where constraints (23) are added to the formulation on the fly, when they are needed. Given the current (possibly fractional) solution x^*, the separation problem associated to a given row i requires to find (if any) a set $S_i \subseteq J_i$ such that $\sum_{j \in S_i} w_j < \overline{W_i}$ and $\sum_{j \in S_i} x_j^* > \sum_{j \in J_i} x_j^* - 1$, and can be solved through the following ILP:

$$\beta(x^*, \overline{W_i}) = \quad \max \sum_{j \in J_i} x_j^* \; d_{ij} \tag{25}$$

$$\sum_{j \in J_i} w_j \; d_{ij} < \overline{W_i}, \tag{26}$$

$$d_{ij} \in \{0, 1\}, \qquad j \in J_i. \tag{27}$$

Then, set S_i if defined as $S_i = \{j \in J_i : d_{ij}^* = 1\}$. If $\beta(x^*, \overline{W_i}) = \sum_{j \in S_i} x_j^* > \sum_{j \in J_i} x_j^* - 1$, then constraint (23) associated with row i and set S_i is violated by x^* and has to be added to the current formulation.

3.1.2 An alternative ILP model for USCP

The separation problem for (23) is a genuine ILP (actually, a 0-1 knapsack problem), so we cannot apply LP duality to fully characterize the solutions x^* that do not lead to violated cuts—as in the BS approach. In other words, a compact LP is not readily available that provides the same LP bound attainable through a pure cutting plane procedure based on the exact separation of cuts (23). However, this does not imply that a compact ILP formulation for USCP does not exist, whose LP relaxation provides a *different* bound than the exponential formulation (22)-(24). As a matter of fact, by noting that constraints (23) require each row $i \in M$ be covered by a suitable subset of columns, a compact ILP model for USCP is as follows:

$$\text{(M2)} \qquad \min \sum_{j \in N} c_j \; x_j \tag{28}$$

$$\sum_{j \in J_i} w_j \; x_j \geq \overline{W_i}, \qquad i \in M, \tag{29}$$

$$x_j \in \{0, 1\}, \qquad j \in N, \tag{30}$$

where each x_j variable has the same meaning as in model M1.

The validity of model M2 heavily depends on the fact that the deterministic SCP only involves constraints with 0-1 coefficients and right-hand sides, and is formally established by the following theorem.

Theorem 1 *Let $S1 = \{x \in \{0,1\}^n : (23)$ hold $\}$ and $S2 = \{x \in \{0,1\}^n : (29)$ hold $\}$ the set of feasible solutions to M1 and to M2, respectively. Then $S1 = S2$.*

Proof. We first prove that any solution x^* feasible for M1 satisfies the i-th constraint (29). Recall that $J_i^* = \{j \in J_i : x_j^* = 1\}$ and assume, by contradiction, that x^* violates the i-th constraint (29) in M2, i.e., $\sum_{j \in J_i^*} w_j < \overline{W_i}$. A feasible (indeed, optimal) solution to the separation problem (25)–(27) associated to solution x^* is $S_i = J_i^*$, thus $\beta(x^*, \overline{W_i}) = \sum_{j \in J_i^*} x_j^*$, which makes constraint (23) associated with set S_i violated. Hence, it is impossible that constraint (29) is violated by solution x^*.

Analogously we prove that, given any feasible solution x^* to M2, the i-th constraint (23) is satisfied by x^* as well. Note that only variables d_{ij} associated with columns $j \in J_i^*$ have to be taken into account in the separation problem (25)–(27). As x^* is feasible for M2, we have $\sum_{j \in J_i^*} w_j \geq \overline{W_i}$, hence constraint (26) imposes that not all the d_{ij} variables are set to 1. This implies $\beta(x^*, \overline{W_i}) < |J_i^*|$, which makes all constraints (23) associated with row i satisfied by x^*. □

Theorem 1 states that the sets of feasible solutions to M1 and M2 coincide. However, as already mentioned, this equivalence does not necessarily hold when solutions to the LP relaxations of the two models are considered. The following example shows that no dominance exists among the optimal solution value of the LP relaxation of M1 and the optimal solution value of the LP relaxation of model M2.

Example 1
We are given the following SCP instance with $m = 1$, $n = 3$, $c_j = 1$, $w_j = 3$ ($j = 1, \ldots, 3$) and $\overline{W_1} = 4$. The LP relaxation of M1 yields solution $x_1 = x_2 = x_3 = 1/2$, with a lower bounds equal to 3/2, while an optimal solution for the LP relaxation of M2 is $x_1 = 1$, $x_2 = 1/3$, $x_3 = 0$, having value 4/3. When considering the same instance with $\overline{W_1} = 5$, the optimal solution of the LP relaxation of M1 is the same, while the optimal solution of the LP relaxation of M2 is $x_1 = 1$, $x_2 = 2/3$, $x_3 = 0$, having value 5/3 (worse than the value of the value of the LP relaxation of M1). □

The LP relaxation of model M2 can be strengthened by exploiting the integrality of the x variables appearing in (29). As a matter of fact, due to the definition of the w_j's, these latter constrains are quite "weird" and typically lead to nasty knapsack conditions that are very challenging for MIP solvers.[1] A first trivial strengthening is obtained by just replacing w_j by $\overline{w}_j := \min\{w_j, \overline{W}_i\}$. In addition, given a positive integer k, a simple rounding argument similar to that used to derive Gomory's fractional cuts allows one to derive the valid inequality

$$\sum_{j \in J_i} \lceil \frac{k-1}{\overline{W}_i - \epsilon} \, \overline{w}_j \rceil \; x_j \geq \lceil \frac{k-1}{\overline{W}_i - \epsilon} \, \overline{W}_i \rceil = k \tag{31}$$

where $k \geq 2$ is an integer parameter giving the desired right-hand side value, and ϵ is a small positive value. Because of their combinatorial nature, constraints (31) are numerically more stable than (29). However, it is easy to see that no dominance between (29) and (31) exists.

