New Facets of the STS Polytope Generated from Known Facets of the ATS Polytope

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Abstract

While it was known for a long time how to transform an asymmetric traveling salesman problem on the complete graph with n vertices into a symmetric traveling salesman problem on an incomplete graph with $2n$ vertices, no method was available until recently for using this correspondence to derive facets of the symmetric traveling salesman polytope from facets of the asymmetric one. In this paper we develop a procedure for accomplishing this task, and use it to obtain several classes of new facet defining inequalities for the symmetric polytope, derived from odd CAT inequalities and lifted cycle inequalities for the asymmetric polytope.

1 Introduction

The *Traveling Salesman Problem* (TSP), one of the earliest, most heavily studied combinatorial optimization problems, has two major variations in its definition. There is the *Asymmetric Traveling Salesman Problem* (ATSP) formulated on a directed graph, and the *Symmetric Traveling Salesman Problem* (STSP) formulated on an undirected graph. Of these two variations, the STSP has gotten much more attention up to now, but we have learned a fair amount regarding the ATSP as well, [4].

Interestingly, relationships between the STSP and ATSP are not well understood, and are seldom exploited for the purposes of better understanding both types of TSP problems. In this paper, we start to better understand these relationships. In our case, we exploit current insights into the ATSP to better understand the STSP.

Let us denote the complete undirected graph whose vertex set is V by $K_V = (V, E(V))$ and the complete directed graph whose vertex set is V by $\vec{K}_V = (V, A(V))$. Notice that we have need to specify the exact vertex set instead of using the usual notation K_n for a complete graph. A *Hamilton cycle* in a graph is a cycle that visits every vertex of the graph exactly once. The input to the ATSP (STSP) is the vertex set V and a cost c_e for each arc $e \in A(V)$ (edge $e \in E(V)$). The ATSP (STSP) consists in finding a minimum cost Hamilton cycle in $\vec{K}_V$ (K_V).

Most methods for solving the ATSP (STSP) exactly involve integer and linear programming. Hence, it is important to study the *ATS polytope* (*STS polytope*), defined as the convex hull of the edge incidence vectors of all the Hamilton cycles in $\vec{K}_V$ (K_V). The ATS polytope for $\vec{K}_V$ will be denoted by ATS(V), whereas the STS polytope will be denoted by STS(V). In particular, we aim at finding *facet-defining* and *valid* inequalities for these polytopes. The goal of this paper is to provide a method of deriving facet-defining STSP inequalities from a facet-defining ATSP inequality, based on the technique introduced in [1].

In order to achieve our goal, we use the idea of *lifting* a valid inequality for a lower dimensional polyhedron to create a valid inequality for a polyhedron of higher dimension. Let P be a polyhedron. If H is a closed half space containing P, and whose boundary is the hyperplane B, then $F := B \cap P$ is said to be a *face* of P. A *facet* of P is a face $F \neq P$ having maximal dimension. A face F of a polyhedron P is itself a polyhedron, with its own facets. We will explain later how a facet-defining inequality $ax \leq a_0$ for F can be lifted to produce a facet-defining inequality $a'x \leq a'_0$ for P.

Our main idea is to first take a facet-defining inequality for the ATS polytope and, exploiting known relationships between the ATS and STS polytopes, produce an inequality that is facet-defining for a particular face F of the STS polytope. Then, we lift this inequality of the STS face F to obtain a new facet-defining inequality for the entire STS polytope.

The paper is organized as follows. Section 2 explains how ATS inequalities can be lifted into STS inequalities (*A2S liftings*) even though the corresponding polytopes live in apparently incomparable spaces. Section 3 gives facet-defining A2S liftings of the *CAT inequalities* of the ATSP [2], including a new facet-defining inequality class that generalizes the *new1 inequality* found in [5]. Section 4 analyzes the properties of lifting the variables fixed to 1 in the A2S lifting procedure. Finally, Section 5 applies our lifting procedure to obtain STSP facets from the *curtain inequalities* [1] of the ATSP.

