How tight is the corner relaxation?

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Abstract

Given a mixed-integer linear programming (MILP) model and an optimal basis of the associated linear programming relaxation, the *Gomory's corner relaxation* is obtained by dropping nonnegativity constraints on the basic variables. Although this relaxation received a considerable attention in the literature in the last 40 years, the crucial issue of evaluating the practical quality of the corner-relaxation bound was not addressed so far. In the present paper we report, for the first time, the optimal value of the corner relaxation (in two possible variants) for a very large set of MILP instances from the literature, thus providing a missing yet very important piece of information about the practical relevance of this relaxation. The outcome of our experiments is that the corner relaxation often gives a tight approximation of the integer hull, the main so for MILPs with general integer variables–the approximation tends to be less satisfactory when a consistent number of binary variables exists.

Key words: Mixed integer linear program, Gomory's corner polyhedron, Computational analysis.

We consider the Mixed-Integer Linear Programming (MILP) model

$$min\{c^T x : Ax \leq b,\ x_j \text{ integer for all } j \in J\} \tag{1}$$

where A is a $m \times n$ rational matrix, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$, and $J \subseteq \{1, \ldots, n\}$ is the (nonempty) index set of the integer-constrained variables.

Given any optimal vertex x^* of the linear programming relaxation $min\{c^T x : Ax \leq b\}$, let $I(x^*) := \{i \in \{1, \ldots, m\} : a_i^T x^* = b_i\}$ denote the index set of the constraints that are binding at x^*. In this paper we address the following *corner relaxation* (called "Gomory integer program" in [13]):

$$min\{c^T x : a_i^T x \leq b_i \text{ for all } i \in I(x^*),\ x_j \text{ integer for all } j \in J\} \tag{2}$$

obtained from (1) by removing all the constraints that are not binding at x^*. This definition of the corner relaxation depends on the choice of vertex x^* but not on the corresponding optimal LP basis. The relaxation is a variant of the well-known *group relaxation* introduced

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by Ralph Gomory [8] for pure integer linear programming (ILP) problems, which is obtained by dropping nonnegativity constraints on the variables that are basic in a given optimal LP basis. By definition, the group relaxation is affected by the presence of primal degeneracy, while definition (2) of the corner relaxation is not. Notice that the group relaxation can be significantly weaker than (2) for highly-degenerate ILPs, as it drops the binding constraints whose slack variable is basic (at level zero) in the chosen optimal basis.

The group relaxation was deeply investigated by Gomory and Johnson [9, 10] who exploited an interpretation in terms of mod-1 equations to obtain its complete facial characterization through subadditive functions. Since then, the *theoretical* study of the group relaxation received a considerable attention in the literature as witnessed, e.g., by the recent papers [11, 12, 15], among others. The *practical* utility of the relaxation remains however a quite unexplored topic; e.g., Gomory, Johnson, and Evans [12] observe that "If we were able to come close to solving the corner polyhedron problem, say by having an adequate supply of cutting planes or perhaps in other ways, such as finding solutions to the group problem, we could come close to a different kind of algorithm—one based on solving a sequence of Corner Polyhedron problems". In particular, no computational study was performed to address a key question: How tight is, in practice, the corner relaxation?

The following result about the computational complexity of optimizing over the corner relaxation was implicit in the previous papers dealing with such a relaxation, but we could not find an easy reference to its proof. E.g., Letchford [16] observed that "the only known algorithms for solving the group relaxation have a running time which is proportional to the determinant of the LP optimal basis (see Chen and Zionts [4]), that is often very large; moreover, it is not difficult to show (for example by reduction from the multi-dimensional knapsack problem) that the standard group relaxation is strongly NP-hard in general". As our definition of the corner relaxation differs from the standard one in case of degeneracy, and for the sake of completeness, we also provide a complexity proof.