In order to accelerate the convergence of MIP solver even further, we implemented an ad-hoc heuristic procedure aimed at producing an initial upper bound value. Our heuristic is a simple greedy that starts from a partial solution and iteratively adds a column to the current solution, until the solution becomes feasible. Given the current solution, we check feasibility by scanning constraints (29). If the current solution is feasible, the algorithm stops. Otherwise, a minimum-cost unselected column appearing in the first violated constraint is added to the solution. The algorithm is executed several times, starting from different partial solutions, and returns the best solution found. In our implementations, we start with (i) an empty solution, or (ii) an optimal solution of the nominal problem, or (iii) an optimal solution of the nominal problem restricted to the columns with $w_j \geq \max_i\{\overline{W}_i : a_{ij} = 1\}$,

[1] Fortunately, as shown in the computational section, the rich arsenal of general-purpose preprocessing and cut-generation procedures embedded in modern MIP solvers turns out to be quite effective for M2.

with high-cost slack variables added for each constraint so as to ensure feasibility (this latter starting solution turns out to be quite effective for problems with small $\overline{P}_i$'s).

3.1.3 Computational experiments on USCP instances

In this section we report two different kinds of experiments on USCP instances derived from SCP instances of the literature.

The first set of experiments focuses on the best method for an efficient solution of USCP.

We considered all the set covering instances publicly available at the ORLIB library (web site `http://people.brunel.ac.uk/~mastjjb/jeb/info.html`) and selected 59 of them, namely those that were solved to proven optimality within 600 CPU seconds using `ILOG-Cplex` version 11.0 on our computer.

Starting from these instances, we set the threshold probability associated with each row $i \in M$ as $\overline{P_i} = P^{\min}$, where $P^{\min}$ is a parameter. Probabilities p_j associated with columns $j \in N$ were randomly generated according to a uniform distribution in $[0, 0.2]$. As to the definition of coefficients w_j (resp. $\overline{W_i}$), we computed the logarithm of each p_j (resp. $\overline{P_i}$), multiplied it by 1,000 and rounded the result to the nearest integer.

Tables 3 and 4 report the outcome of our experiments for different values of $P^{\min} \in \{0.85, 0.90, 0.95, 0.99\}$. In particular, for each instance we report the following information:

- optimal value (z^*) and solution time (T) of the nominal problem;
- the best solution found (z_h) by the initial heuristic and the associated computing time (T_h);
- optimal value (z_u) of the uncertain version of the problem (an asterisk indicates that the optimal value is not known);
- computing times for model M1 (22)–(24), for model M2 (28)–(30), and for model M2', i.e., model M2 when constraints (31) for $k = 2$ and $k = 3$ are added to the initial formulation as "lazy constraints"; a time limit equal to 1,800 seconds was given to each model.

The last rows of each table report, for each solution method, the average computing time, in seconds (for the unsolved instances, the time limit is considered), and the number of instances solved to proven optimality within the given time limit.

Computational results show that model M2 (28)–(30) qualifies as the best method to solve USCP. This compact formulation is able to solve all instances but 18, with a reasonable average computing time. Note that increasing the value of $P^{\min}$ from 0.85 to 0.99 leads to increasingly difficult problems and, as be expected, to worse solution values—being the associated problem more constrained.

Quite surprisingly, the addition of constraints (31) in M2' did not improve the overall performance—actually, it produced a certain slow-down. This behavior is due not only to the extra time required to handle the additional cuts, but also to the effectiveness of the general-purpose preprocessing and cut-generation procedures embedded in `ILOG-Cplex`. Indeed, these procedures were able to "squeeze" automatically most of the information conveyed by the additional cuts (31). As a matter of fact, turning preprocessing and cut-generation off the performance of both M2 and M2' deteriorates considerably, with M2' becoming significantly better than M2. E.g., the very easy instance `scpe2` with $P^{\min} = 0.85$ is solved by M2