2 Exploiting relationships between the ATSP and STSP

Consider the ATSP on the complete directed graph $\vec{K}_V$. We create two copies of each vertex $i \in V$, as in [7, 6], resulting in:

$$\begin{aligned} V^+ &:= \{i^+ : i \in V\}, \\ V^- &:= \{i^- : i \in V\}. \end{aligned}$$

Let the subsets

$$\begin{aligned} E^+ &= E(V^+) \\ E^- &= E(V^-) \\ E^0 &= \{\{i^+, i^-\} : i \in V\} \\ E^{+-} &= \delta(V^+) \setminus E^0 \end{aligned}$$

define a partition of the edges of the complete graph $K_{V^+ \cup V^-}$ on the vertex set $V^+ \cup V^-$. Take any directed Hamilton cycle (V, H) in $\vec{K}_V$. We construct an undirected Hamilton cycle in $K_{V^+ \cup V^-}$ as follows. We define

$$H' := \{\{i^+, j^-\} : (i, j) \in H\} \cup E_0. \tag{1}$$

Then, by construction, $(V^+ \cup V^-, H')$ is a Hamilton cycle in $K_{V^+ \cup V^-}$ such that

$$\begin{aligned} E^0 &\subset H', \\ (E^+ \cup E^-) \cap H' &= \emptyset. \end{aligned} \tag{2}$$

We call such a Hamilton cycle satisfying (2) an *admissable* Hamilton cycle. So, for any directed Hamilton cycle in $\vec{K}_V$, there is a corresponding admissable Hamilton cycle in $K_{V^+ \cup V^-}$ given by (1). Conversely, for any admissable Hamilton cycle in $K_{V^+ \cup V^-}$, there is a corresponding directed Hamilton cycle in $\vec{K}_V$, defined so as to give this admissible Hamilton cycle via (1). Hence, we have a bijection ϕ between directed Hamilton cycles in $\vec{K}_V$ and admissable undirected Hamilton cycles in $K_{V^+ \cup V^-}$.

Define $F(V^+ \cup V^-)$ to be the convex hull of the edge incidence vectors for all the admissable Hamilton cycles in $K_{V^+ \cup V^-}$. It is fairly easy to see that $F(V^+ \cup V^-)$ is the face of $STS(V^+ \cup V^-)$ obtained by fixing the edge variables of the E^+ and E^- edges to be 0 and fixing the edge variables of the E_0 edges to be 1.

Because of our bijection ϕ, we can determine each extreme point x' of $F(V^+ \cup V^-)$ from an extreme point x^* of $ATS(V)$ by $x' = \phi(x^*)$. In fact, we will see that when ϕ is extended linearly to all of $\mathbf{R}^{A(V)}$, we get that $F(V^+ \cup V^-) = \phi(ATS(V))$. We further aim at determining the *facets* of $F(V^+ \cup V^-)$ from the facets of $ATS(V)$. We do this by breaking down ϕ from $x \in \mathbf{R}^{A(V)}$ to $\phi(x) \in \mathbf{R}^{E(V^+ \cup V^-)}$ into $\phi = \phi_3 \circ \phi_2 \circ \phi_1$, where

$$\begin{array}{lcll} \phi_1 & : & \mathbf{R}^{A(V)} & \longrightarrow & \mathbf{R}^{E^{+-}(V)} \\ \phi_2 & : & \mathbf{R}^{E^{+-}(V)} & \longrightarrow & \mathbf{R}^{E(V^+ \cup V^-)} \\ \phi_3 & : & \mathbf{R}^{E(V^+ \cup V^-)} & \longrightarrow & \mathbf{R}^{E(V^+ \cup V^-)} \end{array}$$

are defined as follows:

$$\phi_1(x)_{\{i^+ j^-\}} := x_{(i,j)},$$

$$\phi_2(x')_e := \begin{cases} x'_e & \text{if } e \in E^{+-}, \\ 0 & \text{if } e \in E^+ \cup E^- \cup E^0, \end{cases}$$

and

$$\phi_3(x'') := x'' + v^{shift},$$

with

$$v_e^{shift} := \begin{cases} 1 & \text{if } e \in E^0, \\ 0 & \text{otherwise.} \end{cases}$$