Proposition 1 *The corner relaxation (2) is strongly NP-hard.*

Proof: We provide a simple reduction from the following strongly NP-complete [7] *Decoding of Linear Codes* (DLC) problem, in recognition form: Given a $r \times (t+1)$ 0-1 matrix (Q, d) and a positive integer k, decide whether there exist a vector $y \in \{0,1\}^t$ with $Qy \equiv d\ (mod\ 2)$ and $\mathbf{1}^T y \leq k$, where $\mathbf{1} := (1, 1, \ldots, 1)$ denotes a vector of 1's of appropriate size. Without loss of generality, we can assume $d = \mathbf{1}$, since one can always increase by 1 the threshold k and replace the congruence system $Qy \equiv d\ (mod\ 2)$ by $Qy + (\mathbf{1} - d)z \equiv \mathbf{1}\ (mod\ 2), z \equiv 1\ (mod\ 2), z \in \{0,1\}$. In addition, condition $y \in \{0,1\}^t$ can be replaced by "$y \geq 0$ integer", since one can iteratively subtract 2 from the value of any component $y_j \geq 2$ without affecting the congruence system while favoring the condition $\mathbf{1}^T y \leq k$. Hence, given a 0-1 matrix Q and an integer k, it is strongly NP-complete to decide whether

$$min\{\mathbf{1}^T y : Q\ y \equiv \mathbf{1}\ (\text{mod } 2),\ y \geq 0 \text{ integer}\} \leq k. \tag{3}$$

Given Q, we define our MILP model (1) as $min\{0^T z + \mathbf{1}^T y\ :\ z + Q/2\ y = \mathbf{1}/2,\ (z, y) \geq 0 \text{ integer}\}$. By construction, the LP relaxation of this model has a unique and nondegenerate optimal vertex (z^*, y^*), where $z^* = \mathbf{1}/2$ and $y^* = 0$. Removing all nonbinding constraints one therefore gets a corner relaxation $min\{\mathbf{1}^T y : z + Q/2\ y = \mathbf{1}/2,\ y \geq 0,\ (z, y) \text{ integer}\} = min\{\mathbf{1}^T y\ :\ Q/2\ y \equiv \mathbf{1}/2\ (\text{mod } 1),\ y \geq 0 \text{ integer}\}$ that is equivalent to the minimization problem in (3), hence the claim follows. $\square$

Note that the above proof applies to Gomory's group relaxation as well, in that its constructions refer to a unique and nondegenerate optimal LP basis.

In the present paper we report the optimal value (or a lower bound) of the corner and group relaxations, computed through a commercial MILP software based on a standard branch-and-cut solution method. From the point of view of computing time, this naive approach is far from satisfactory, since the structure of the corner/group MILP (that involves integer variables with no bounds) seems to be intrinsically unappropriate to be dealt with by branch-and-cut methods. As a matter of fact, the computing time needed to solve the relaxations was almost always much larger than the one needed for the original problem—often by 1-2 orders of magnitude. This suggests that different techniques based, e.g., on dynamic programming or basis-reduction methods, should be applied. Nevertheless, we believe that knowing the quality of the bound that can be achieved on a large set of test cases is important to give the researchers further motivations to study (or not to study) the corner and group relaxations.

In our experiments, the corner and group relaxations were solved with the commercial MILP solver `ILOG-Cplex` 9.1 [14], using its default parameter setting. For each instance, the relaxations were defined with respect to the original formulation, with the `ILOG-Cplex` presolver turned off. All computing times are expressed in CPU seconds of a PC AMD Athlon 4200+ with 4 GB ram.

Our test-bed is taken from Fischetti and Saturni [6] and is composed of two sets of instances:

- all MIPLIB 3.0 and 2003 instances [17], except those with unknown optimal solution or having some variables with negative lower bound; also excluded from our analysis are some very large MIPLIB instances with an LP file larger than than 1.7 MB;
- the hard MILP instances available at the Alper Atamtürk's home page [1], associated with multiple-knapsack problems involving both binary and general-integer (either bounded or unbounded) variables (see [2]).