Nominal problem			$P^{\min} = 0.85$						$P^{\min} = 0.90$					
name	z^*	T	z_h	T_h	z_u	T_{M1}	T_{M2}	$T_{M2'}$	z_h	T_h	z_u	T_{M1}	T_{M2}	$T_{M2'}$
scp41	429	0.01	610	0.04	592	0.04	0.08	0.06	824	0.02	701	0.06	0.05	0.09
scp42	512	0.02	623	0.08	612	0.08	0.08	0.07	830	0.02	782	0.14	0.07	0.10
scp43	516	0.01	662	0.11	662	0.10	0.17	0.10	967	0.03	830	0.64	0.30	0.38
scp44	494	0.04	660	0.09	602	0.07	0.08	0.08	878	0.11	740	0.40	0.20	0.24
scp45	512	0.01	691	0.24	656	0.15	0.13	0.18	907	0.02	797	0.63	0.24	0.34
scp46	560	0.03	674	0.05	655	0.03	0.03	0.07	922	0.09	796	0.48	0.11	0.16
scp47	430	0.02	608	0.02	579	0.05	0.06	0.08	789	0.03	695	0.07	0.06	0.10
scp48	492	0.05	678	0.10	647	0.13	0.08	0.11	900	0.05	841	0.97	0.42	0.27
scp49	641	0.04	813	0.07	809	0.15	0.15	0.18	1070	0.15	984	1.03	0.15	0.62
scp410	514	0.02	715	0.04	708	0.09	0.07	0.11	880	0.02	811	0.17	0.06	0.11
scp51	253	0.10	307	0.13	302	0.10	0.12	0.13	436	0.11	391	0.95	0.27	0.24
scp52	302	0.16	371	0.25	371	0.19	0.25	0.24	501	0.17	459	1.44	1.33	1.03
scp53	226	0.02	303	0.03	300	0.03	0.04	0.07	408	0.04	344	0.16	0.08	0.09
scp54	242	0.07	321	0.08	321	0.09	0.09	0.15	435	0.10	370	0.41	0.13	0.17
scp55	211	0.06	308	0.06	300	0.05	0.09	0.07	420	0.09	350	0.40	0.19	0.18
scp56	213	0.03	300	0.05	291	0.09	0.06	0.12	447	0.05	369	0.66	0.13	0.18
scp57	293	0.09	395	0.14	386	0.09	0.14	0.14	560	0.14	480	0.82	0.37	0.43
scp58	288	0.08	351	0.10	351	0.14	0.09	0.15	433	0.08	415	0.50	0.21	0.15
scp59	279	0.06	370	0.07	370	0.05	0.06	0.08	606	0.13	477	1.71	0.75	0.74
scp510	265	0.03	366	0.07	360	0.17	0.14	0.16	513	0.04	429	0.51	0.21	0.27
scp61	138	0.45	156	0.58	153	0.16	0.29	0.27	212	0.66	199	0.84	0.72	0.71
scp62	146	0.51	181	1.05	174	0.49	0.65	0.54	212	0.61	199	0.80	0.77	0.92
scp63	145	0.37	169	0.48	169	0.21	0.28	0.28	225	0.59	218	2.54	0.90	0.86
scp64	131	0.07	165	0.11	157	0.20	0.20	0.23	213	0.12	196	0.90	0.93	1.22
scp65	161	0.65	220	1.11	219	0.81	0.68	0.78	260	0.66	251	2.47	1.19	1.25
scpa1	253	1.47	299	1.60	291	0.46	0.37	0.48	442	1.79	385	8.68	2.72	3.88
scpa2	252	1.08	318	1.09	300	0.28	0.22	0.27	415	1.06	368	2.06	1.28	1.20
scpa3	232	0.73	288	1.00	278	0.59	0.88	0.59	373	0.72	343	1.85	0.54	0.76
scpa4	234	0.47	287	1.49	284	0.46	0.89	0.65	379	1.75	359	9.68	2.59	3.47
scpa5	236	0.18	328	0.25	312	0.77	0.81	0.94	416	0.22	359	2.87	1.13	1.22
scpb1	69	1.57	97	3.12	97	15.38	9.33	9.69	118	2.49	109	77.24	14.71	33.21
scpb2	76	2.53	90	5.05	89	14.92	3.52	18.65	110	3.47	101	42.92	7.20	21.35
scpb3	80	1.57	101	3.58	100	12.39	3.04	10.03	128	2.73	125	584.20	30.44	45.14
scpb4	79	3.10	92	4.21	91	1.34	1.68	1.54	117	3.58	109	97.96	15.85	14.77
scpb5	72	1.36	88	2.93	88	3.41	1.73	4.44	104	1.60	99	3.94	2.26	2.50
scpc1	227	1.12	288	2.90	287	1.23	1.60	2.38	382	2.74	360	54.56	9.18	16.06
scpc2	219	2.50	275	4.35	275	5.24	3.24	5.83	385	5.09	340	556.18	11.07	26.48
scpc3	243	2.80	311	4.56	310	5.43	2.96	3.26	413	3.60	373	301.95	32.11	32.54
scpc4	219	1.48	275	1.81	275	2.49	2.21	3.60	374	2.82	328	22.91	5.52	5.89
scpc5	215	1.36	278	2.37	278	2.52	1.66	2.87	355	1.94	333	100.92	8.73	11.01
scpd1	60	2.64	79	14.88	76	17.49	13.54	27.42	105	5.43	89	129.47	9.21	21.57
scpd2	66	11.63	82	18.81	82	99.34	32.98	141.47	103	12.78	96	883.13	35.46	60.70
scpd3	72	11.43	90	25.43	90	376.81	55.10	164.62	108	13.62	100	1128.05	45.03	65.99
scpd4	62	5.02	78	7.99	78	29.48	17.79	43.45	101	7.07	91	1086.46	80.37	95.08
scpd5	61	1.91	81	4.39	81	59.84	13.90	21.27	113	4.20	91	38.92	8.34	13.77
scpe1	5	0.39	5	0.61	5	63.80	1.22	0.28	5	1.29	5	74.92	2.34	5.40
scpe2	5	0.44	5	0.70	5	53.17	5.74	8.06	5	0.62	5	255.68	3.85	7.59
scpe3	5	0.46	5	0.84	5	29.52	0.37	0.56	5	0.64	5	189.53	2.93	4.82
scpe4	5	0.41	5	0.79	5	66.04	0.37	13.15	5	1.36	5	102.34	2.44	3.90
scpe5	5	0.42	5	0.80	5	1.00	0.38	7.69	5	0.63	5	4.10	3.35	6.25
scpnre1	29	94.62	33	173.31	33	>1800.00	566.03	1155.20	40	140.98	*	>1800.00	>1800.00	>1800.00
scpnre2	30	416.43	32	525.58	32	>1800.00	222.82	296.41	38	531.79	*	>1800.00	>1800.00	>1800.00
scpnre3	27	100.21	31	112.86	31	>1800.00	298.71	1192.15	36	99.92	35	>1800.00	416.65	795.02
scpnre4	28	164.20	33	223.15	33	>1800.00	334.70	960.96	38	168.57	38	>1800.00	1543.10	>1800.00
scpnre5	28	66.98	32	82.47	32	1160.49	122.07	272.46	40	68.67	*	>1800.00	>1800.00	>1800.00
scpnrf1	14	113.12	14	133.39	14	243.50	59.72	188.70	16	124.73	16	>1800.00	114.98	291.87
scpnrf2	15	88.24	16	103.17	16	274.09	43.24	117.31	19	101.80	19	>1800.00	1552.88	>1800.00
scpnrf3	14	53.89	15	57.05	15	188.84	26.18	226.03	19	74.95	*	>1800.00	>1800.00	>1800.00
scpnrf4	14	156.36	19	239.12	15	>1800.00	132.21	750.47	17	221.11	*	>1800.00	>1800.00	>1800.00
avg.		22.29		30.01		> 198.88	33.65	95.89		27.47		> 372.56	> 219.96	> 240.77
# solved		59				54	59	59				50	54	52