We first obtain the facet-defining inequalities for $ATS'(V) := \phi_1(ATS(V))$ from those of $ATS(V)$. Consider an inequality

$$ax \leq a_0$$

defining a facet of $ATS(V)$. Define

$$a'_{\{i^+,j^-\}} := a_{(i,j)}.$$

Since ϕ_1 just relabels indices, the corresponding facet of $ATS'(V)$ is clearly defined by the inequality

$$a'x' \leq a_0.$$

We next obtain the facet-defining inequalities for $ATS''(V) := \phi_2(ATS'(V))$ from those of $ATS'(V)$. Consider an inequality

$$a'x' \leq a_0$$

defining a facet of $ATS'(V)$. Define

$$a''_e := \begin{cases} a'_e & \text{if } e \in E^{+-}, \\ 0 & \text{if } e \in E^+ \cup E^- \cup E^0. \end{cases}$$

Since ϕ_2 just adds components of value 0 to the point x', the corresponding facet of $ATS''(V)$ is then defined by the inequality

$$a''x'' \leq a_0.$$

We finally obtain the facet-defining inequalities for $F(V^+ \cup V^-) = \phi_3(ATS''(V))$ from those of $ATS''(V)$. Since ϕ_3 just translates every point by a fixed vector, the normal vector a'' of the facet-defining $ATS''(V)$ inequality

$$a''x'' \leq a_0$$

remains unchanged in the corresponding facet-defining inequality for $F(V^+ \cup V^-)$, with only the right hand side a_0 being possibly affected. But since the translation ϕ_3 is perpendicular to this normal vector, even the right hand side a_0 remains the same.

Thus, corresponding to each facet-defining ATS(V) inequality

$$ax \leq a_0$$

is the inequality

$$a''x'' \leq a_0$$

that is facet-defining for $F(V^+ \cup V^-)$.

2.1 Asymmetric to Symmetric Lifting

Consider a particular vertex set V, and any facet-defining inequality for $ATS(V)$. By the previous analysis, we can easily find a corresponding facet-defining inequality for $F(V^+ \cup V^-)$. Since $F(V^+ \cup V^-)$ is a face of $STS(V^+ \cup V^-)$, we can lift it to a facet-defining inequality for $STS(V^+ \cup V^-)$ by using well-known *sequential lifting* techniques,[9]. We call our procedure of taking a facet-defining ATS inequality and producing facet-defining STS inequalities in this manner an *A2S lifting* (Asymmetric to Symmetric lifting). This type of lifting was first described in [1].

In sequential lifting, one creates a *lifting sequence* for the variables fixed at 0 or 1. Let us first consider the case where our variables are only fixed at 0, and we have a $\leq$ inequality. Going one-by-one through the lifting sequence, we calculate the largest possible value for the coefficient of our current variable, so that our inequality remains valid when the current variable is no longer fixed. At the end of this process, we will have a facet-defining inequality for the larger dimension polytope, assuming the polytope dimension increases by exactly one unit at each lifting step, as is the case in our application. Having variables fixed at 1 essentially does not change the procedure, but one must first complement these variables and then calculate the value of the lifting coefficient. As a result, the right hand side of our inequality can change in this case.

We currently create our lifting sequence so that we lift first all the variables fixed at 0, and then lift those fixed at 1. If we stop the lifting once all the variables fixed at 0 have been lifted, we are left with a facet-defining inequality on a polytope that includes $F(V^+ \cup V^-)$ and is included in STS$(V^+ \cup V^-)$. We name this polytope the *Twin Traveling Salesman Polytope* TTS$(V^+ \cup V^-)$, which can be defined as the convex hull of all Hamilton cycles that use all edges of E^0.