For each instance in our test-bed, we report in the tables the number of general-integer (column I), binary (B) and continuous (C) variables; the percentage integrality gap (*LP %gap*) computed as

$$100 * |(optimal\_integer\_value - LP\_bound)/optimal\_integer\_value|;$$

and the percentage of gap closed (*%gc*) defined as

$$100 * (lower\_bound - LP\_bound)/(optimal\_integer\_value - LP\_bound),$$

where the lower bounds are computed by using six alternative bounding procedures. More specifically, column *GMI* refers to the lower bound obtained by adding to the original formulation all the Gomory mixed-integer cuts [18] that can be derived from the fractional tableau rows (i.e., from the rows of the first LP optimal tableau with fractional right-hand-side value). Column *1:50-c* is computed in a similar way, by adding for each fractional tableau row all the Cornuejols-Li-Vandenbussche [5] k-cuts for $k = 1, \ldots, 50$. Column $k = 60$ exploits (implicitly), for each fractional tableau row, all the so-called *interpolated subadditive cuts* with group-order parameter $k = 60$, as recently proposed by Fischetti and Saturni [6]. All entries in columns *1:50-c* and $k = 60$ are taken from [6]. Columns *Corner* and *Group* refer to the optimal solution of the corner and group relaxation, respectively. For the cases where our solution approach did not reach convergence within the imposed time limit of 10 hours, we report the best lower bound available at the end of the computation.

Finally, column GMI' is intended to address the following important issue. In our experiments, we compare the bound obtained by solving the corner/group relaxation with the bound obtained by adding to the LP relaxation one round of GMI cuts. This comparison is however biased in favor of the GMI cuts, since they are used in conjunction with the other inequalities of the LP relaxation. Indeed, as shown in the tables below, for some instances one round of GMI cuts produces a much larger improvement than solving the corner/group relaxation. As a matter of fact, the *GMI* (as well as the *1:50-c* and $k = 60$) bounds could only be dominated by a bound obtained by optimizing over the intersection of the corner/group polyhedron (defined as the convex hull of the relaxation feasible points) with the LP relaxation polyhedron, a task that would require a "dual" (cutting-plane or Lagrangian) solution approach to the corner/group relaxation, whereas our approach is heavily based on enumeration. As suggest by one of the referees, a more unbiased comparison between the strengths of the corner/group relaxations and that of one round of GMI cuts can be obtained by solving an LP relaxation of the original MILP model defined by the following constraints (i) all the original constraints that define the Gomory's group relaxation, and (ii) one round of GMI cuts derived from the optimal LP tableau (that are easily seen to be also valid for the corner/group relaxation as well). By construction, the optimal value of this relaxation (reported in column GMI') cannot exceed the value reported in column GMI, as the latter uses the same GMI cuts but does not remove any constraint from the LP. Hence, comparing columns GMI and GMI' shows the bound deterioration when removing the nonnegativity constraints on the variables that are basic in a given optimal LP basis, whereas comparing GMI' and *Group* gives an idea of the marginal benefit of exploiting valid inequalities of the group relaxation other than the GMI cuts read from the first LP tableau.

Table 1 includes all those instances for which the optimal value of both the corner relaxation and of the group relaxation have been computed within a time limit of 10 hours. For these instances we report the computing times required to obtain the corner and group relaxation as well. The first part of the table refers to pure 0-1 instances, while the second one addresses the problems with general-integer and/or continuous variables. As expected, the corner and group bounds differ in a significant way for highly-degenerate problems such as mod010 or fixnet6.

Table 2 addresses all the instances for which the optimal value of either the corner relaxation or of the group relaxation (or both) cannot be computed in a provable way within the 10-hour time limit. In this table, the entries refer to the best lower bound achieved at the time limit, thus in principle they give just a lower bound on the percentage gap closed by the relaxations. However, this lower bound turns out to be a quite good estimate of the optimal value of the relaxation, as confirmed by Table 3 where we report the percentage gap closed after 1, 3, 5 and 10 hours by both the corner and the group relaxation (for this latter relaxation, 24 hours of computation were even allowed). The table shows that, for most instances, the percentage of gap closed is not improved significantly after the first hour of computing time, so the 10-hour bounds reported in Table 2 are likely to be very tight.