Table 3: Results on uncertain set covering instances for $P^{\min} \in \{0.85, 0.90\}$ (* indicates that the optimal value is not known).

Nominal problem			$P^{\min} = 0.95$						$P^{\min} = 0.99$					
name	z^*	T	z_h	T_h	z_u	T_{M1}	T_{M2}	$T_{M2'}$	z_h	T_h	z_u	T_{M1}	T_{M2}	$T_{M2'}$
scp41	429	0.01	1105	0.07	921	1.32	0.09	0.17	1820	0.01	1421	>1800.00	0.91	0.74
scp42	512	0.02	1234	0.02	983	3.61	0.15	0.22	1763	0.01	1379	>1800.00	1.03	0.89
scp43	516	0.01	1229	0.02	1048	80.74	1.63	1.89	2157	0.02	1526	>1800.00	2.24	2.10
scp44	494	0.04	1213	0.05	977	5.87	0.48	0.43	1927	0.04	1388	1021.33	0.82	0.54
scp45	512	0.01	1309	0.02	1065	6.21	1.03	0.93	1970	0.02	1429	529.12	1.75	1.40
scp46	560	0.03	1371	0.04	1113	63.21	0.65	0.57	2026	0.03	1522	>1800.00	3.43	3.21
scp47	430	0.02	1133	0.02	975	2.58	0.12	0.22	1677	0.01	1337	434.94	0.96	1.08
scp48	492	0.05	1292	0.07	1054	13.79	0.69	0.68	2005	0.05	1462	>1800.00	1.91	2.27
scp49	641	0.04	1460	0.06	1222	162.20	1.47	1.47	2221	0.03	1647	>1800.00	1.96	1.34
scp410	514	0.02	1375	0.03	1117	4.03	0.37	0.35	2265	0.02	1638	>1800.00	3.47	3.94
scp51	253	0.10	564	0.11	467	20.76	0.77	0.60	892	0.10	682	>1800.00	1.19	0.87
scp52	302	0.16	725	0.17	567	466.65	0.89	1.23	1030	0.15	786	>1800.00	3.12	2.07
scp53	226	0.02	589	0.04	446	1.59	0.08	0.11	901	0.04	696	>1800.00	1.62	2.00
scp54	242	0.07	604	0.10	466	21.21	0.43	0.48	926	0.08	672	>1800.00	2.32	1.67
scp55	211	0.06	515	0.07	425	1.30	0.24	0.21	811	0.06	640	311.69	1.03	0.80
scp56	213	0.03	554	0.04	465	8.95	0.58	0.49	940	0.04	723	>1800.00	4.81	6.70
scp57	293	0.09	739	0.15	597	14.38	0.20	0.31	1166	0.09	873	>1800.00	2.53	2.58
scp58	288	0.08	648	0.11	527	54.13	0.61	0.51	978	0.07	753	>1800.00	2.51	2.16
scp59	279	0.06	742	0.09	578	50.68	1.81	1.77	1143	0.07	823	>1800.00	4.84	3.78
scp510	265	0.03	614	0.06	537	19.44	1.04	1.04	1023	0.03	762	>1800.00	1.96	0.98
scp61	138	0.45	303	0.50	236	11.86	0.97	1.05	445	0.43	340	>1800.00	3.27	4.79
scp62	146	0.51	301	0.51	253	48.57	1.46	1.77	447	0.48	341	1741.75	3.03	2.72
scp63	145	0.37	344	0.68	275	197.92	2.08	2.38	457	0.35	370	>1800.00	13.47	12.37
scp64	131	0.07	308	0.09	235	9.92	1.30	1.66	456	0.06	338	>1800.00	5.02	6.49
scp65	161	0.65	362	0.85	311	271.40	2.48	2.92	559	0.62	441	>1800.00	36.31	44.61
scpa1	253	1.47	605	1.44	472	1152.59	5.03	11.60	915	1.36	676	>1800.00	103.26	101.57
scpa2	252	1.08	555	1.43	452	816.09	2.92	4.40	859	0.99	661	>1800.00	10.85	6.57
scpa3	232	0.73	552	1.18	442	>1800.00	4.09	4.60	830	0.67	630	>1800.00	34.10	43.10
scpa4	234	0.47	565	0.52	448	>1800.00	5.67	9.51	806	0.41	617	>1800.00	39.11	29.69
scpa5	236	0.18	577	0.22	447	>1800.00	2.62	3.37	882	0.16	658	>1800.00	29.52	22.89
scpb1	69	1.57	161	1.84	125	1455.35	7.03	12.85	230	2.08	176	>1800.00	976.58	1715.20
scpb2	76	2.53	153	3.53	118	981.15	6.28	11.12	232	2.60	167	>1800.00	152.33	329.05
scpb3	80	1.57	177	2.66	141	>1800.00	19.32	40.61	247	2.01	191	>1800.00	94.26	142.84