3 STSP Analogues of odd CAT Inequalities

We first did our lifting methods for a subclass of the odd closed alternating trail (odd CAT) inequalities [2] of the ATS polytope. The first odd CAT inequality we look at comes from the odd closed alternating trail shown in Figure 1.

The odd CAT inequality corresponding to this is denoted by $ax \leq 2$,

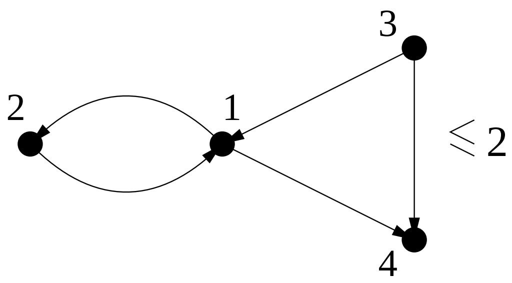


Figure 1: An odd closed alternating trail.

where the coefficients a_{ij} are as follows.

$$
\begin{aligned}
&a_{12} = a_{21} = a_{31} = a_{34} = a_{14} = 1,\\
&a_{ij} = 0 \qquad \text{otherwise.}
\end{aligned}
$$

This odd CAT inequality is facet-defining for $ATS(V)$ for $|V| \geq 4$, [2]. Through our lifting methods, we obtain inequalities which are facet-defining for $STS(V^+ \cup V^-)$. The inequalities we obtain are, of course, well-known since a complete description of STS(K_8) is known, [5]. On 500 randomly chosen lifting sequences, we obtained the following:

(i) a three-tooth comb inequality on 213 cases,

(ii) a four-tooth ladder inequality on 33 cases,

(iii) a new1 inequality on 254 cases.

The *new1 inequality* was discovered in [5], and along with two other inequalities, completed the polyhedral description of $STS(V)$ for $|V| = 8$. Figure 2 shows the standard form for the new1 inequality that our procedure produced. Figure 3 displays the skeleton of the tight-triangular form of this inequality [8].

3.1 SymCAT Inequalities

This prompted us to investigate what STS inequalities we could produce from other odd CAT inequalities. Here, we consider odd CAT inequalities formed from one alternating cycle and one two-cycle. An example on six nodes is seen in Figure 4.

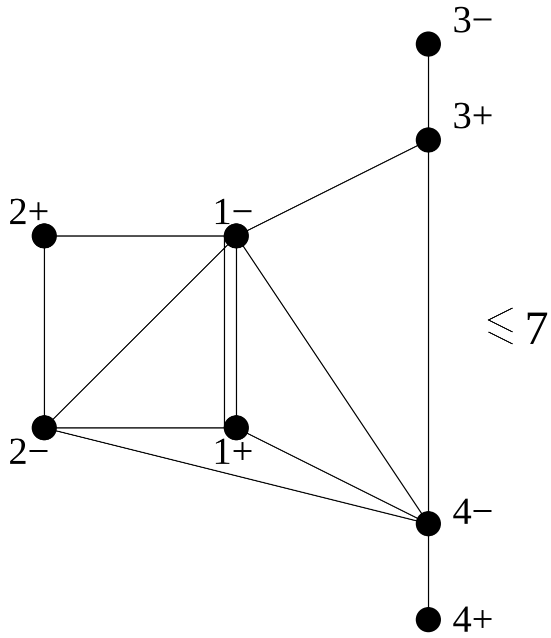


Figure 2: The new1 inequality in standard form.