Finally, Tables 4 and 5 address the Atamtürk (bounded and unbounded, respectively) multiple-knapsack problems. For those instances we report the same information as in Tables 1-2, the only difference being that the time limit for solving the corner/group relaxation was set to 1 hour.

According to our computational experiments, the pure 0-1 ILP models typically have a corner/group bound that is not significantly better (and sometimes quite worse) than the GMI one. This is not surprising, since for these problems removing the nonbinding bounds on the binary variables is likely to produce a weak relaxation, and confirms experimentally the theoretical observations in Balas [3]. The behavior is confirmed by the comparison of columns GMI and GMI', showing that the GMI bound is considerably stronger than the GMI' one, thus stressing the importance of the nonbinding constraints after the addition of GMI cuts.

On the other hand, for many problems the corner and group relaxations do improve the GMI bound considerably as a result of their ability of taking into account all the fractional rows of the optimal LP tableau simultaneously. A comparison between the corner and group bounds shows that the latter can be significantly weaker, the average gap closed (over all the instances of Tables 1 and 2) being 34.48% for the corner, and 23.61% for the group relaxation.

Finally, our experiments stress the role of the nonbinding constraints in producing tight LP bounds: though these constraints are, by definition, completely useless to improve the initial LP relaxation bound, they become quite relevant even after just one round of GMI cuts, due to their capability of cutting the optimal LP solution resulting from the addition of the new cuts. As a matter of fact, the average gap closed (over all the instances of Tables 1 and 2) goes from 9.67% for GMI' (i.e., for GMI cuts without nonbinding constraints) to 25.32% for GMI (same GMI cuts with nonbinding constrains still in the LP model).

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Table 1: MIPLIB 3.0 and 2003 instances for which both the corner and the group relaxations can be solved in a provable way within 10 hours of computing time.

					GMI	1:50-c	$k = 60$	GMI'	Corner		Group	
Name	I	B	C	LP %gap	%gc	%gc	%gc	%gc	%gc	Time	%gc	Time
air03	0	10757	0	0.38	100.00	100.00	100.00	100.00	100.00	0.66	100.00	0.52
air04	0	8904	0	1.07	7.22	9.23	8.44	0.47	4.58	21.52	1.76	42.13
air05	0	7195	0	1.88	4.54	5.24	4.92	0.84	4.51	45.73	2.29	27.67
cap6000	0	6000	0	0.01	41.65	41.65	40.44	12.18	34.51	23.75	34.51	26.08
disctom	0	10000	0	0.00	-	-	-	-	-	502.01	-	1016.89
enigma	0	100	0	0.00	-	-	-	-	-	0.21	-	0.04
l152lav	0	1989	0	1.39	9.31	13.40	14.48	2.39	14.68	0.14	4.02	0.35
mod010	0	2655	0	0.24	100.00	21.47	24.61	6.54	100.00	0.70	37.17	1.43
p0282	0	282	0	31.56	3.70	3.70	3.63	1.05	9.28	9.21	9.27	7.89
p0548	0	548	0	96.37	60.59	40.04	37.94	0.00	0.02	0.05	0.02	0.05
seymour	0	1372	0	4.53	8.33	6.75	6.75	2.69	11.24	478.77	6.02	335.50
stein27	0	27	0	27.78	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
stein45	0	45	0	26.67	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
aflow30a	0	421	421	15.10	10.87	11.15	11.08	1.44	37.08	3.94	13.06	7.03
aflow40b	0	1364	1364	13.90	9.79	5.48	5.45	1.21	26.08	2090.51	8.21	108.30
bell3a	32	39	62	1.80	52.92	60.43	60.32	20.40	83.79	3.94	66.84	0.21
bell5	28	30	46	3.99	84.90	14.53	14.73	1.92	5.05	1.13	5.05	1.42
blend2	33	231	89	9.00	16.48	0.00	0.00	1.11	37.93	0.48	10.46	5.04
danoint	0	56	465	4.61	0.24	1.74	0.96	0.03	1.74	53.98	1.74	1316.85
egout	0	55	86	73.67	59.87	57.47	40.58	7.56	100.00	0.29	70.67	0.04
fixnet6	0	378	500	69.85	11.49	10.65	10.52	6.81	80.63	1028.28	13.77	1.02
flugpl	11	0	7	2.86	11.74	11.74	11.69	11.39	97.38	0.16	97.38	0.16
gesa2_o	336	384	504	1.18	31.03	30.29	30.19	23.43	97.64	1346.02	44.36	46.50
gesa3	168	216	768	0.56	53.26	47.56	47.50	39.47	70.19	0.98	59.85	3.38
gesa3_o	336	336	480	0.56	54.30	60.53	60.35	40.36	70.19	1.02	58.61	1.68
noswot	25	75	28	4.88	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
qiu	0	48	792	601.15	2.36	0.93	0.90	0.00	0.00	0.31	0.00	0.13
qnet1	129	1288	124	10.95	7.44	9.87	9.85	0.54	64.31	465.34	8.91	796.32
rentacar	0	55	9502	5.11	34.62	15.53	4.58	0.00	0.00	0.44	0.00	0.44