scpb4	79	3.10	174	3.82	135	>1800.00	40.74	95.85	260	3.35	185	>1800.00	108.63	124.74
scpb5	72	1.36	150	2.26	122	390.71	3.46	6.73	226	1.62	170	>1800.00	42.10	77.12
scpc1	227	1.12	552	1.19	442	>1800.00	122.30	63.91	896	0.98	*	>1800.00	>1800.00	>1800.00
scpc2	219	2.50	523	2.98	404	>1800.00	60.12	84.76	781	2.27	*	>1800.00	>1800.00	>1800.00
scpc3	243	2.80	575	4.79	455	>1800.00	163.70	119.57	817	2.51	*	>1800.00	>1800.00	>1800.00
scpc4	219	1.48	523	2.10	417	>1800.00	122.68	200.73	728	1.36	561	>1800.00	364.49	334.86
scpc5	215	1.36	548	2.09	422	>1800.00	103.41	110.41	806	1.23	565	>1800.00	49.40	70.53
scpd1	60	2.64	129	4.13	106	>1800.00	35.70	70.01	187	3.75	145	>1800.00	445.02	800.93
scpd2	66	11.63	137	12.94	116	>1800.00	141.45	200.22	203	11.36	*	>1800.00	>1800.00	>1800.00
scpd3	72	11.43	157	11.77	124	>1800.00	243.09	1246.25	219	11.56	*	>1800.00	>1800.00	>1800.00
scpd4	62	5.02	138	6.58	108	>1800.00	91.75	96.33	194	6.14	*	>1800.00	>1800.00	>1800.00
scpd5	61	1.91	152	3.80	118	>1800.00	73.40	90.78	204	3.63	156	>1800.00	313.21	809.73
scpe1	5	0.39	6	0.65	6	110.30	4.09	9.72	8	0.36	8	>1800.00	4.87	6.93
scpe2	5	0.44	6	0.72	6	218.22	5.67	17.24	10	0.40	7	>1800.00	1.14	0.30
scpe3	5	0.46	5	0.48	5	95.56	0.25	0.56	7	0.42	7	>1800.00	2.75	1.89
scpe4	5	0.41	6	0.72	6	157.53	0.33	0.96	7	0.38	7	>1800.00	1.27	1.53
scpe5	5	0.42	6	0.76	6	353.88	0.40	5.47	7	0.39	7	>1800.00	1.29	1.46
scpnre1	29	94.62	61	120.78	45	>1800.00	1017.23	>1800.00	82	113.14	*	>1800.00	>1800.00	>1800.00
scpnre2	30	416.43	51	499.74	44	>1800.00	1166.51	>1800.00	81	509.16	*	>1800.00	>1800.00	>1800.00
scpnre3	27	100.21	51	97.47	40	>1800.00	135.32	648.17	67	83.94	*	>1800.00	>1800.00	>1800.00
scpnre4	28	164.20	57	158.24	44	>1800.00	1335.58	>1800.00	73	135.57	*	>1800.00	>1800.00	>1800.00
scpnre5	28	66.98	60	68.08	*	>1800.00	>1800.00	>1800.00	84	56.18	*	>1800.00	>1800.00	>1800.00
scpnrf1	14	113.12	23	122.92	19	>1800.00	69.21	184.62	30	93.29	23	>1800.00	466.14	1063.87
scpnrf2	15	88.24	25	95.52	21	>1800.00	76.52	696.33	35	71.18	26	>1800.00	954.22	1364.97
scpnrf3	14	53.89	25	59.66	21	>1800.00	127.28	383.97	34	44.56	26	>1800.00	1229.53	>1800.00
scpnrf4	14	156.36	21	200.22	20	>1800.00	756.29	>1800.00	34	209.79	*	>1800.00	>1800.00	>1800.00
avg.		22.29		25.44		> 855.51	> 131.72	> 228.08		23.42		>1717.99	> 459.86	> 518.06
# solved		59				35	58	54				5	47	46

Table 4: Results on uncertain set covering instances for $P^{\min} \in \{0.95, 0.99\}$ (* indicates that the optimal value is not known).

(resp., M2') in just 5.74 seconds and 4,299 nodes (resp., 8.06 seconds and 3,355 nodes), but deactivating `ILOG-Cplex` preprocessing and cut generation these figures become 15.67 seconds and 5,842 nodes for M2, and 12.08 seconds and 4,985 nodes for M2'.

As to the quality of our simple initial heuristic, it turns out to be very satisfactory for problems with $P^{\mathtt{min}} = 0.85$, and tends to deteriorate when $P^{\mathtt{min}}$ increases.