The odd CAT inequality is denoted by $ax \leq 3$, where the coefficients a_{ij} are as follows:

$$
\begin{aligned}
&a_{12} = a_{21} = a_{31} = a_{34} = a_{54} = a_{56} = a_{16} = 1, \\
&a_{14} = a_{36} = a_{51} = 1, \\
&a_{ij} = 0 \qquad \text{for all other arcs } (i, j).
\end{aligned}
$$

We again used our lifting procedure with random lifting sequences, obtaining facet-defining inequalities for $STS(V^+ \cup V^-)$ from the above odd CAT inequality on $ATS(V)$ for $|V| \geq 6$. On some of these lifting sequences, we obtained just comb inequalities. However, on most of the lifting sequences, we encountered a $STS(V^+ \cup V^-)$ facet-defining inequality, with $|V^+ \cup V^-| = 12$, which we could not identify as a known STS inequality. The support of this inequality is shown in Figure 5. Figure 6 shows this inequality in tight-triangular form.

We studied this STS inequality in the attempt of generalizing it, and as a result we inferred the following class of STS inequalities, that we call *symCAT* inequalities. Let $V = \{1, 2, \ldots, n\}$, where n is an even integer. Let $ax \leq \frac{n}{2}$ be the odd CAT inequality for $ATS(V)$ corresponding to an

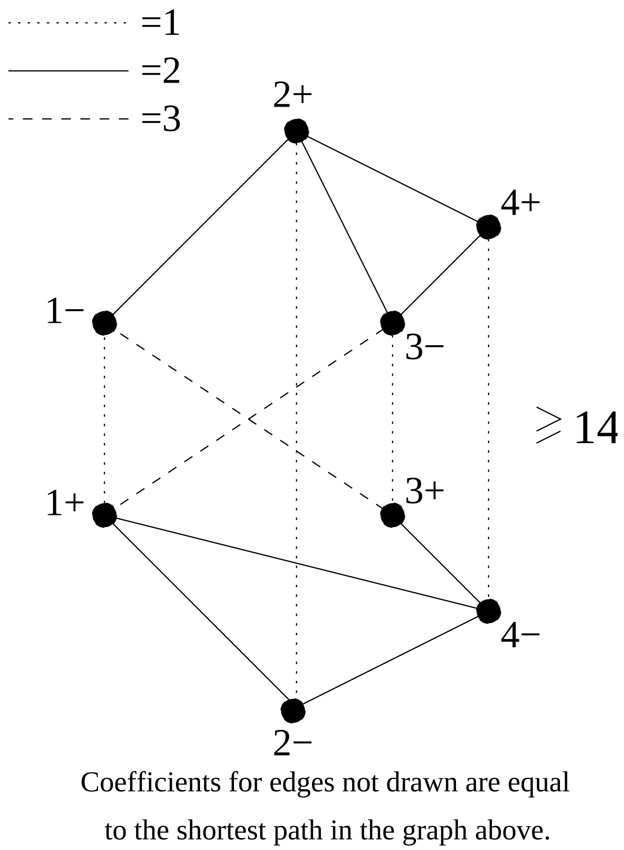


Figure 3: The new1 inequality in tight-triangular form.

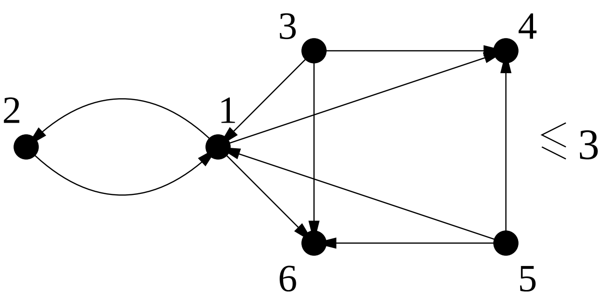


Figure 4: Support of a 6-node odd CAT inequality.