Table 2: MIPLIB 3.0 and 2003 instances for which either the corner or the group relaxation (or both) cannot be solved in a provable way within 10 hours of computing time. An asterisk * indicates that the relaxation has been solved in a provable way, while ✠ indicates that the exact bound cannot be computed because of numerical problems.

					GMI	1:50-c	$k = 60$	GMI'	Corner	Group
Name	I	B	C	LP %gap	%gc	%gc	%gc	%gc	%gc	%gc
harp2	0	2993	0	0.61	28.62	29.43	28.09	11.11	15.69	13.93
lseu	0	89	0	25.47	57.82	56.88	58.04	✠	4.58	4.58
mitre	0	10724	0	0.36	95.17	82.20	82.20	6.25	77.32	77.32
mod008	0	319	0	5.23	20.10	20.92	20.83	2.69	41.96	56.44
p0033	0	33	0	18.40	56.82	56.85	56.59	31.51	31.86	31.51
p0201	0	201	0	9.72	20.27	17.96	17.56	0.00	0.00	0.00 *
p2756	0	2756	0	13.93	3.20	0.61	0.29	✠	0.00	0.00
10teams	0	1800	225	0.76	57.14	100.00	100.00	14.29	0.00	0.00
fiber	0	1254	44	61.55	57.68	54.28	51.98	31.76	80.22	46.02
gen	6	144	720	0.16	64.97	59.77	59.75	0.10	38.26	27.81
gesa2	168	240	816	1.18	30.83	30.17	30.02	21.85	97.64 *	31.34
gt2	164	24	0	36.41	25.32	73.09	58.74	3.40	46.79 *	27.99
manna81	3303	18	0	1.01	25.19	100.00	100.00	25.19	57.14	57.14
markshare1	0	50	12	100.00	0.00	0.00	0.00	0.00	0.00	0.00
markshare2	0	60	14	100.00	0.00	0.00	0.00	0.00	0.00	0.00
mas74	0	150	1	11.17	6.67	7.27	7.82	5.90	33.14	33.14
mas76	0	150	1	2.78	6.42	7.02	7.55	2.38	33.53	33.52
mkc	0	5323	2	8.51	7.96	6.62	6.11	0.00	0.00	0.00
modglob	0	98	324	1.49	17.28	17.28	16.75	9.48	62.88	31.47 *
net12	0	1603	12512	91.94	27.89	7.07	7.04	3.47	6.37	2.67
opt1217	0	768	1	25.13	19.61	19.74	19.68	14.24	0.00	0.00
pk1	0	55	31	100.00	0.00	0.00	0.00	0.00	0.00	0.00 *
pp08a	0	64	176	62.61	53.48	52.22	52.08	21.68	24.30	23.49
pp08aCUTS	0	64	176	25.43	32.83	31.32	31.17	14.39	40.32	33.13 *
qnet1_o	129	1288	124	24.54	37.77	42.02	41.76	19.39	64.80 *	23.26
rgn	0	100	80	40.63	5.02	10.37	10.08	✠	0.00 *	0.00
set1ch	0	240	472	41.31	38.91	39.16	38.92	23.06	71.60	50.90
timtab1	107	64	226	96.25	24.08	23.51	23.51	17.56	28.17	26.56
tr12-30	0	360	720	89.12	45.52	60.27	59.81	42.09	69.24	68.54
vpm1	0	168	210	22.92	17.09	15.86	15.86	17.09	5.45	5.45
vpm2	0	168	210	28.08	8.25	10.36	10.36	2.59	17.35	2.61