Finally, we performed a set of experiments aimed at evaluating the quality of the USCP solutions with respect to the nominal SCP ones for a given real-world problem arising in crew scheduling in railways. Problem `rail2536c` has 2,536 rows, each corresponding to a duty to be performed by a crew, and 15,284 columns, each corresponding to a feasible pairing (this instance corresponds to a suitable "core" problem of the original railway instance; see [8] for more details). Costs of columns are either 1 or 2. We eliminated 6 rows which can be covered by one or two columns, since the associated columns should be active in each feasible solution.

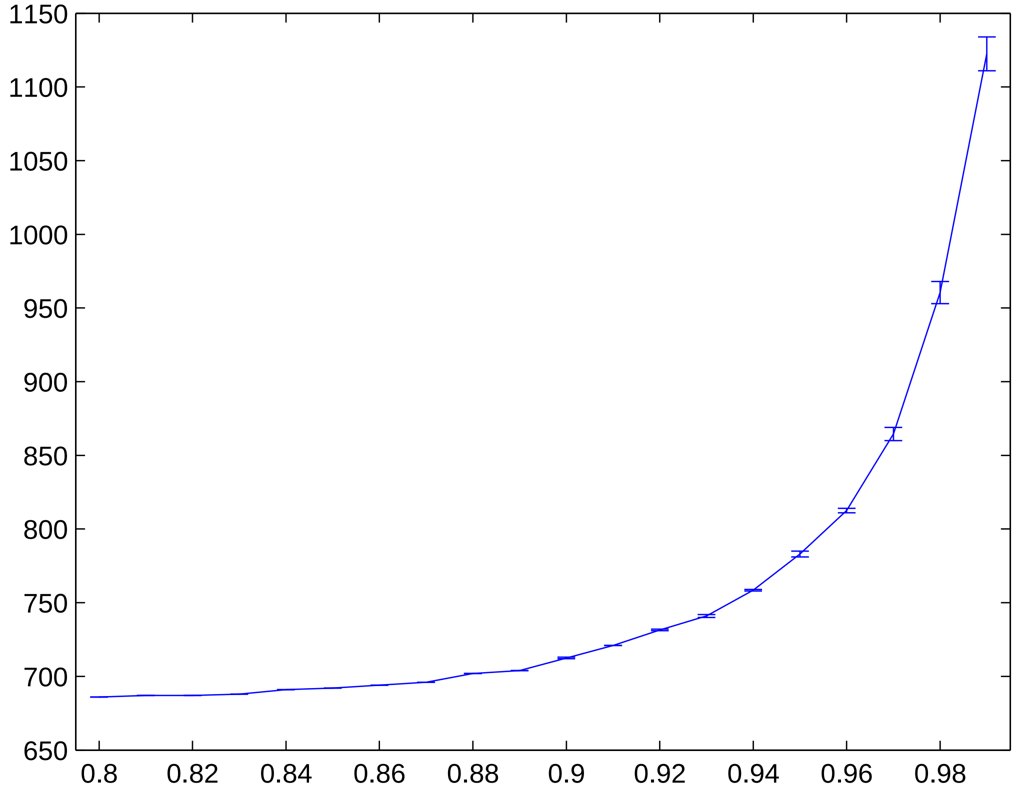

Figure 1: Lower and upper bounds for different values of $P^{\mathtt{min}}$ on a real-world set covering instance.

We randomly generated probabilities p_j in $[0, 0.2]$ and solved the uncertain problem with different values of the threshold probability $P^{\mathtt{min}}$ (again, we require the same probability for each row $i \in M$). Figure 1 reports the value of the optimal uncertain solution as a function of parameter $P^{\mathtt{min}}$. For each value of $P^{\mathtt{min}}$ the associated instance was solved using model M2 (28)–(30) with a time limit of 10 hours.

3.2 Uncertain graph connectivity

Given an undirected graph $G = (V, E)$ with nonnegative edge costs c_e, the classical Minimum Spanning Tree Problem (MSTP) requires to find a minimum-cost set of edges so that the associated subgraph is connected. In many practical applications arising in communications networks, survivability of the network is a major issue. Hence, one has to take care of possible link failures, which is often modeled by solving a variant of MSTP in which the set of selected edges must provide (at least) k edge-disjoint paths between each pair of nodes, k being an input parameter; see, e.g., Monma, Munson and Pulleyblank [19], Grötschel and Monma [17], and Grötschel, Monma and Stoer [18].

Despite the deterministic version of the problem is polynomially solvable, the uncertain graph connectivity problem turns out to be strongly NP-hard as stated by the following theorem.

Theorem 2 *The uncertain graph connectivity problem is strongly NP-hard, even if edge costs satisfy the triangular condition.*

Proof. We prove NP-hardness of our problem through a reduction from the min-cost 2-edge connected subgraph problem (2ECSP). 2ECSP is known to be NP-hard even when edge costs satisfy the triangular condition [16]. Given a 2ECSP instance, define an instance of uncertain graph connectivity as follows: each edge has probability 1/2 of disappearing, and the required probability for connection is equal to 3/4. This imposes that at least two edges are selected for each cut, hence the set of feasible solutions of the two problems coincide. □

By associating to each edge $e \in E$ a binary variable x_e taking value 1 iff edge e is selected, a set covering model for MSTP is as follows:

$$\min \sum_{e \in E} c_e \; x_e \tag{32}$$

$$\sum_{e \in \delta(S)} x_e \geq 1, \qquad S \subset V,\ S \neq \emptyset, \tag{33}$$

$$x_e \in \{0, 1\}, \qquad e \in E. \tag{34}$$

Possible failures in the connections can be modeled by associating each edge $e \in E$ with a failure probability p_e and requiring a minimum probability $\overline{P}$ for the connection to be provided. The same reasoning used in Theorem 1 shows that, if probabilities are independent each other, the uncertain version of the problem can be modeled as:

$$\min \sum_{e \in E} c_e \; x_e \tag{35}$$

$$\sum_{e \in \delta(S)} w_e x_e \geq \overline{W}, \qquad S \subset V,\ S \neq \emptyset, \tag{36}$$

$$x_e \in \{0, 1\}, \qquad e \in E, \tag{37}$$

where $w_e = -\ln(p_e)$ $(e \in E)$ and $\overline{W} = -\ln(1 - \overline{P})$.