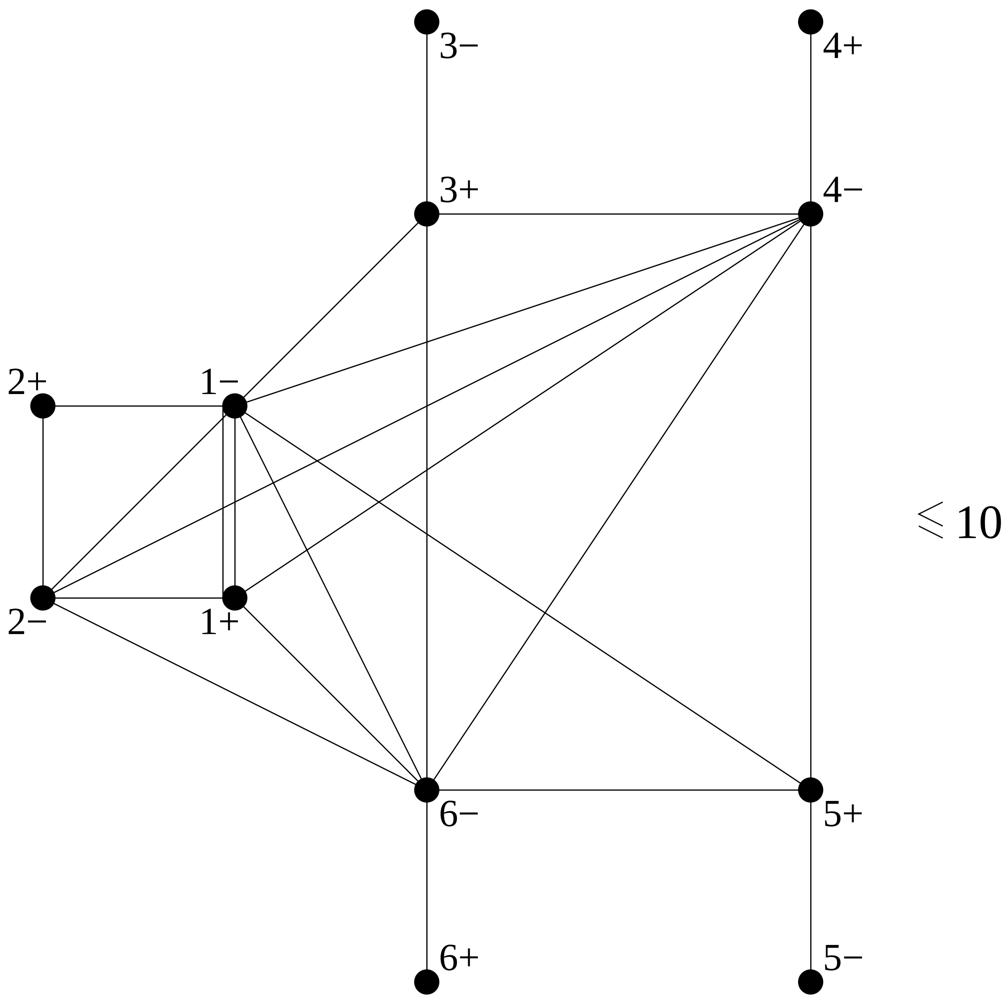


Figure 5: Support of the 12 node SymCAT inequality

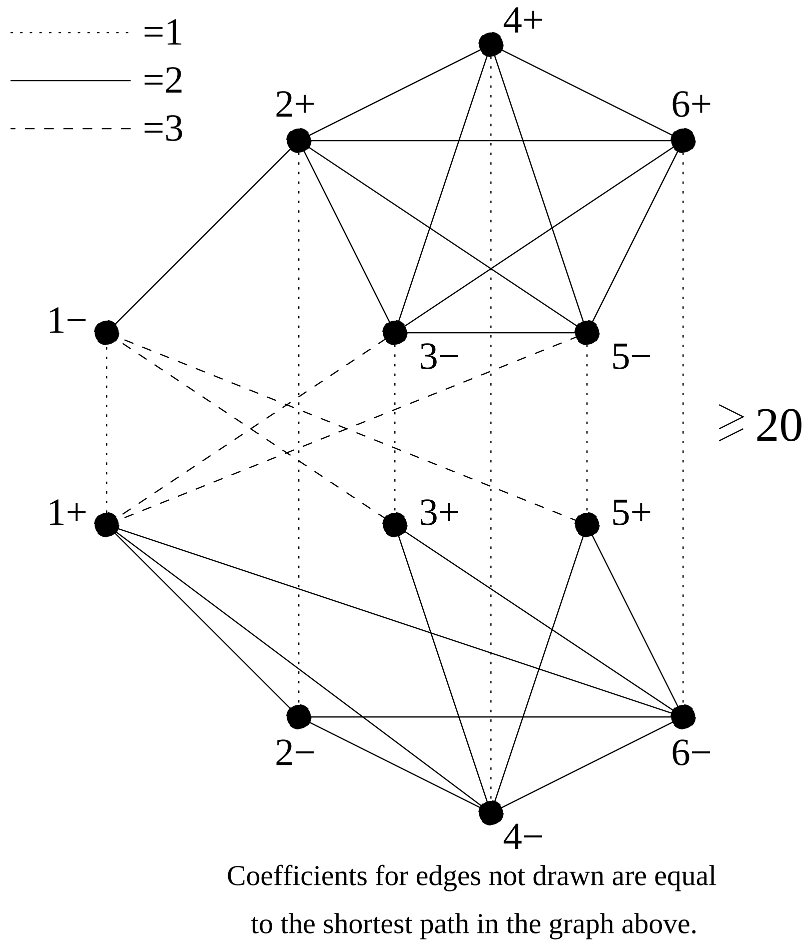


Figure 6: The 12 node SymCAT inequality in tight-triangular form

odd closed alternating trail on n vertices which, when directions are ignored, has a cycle on vertices 1 and 2 and another cycle of vertex 1 and all the other vertices except 2, such as is shown in Figure 4. We then have the $STS(V^+ \cup V^-)$ inequality $\overline{a}x \leq \frac{3n}{2} + 1$, where the coefficients $\overline{a}_{ij}$ for edges $\{i, j\}$ are given by:

$$
\begin{array}{lcll}
\overline{a}_{i^+j^-} & = & a_{ij} & \text{for all } i \neq j \in \{1, 2, \ldots, n\}, \\
\overline{a}_{i^+j^+} & = & 0 & \text{for all } i \neq j \in \{1, 2, \ldots, n\}, \\
\overline{a}_{i^-j^-} & = & 1 & \text{for all } i \neq j \in \{2, 4, \ldots, n\} \cup \{1\}, \\
\overline{a}_{i^-j^-} & = & 0 & \text{otherwise}, \\
\overline{a}_{1^+1^-} & = & 2, & \\
\overline{a}_{i^+i^-} & = & 1 & \text{for all } i \in \{2, 3, \ldots, n\}.
\end{array} \tag{3}
$$

Figure 7 gives an illustration of the general SymCAT inequality. Figure 8 has this in tight-triangular form.