Table 3: Percentage gap closed by the corner and group relaxations with different time limits for the instances of Table 2. An asterisk * indicates that the relaxation has been solved in a provable way, while † indicates that the exact bound cannot be computed because of memory fault.

	Corner Relaxation				Group Relaxation				
Name	1 hour	3 hours	5 hours	10 hours	1 hour	3 hours	5 hours	10 hours	1 day
harp2	14.09	15.26	15.69	†	13.26	13.47	13.90	13.93	13.93
lseu	4.58	4.58	4.58	4.58	4.58	4.58	4.58	4.58	†
mitre	77.32	77.32	77.32	†	77.32	77.32	77.32	†	
mod008	29.85	35.14	37.69	41.96	31.46	39.89	44.34	56.44 *	
p0033	31.86	31.86	31.86	31.86	31.51	31.51	31.51	31.51	†
p0201	0.00	0.00	0.00	0.00	0.00 *				
p2756	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
10teams	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
fiber	80.22	80.22	80.22	80.22	46.02	46.02	46.02	46.02	†
gen	38.21	38.21	38.21	38.26	27.81	27.81	27.81	27.81	27.81
gesa2	97.64 *				30.03	30.49	30.88	31.34	31.77
gt2	46.79 *				27.99	27.99	27.99	27.99	27.99
manna81	56.02	56.39	56.77	57.14	56.02	56.39	56.77	57.14	57.52
markshare1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
markshare2	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
mas74	26.73	29.69	31.07	33.14	26.71	29.70	31.07	33.14	36.12
mas76	26.33	29.46	31.07	33.53	26.31	29.46	31.07	33.52	36.77
mkc	0.00	0.00	0.00	0.00	0.00	0.00	0.00	†	
modglob	55.62	59.23	60.89	62.88	31.47 *				
net12	6.23	6.34	6.37	6.37	2.67	2.67	2.67	2.67	†
opt1217	0.00	0.00	0.00	†	0.00	0.00	0.00	†	
pk1	0.00	0.00	0.00	0.00	0.00 *				
pp08a	24.30	24.30	24.30	24.30	23.49	23.49	23.49	23.49	†
pp08aCUTS	36.15	38.14	39.05	40.32	33.13 *				
qnet1_o	64.80 *				23.21	23.26	23.26	23.26	23.26
rgn	0.00 *				0.00	0.00	0.00	0.00	0.00
set1ch	69.38	70.45	70.83	71.60	49.41	50.05	50.38	50.90	†
timtab1	28.17	28.17	28.17	28.17	26.56	26.56	26.56	26.56	26.56
tr12-30	67.97	68.57	68.86	69.24	67.02	67.78	68.10	68.54	†
vpm1	5.45	5.45	5.45	5.45	5.45	5.45	5.45	†	
vpm2	17.35	17.35	17.35	17.35	2.61	2.61	2.61	2.61	†

Table 4: Atamtürk's bounded multiple-knapsack problems.