The exponential number of constraints (36) makes it natural to approach the above ILP by means of a branch-and-cut algorithm based on cutting planes. Given a solution x^*, the separation problem for constraints (36) calls for the determination of a subset S^* of nodes for which $\sum_{e \in \delta(S^*)} w_e x_e^*$ is a minimum: if such value is smaller than $\overline{W}$, a violated constraint is found and the process is iterated. Otherwise x^* is the optimal solution of the LP relaxation of the current subproblem. Thus, the separation problem amounts to finding a minimum-capacity cut in an undirected graph with edge capacities $w_e x_e^*$, and can be solved in polynomial time through max-flow techniques—very much in the spirit of separation of subtour elimination constraints for the TSP (see, e.g., Crowder and Padberg [12]).

3.2.1 Computational experiments on Uncertain Graph Connectivity

In our computational experiments, we considered all the instances for the Symmetric Traveling Salesman Problem in the TSPLIB with at most 50 nodes. In order to simulate practical networks, for each instance we considered a sparse graph containing only the edges belonging to the 5 disjoint minimum-cost spanning trees.

Probabilities p_j associated with edges were randomly generated as in Section 3.1.3, i.e., according to a uniform distribution in $[0, 0.2]$, and were used to determine coefficients w_j and $\overline{W}$ in the same way.

Tables 5 and 6 give, for each instance and for each value of $P^{\mathtt{min}} \in \{0.85, 0.90, 0.95, 0.99\}$, the following information:

- the best solution found (z_h) by the initial heuristic—the associated computing times are omitted since they are always negligible;
- best solution found, best lower bound and computing times for model M2 defined by (35)–(37);
- best solution found, best lower bound and computing times for model M2' defined by (35)–(37) in which constraints (31) with $k = 2$ and $k = 3$ are separated before (36) are considered; for each value of k, separation for constraints (31) requires the solution of an additional min-cut problem with modified edge capacities.

A time limit of 10,000 seconds was given to each model. The last row of the table reports, for each solution method, the average computing time, in seconds (for the unsolved instances, the time limit is considered), and the number of instances solved to proven optimality within the given time limit.

Computational results clearly show that the separation of constraints (31) before (36) plays a relevant role for this problem. Indeed, model M2' is able to solve all the instances with $n \leq 26$ within the given time limit. In addition, for the instances that are not solved to proven optimality, model M2' finds better solutions and lower bounds than model M2. This is not surprising, since the robust connectivity constraints are generated at run time, so the initial MIP preprocessing and cut generation procedures are ineffective.

As in the USCP case, the initial heuristic is very tight for $P^{\mathtt{min}} = 0.85$, with a performance worsening for larger values of $P^{\mathtt{min}}$.

Figure 2 gives a graphical illustration of the optimal robust solutions for instance `burma14` for different values of $P^{\mathtt{min}}$.

4 Conclusions

In this paper we considered optimization problems where the exact value of some input data is not known in advance. Based on the well-known concept of robustness as defined

Nominal problem			$P^{\mathtt{min}} = 0.85$							$P^{\mathtt{min}} = 0.90$						
				Model M2			Model M2'				Model M2			Model M2'		
name	n	m	z_h	z	LB	T	z	LB	T	z_h	z	LB	T	z	LB	T
att48	48	235	9696	9696	6704	>10000.00	9696	7563	>10000.00	10997	10997	7021	>10000.00	10454	8458	>10000.00
bayg29	29	140	1373	1373	1059	>10000.00	1373	1110	>10000.00	1566	1566	1105	>10000.00	1489	1337	>10000.00
bays29	29	140	1728	1728	1319	>10000.00	1728	1480	>10000.00	1899	1899	1459	>10000.00	1789	1696	>10000.00
burma14	14	65	2671	2671	2671	142.67	2671	2671	1.06	2850	2811	2811	80.53	2811	2811	0.49
dantzig42	42	205	642	642	481	>10000.00	642	514	>10000.00	729	729	499	>10000.00	668	595	>10000.00
fri26	26	125	802	802	692	>10000.00	780	780	8714.49	894	894	713	>10000.00	841	841	1548.62
gr17	17	80	1496	1496	1496	1.56	1496	1496	0.25	1693	1599	1599	28.89	1599	1599	0.21
gr21	21	100	2330	2330	2330	945.45	2330	2330	5.42	2701	2701	2416	>10000.00	2554	2554	58.67
gr24	24	115	1124	1124	948	>10000.00	1124	1076	>10000.00	1233	1233	998	>10000.00	1142	1142	136.62
gr48	48	235	4365	4365	3128	>10000.00	4365	3420	>10000.00	5346	5346	3344	>10000.00	4603	3984	>10000.00
hk48	48	235	11013	11013	7389	>10000.00	1013	8525	>10000.00	12910	12910	8069	>10000.00	11362	9432	>10000.00
swiss42	42	205	1199	1199	841	>10000.00	1199	935	>10000.00	1349	1349	908	>10000.00	1226	1052	>10000.00
ulysses16	16	75	4985	4985	4985	372.67	4985	4985	1.97	5544	5540	5540	2120.25	5540	5540	10.53
ulysses22	22	105	4952	4952	4952	619.95	4952	4952	18.93	5557	5557	5269	>10000.00	5410	5410	18.22
avg.						> 6577.32			> 6338.74				> 8016.42			> 5126.68
# solved						5			6				3			7

Table 5: Results on uncertain graph connectivity for $P^{\mathtt{min}} \in \{0.85, 0.90\}$.