Name	I	B	C	LP %gap	GMI %gc	1:50-c %gc	$k=60$ %gc	GMI' %gc	Corner %gc	Group %gc
mik.250-5-100.1	150	100	5	17.90	57.61	69.73	64.90	55.60	66.67	66.67
mik.250-5-100.2	150	100	5	19.31	61.45	69.23	66.55	59.63	67.16	67.16
mik.250-5-100.3	150	100	5	15.63	60.97	73.27	68.77	59.94	76.09	76.09
mik.250-5-100.4	150	100	5	16.02	61.16	68.39	64.12	60.34	67.25	67.25
mik.250-5-100.5	150	100	5	16.88	65.26	75.21	72.42	64.32	80.25	80.25
mik.250-10-50.1	200	50	10	21.15	48.71	77.75	60.01	48.11	80.15	80.13
mik.250-10-50.2	200	50	10	20.07	48.38	74.53	64.04	47.90	82.06	82.04
mik.250-10-50.3	200	50	10	22.13	52.63	74.66	67.42	51.91	83.94	83.93
mik.250-10-50.4	200	50	10	19.79	48.88	75.63	65.56	48.16	78.85	78.85
mik.250-10-50.5	200	50	10	20.15	55.77	79.14	73.18	53.75	94.91	94.89
mik.250-10-75.1	175	75	10	21.02	59.96	72.66	65.99	58.76	77.38	77.38
mik.250-10-75.2	175	75	10	21.04	55.25	73.59	67.52	53.52	76.71	76.71
mik.250-10-75.3	175	75	10	17.21	58.28	69.88	66.64	55.59	76.20	76.19
mik.250-10-75.4	175	75	10	19.05	51.03	72.33	63.23	50.74	72.22	72.22
mik.250-10-75.5	175	75	10	20.05	56.45	74.26	67.72	55.42	75.52	75.53
mik.250-10-100.1	150	100	10	13.92	71.20	74.50	73.50	69.08	82.79	82.79
mik.250-10-100.2	150	100	10	16.03	71.23	77.86	76.10	69.86	78.72	78.72
mik.250-10-100.3	150	100	10	16.30	58.80	73.70	68.77	57.80	73.38	73.38
mik.250-10-100.4	150	100	10	14.48	66.82	71.82	68.80	65.89	73.66	73.66
mik.250-10-100.5	150	100	10	15.82	69.03	75.60	72.91	68.04	82.37	82.36
mik.250-20-50.1	200	50	20	20.85	49.29	72.18	60.42	48.69	78.53	78.54
mik.250-20-50.2	200	50	20	20.04	48.44	74.33	63.95	47.97	81.42	81.42
mik.250-20-50.3	200	50	20	22.14	52.62	74.78	67.45	51.90	84.05	84.04
mik.250-20-50.4	200	50	20	19.25	50.02	74.93	66.25	49.28	88.04	88.04
mik.250-20-50.5	200	50	20	19.38	57.61	77.60	72.51	55.53	95.83	95.79
mik.250-20-75.1	175	75	20	18.99	65.25	77.13	71.80	63.94	82.94	82.93
mik.250-20-75.2	175	75	20	18.16	62.32	75.60	71.74	60.53	83.80	83.79
mik.250-20-75.3	175	75	20	16.13	61.59	73.06	70.39	58.75	78.82	78.82
mik.250-20-75.4	175	75	20	17.88	53.83	70.74	64.27	53.53	74.35	74.35
mik.250-20-75.5	175	75	20	17.46	63.44	75.31	72.48	62.27	86.20	86.20
mik.250-20-100.1	150	100	20	13.65	71.65	74.53	73.54	70.00	82.15	82.14
mik.250-20-100.2	150	100	20	15.53	72.99	78.25	76.95	71.75	80.42	80.41
mik.250-20-100.3	150	100	20	13.34	69.84	74.29	73.29	68.83	77.42	77.41
mik.250-20-100.4	150	100	20	13.77	69.79	73.33	71.20	68.86	78.04	78.05
mik.250-20-100.5	150	100	20	16.09	68.03	75.95	73.42	67.06	81.77	81.77
Average				17.90	59.87	74.16	68.79	58.66	79.43	79.43

Table 5: Atamtürk's unbounded multiple-knapsack problems.

					GMI	1:50-c	$k = 60$	GMI'	Corner	Group
Name	I	B	C	LP %gap	%gc	%gc	%gc	%gc	%gc	%gc
mik.250-1-50.1	200	50	1	21.16	48.69	84.62	59.98	48.09	88.79	88.77
mik.250-1-50.2	200	50	1	20.12	48.29	75.17	64.19	47.82	82.54	82.53
mik.250-1-50.3	200	50	1	22.43	52.07	75.98	67.79	51.35	84.43	84.41
mik.250-1-50.4	200	50	1	20.83	46.85	77.21	63.11	46.16	81.35	81.33
mik.250-1-50.5	200	50	1	20.73	54.46	79.63	72.53	52.49	90.66	90.67
mik.250-1-75.1	175	75	1	24.20	53.43	77.71	64.85	52.37	76.55	76.55
mik.250-1-75.2	175	75	1	22.21	52.84	79.50	68.64	51.19	74.10	74.09
mik.250-1-75.3	175	75	1	15.55	63.59	74.92	72.79	60.65	81.37	81.37
mik.250-1-75.4	175	75	1	19.74	49.53	73.14	62.28	49.25	71.39	71.38
mik.250-1-75.5	175	75	1	20.80	54.76	75.41	68.69	53.75	76.44	76.45
mik.250-1-100.1	150	100	1	19.65	53.52	71.86	65.88	51.41	70.79	70.79
mik.250-1-100.2	150	100	1	14.24	78.07	78.46	78.38	77.32	87.25	87.25
mik.250-1-100.3	150	100	1	17.01	56.70	74.63	68.61	55.74	73.81	73.80
mik.250-1-100.4	150	100	1	12.94	73.31	73.73	73.55	72.74	81.18	81.18
mik.250-1-100.5	150	100	1	20.83	54.68	74.09	67.71	53.90	72.54	72.54
mik.500-1-50.1	450	50	1	20.22	50.55	73.09	61.82	49.93	88.51	88.52
mik.500-1-50.2	450	50	1	20.36	47.82	78.56	63.94	47.35	88.25	88.28
mik.500-1-50.3	450	50	1	22.69	51.57	77.35	67.64	50.87	87.56	87.57
mik.500-1-50.4	450	50	1	19.11	50.33	74.25	66.32	49.59	92.19	92.19
mik.500-1-50.5	450	50	1	21.35	53.14	80.12	70.77	51.22	90.98	91.00
mik.500-1-75.1	425	75	1	23.89	54.00	73.68	64.48	52.92	73.61	73.61
mik.500-1-75.2	425	75	1	22.15	52.96	78.72	68.57	51.31	72.76	72.75
mik.500-1-75.3	425	75	1	15.76	62.86	74.77	72.42	59.96	81.19	81.19
mik.500-1-75.4	425	75	1	19.11	50.89	72.40	63.33	50.60	72.03	72.02
mik.500-1-75.5	425	75	1	20.71	54.95	75.02	68.58	53.94	76.78	76.77
mik.500-1-100.1	400	100	1	17.39	59.35	73.23	69.02	57.01	76.15	76.16
mik.500-1-100.2	400	100	1	17.11	68.38	79.15	76.79	66.05	85.62	85.61
mik.500-1-100.3	400	100	1	14.95	63.38	75.47	72.59	62.30	79.83	79.83
mik.500-1-100.4	400	100	1	13.55	71.38	74.40	73.66	69.83	81.08	81.07
mik.500-1-100.5	400	100	1	20.71	54.94	73.42	67.53	54.15	71.38	71.37
Average				19.38	59.87	75.99	68.21	58.66	80.37	80.37

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