Nominal problem			$P^{\mathtt{min}} = 0.95$							$P^{\mathtt{min}} = 0.99$						
				Model M2			Model M2'				Model M2			Model M2'		
name	n	m	z_h	z	LB	T	z	LB	T	z_h	z	LB	T	z	LB	T
att48	48	235	12332	11382	8093	>10000.00	10246	9985	>10000.00	16740	15483	10892	>10000.00	13959	11706	>10000.00
bayg29	29	140	1905	1751	1321	>10000.00	1570	1570	92.35	2569	2070	1712	>10000.00	1875	1875	1244.62
bays29	29	140	2391	2099	1661	>10000.00	1945	1945	7.30	3281	2619	2232	>10000.00	2499	2283	>10000.00
burma14	14	65	4039	3182	3182	50.89	3182	3182	0.14	4859	3881	3881	3.71	3881	3881	1.94
dantzig42	42	205	911	760	577	>10000.00	675	675	182.65	1134	933	742	>10000.00	870	783	>10000.00
fri26	26	125	1180	890	855	>10000.00	874	874	2.80	1445	1081	1058	>10000.00	1080	1080	281.15
gr17	17	80	2197	1807	1807	5.95	1807	1807	0.12	2943	2495	2426	>10000.00	2466	2466	925.66
gr21	21	100	3258	2584	2584	267.82	2584	2584	0.29	3716	3113	3113	267.64	3113	3113	23.15
gr24	24	115	1503	1284	1148	>10000.00	1255	1255	5.56	2128	1677	1453	>10000.00	1592	1592	7418.44
gr48	48	235	5410	4663	3768	>10000.00	4528	4528	1873.00	7618	7618	4889	>10000.00	6587	5871	>10000.00
hk48	48	235	15754	12871	9018	>10000.00	11114	11114	656.09	20190	17575	11812	>10000.00	15770	13455	>10000.00
swiss42	42	205	1652	1425	1043	>10000.00	1237	1237	25.22	2116	1848	1339	>10000.00	1658	1447	>10000.00
ulysses16	16	75	7676	6414	6414	8.49	6414	6414	0.21	9766	7770	7770	17.21	7770	7770	0.63
ulysses22	22	105	6732	6241	5797	>10000.00	5999	5999	0.30	9621	7061	7061	2982.58	7061	7061	133.77
avg.						> 7166.70			> 917.57				> 7376.64			> 5002.12
# solved						4			13				4			8

Table 6: Results on uncertain graph connectivity for $P^{\mathtt{min}} \in \{0.95, 0.99\}$.

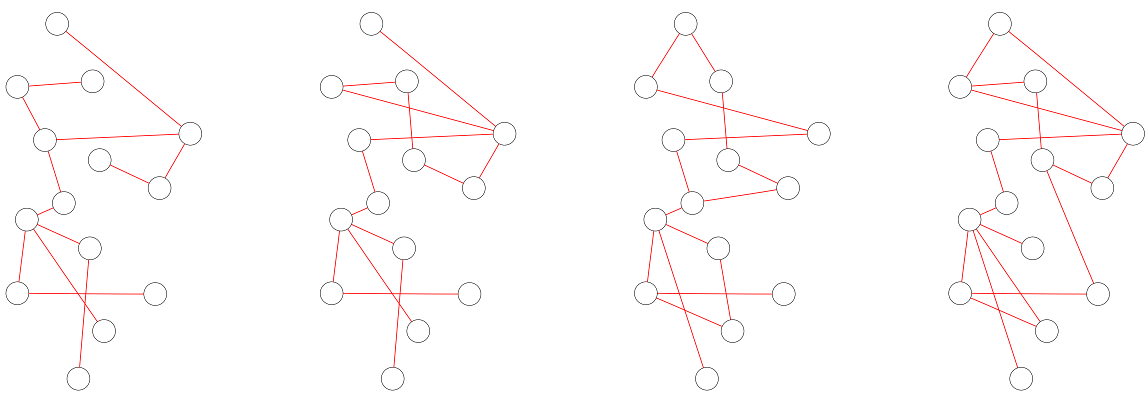

Figure 2: Optimal solutions of instance `burma14` for $P^{\mathtt{min}} \in \{0.85, 0.90, 0.95, 0.99\}$.

by Bertsimas and Sim [7], we proposed a cutting planes approach for robust optimization, pointing out situations where this method has practical performances comparable or better to those of the Bertsimas-Sim method. Moreover, we pointed out that our cutting plane approach has some important features that can make it the most natural (or even the only available) option to face uncertainty in important applications. Indeed, we considered problems in which uncertainty domain involves yes-no decisions that cannot be modeled by continuous variables, and proposed mathematical formulations for these uncertain problems. In particular, we introduced an uncertain version of the well-known Set Covering Problem, arising when each column has a positive probability of disappearing and each row must be covered with a given probability. We also studied an uncertain version of the classical minimum-cost graph connectivity problem, arising when edge failures are taken into account and connectivity has to be guaranteed with a certain probability.

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