3.2 Proof that SymCATs are valid

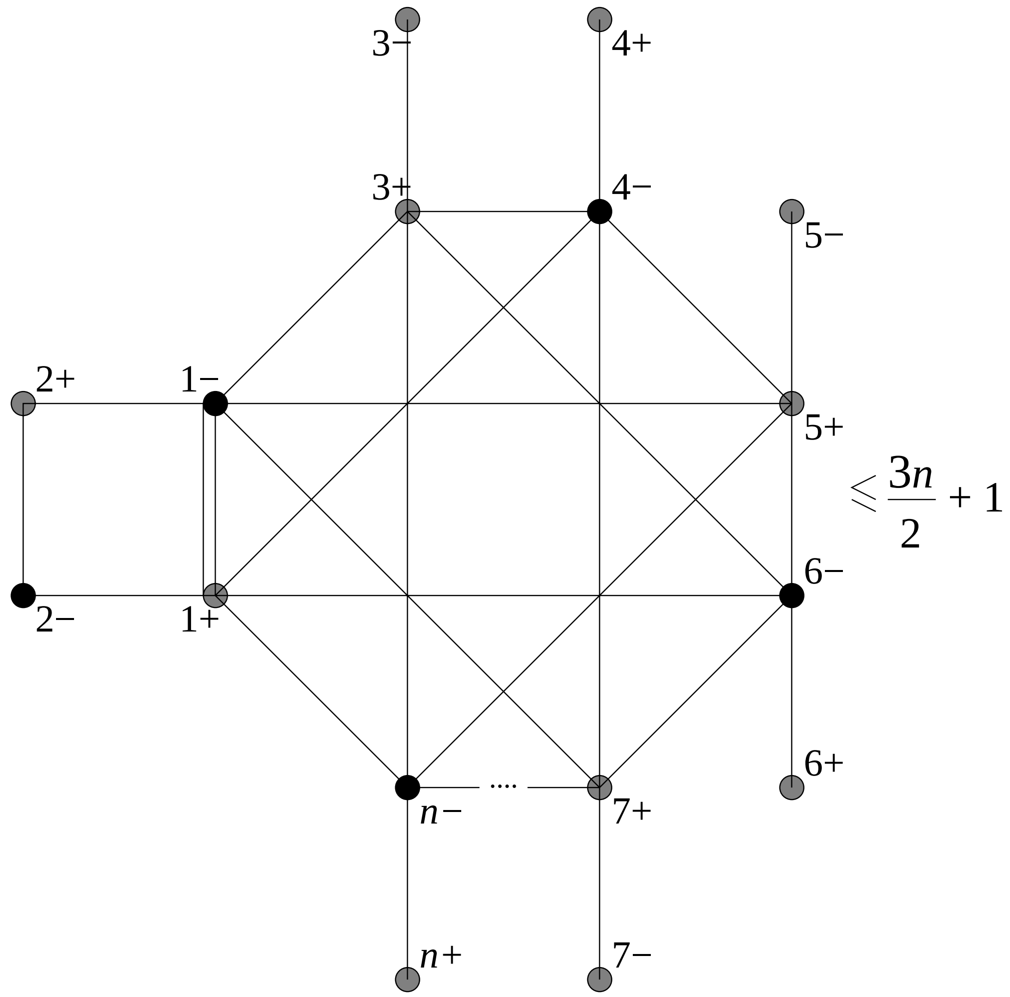
Theorem 1 *The inequality* $\overline{a}x \leq \frac{3n}{2} + 1$ *is valid for* $STS(V^+ \cup V^-)$.

Proof: Consider the comb shown in Figure 9, where the handle is $\{1^-\} \cup \{1^+, 2^-, 3^+, 4^-, \ldots, n^-\}$ and the teeth are $\{i^+, i^-\}$ for $i = 2, 3, \ldots, n$. Denote the corresponding comb inequality by $bx \leq \frac{3(n-2)}{2} + 3 = \frac{3n}{2}$.

Define $S := \{1^+, 2^+, 1^-, 2^-\}$. Consider adding up the following inequalities, weighted by $\frac{1}{2}$:

$$
\begin{array}{lcl}
(x(\delta(1^-)) & \leq & 2) \\
(x(\delta(4^-)) & \leq & 2) \\
(x(\delta(6^-)) & \leq & 2) \\
 & \cdots & \\
(x(\delta(n^-)) & \leq & 2) \\
(bx & \leq & \frac{3n}{2}) \\
(x_{1^+1^-} & \leq & 1) \\
(x_{3^+3^-} & \leq & 1) \\
 & \cdots & \\
(x_{(n-1)^+(n-1)^-} & \leq & 1) \\
(x(E(S)) & \leq & 3).
\end{array} \tag{4}
$$

When these are all added up, one obtains $\overline{a}x + ux \leq \frac{3n}{2} + \frac{3}{2}$, where u is a non-negative vector. By performing Chvatal rounding, one obtains $\overline{a}x \leq \frac{3n}{2} + 1$. This proves our theorem. $\square$



Edges between all pairs of black nodes also have coefficient 1 but are not drawn

Figure 7: The $2n$-node SymCAT inequality

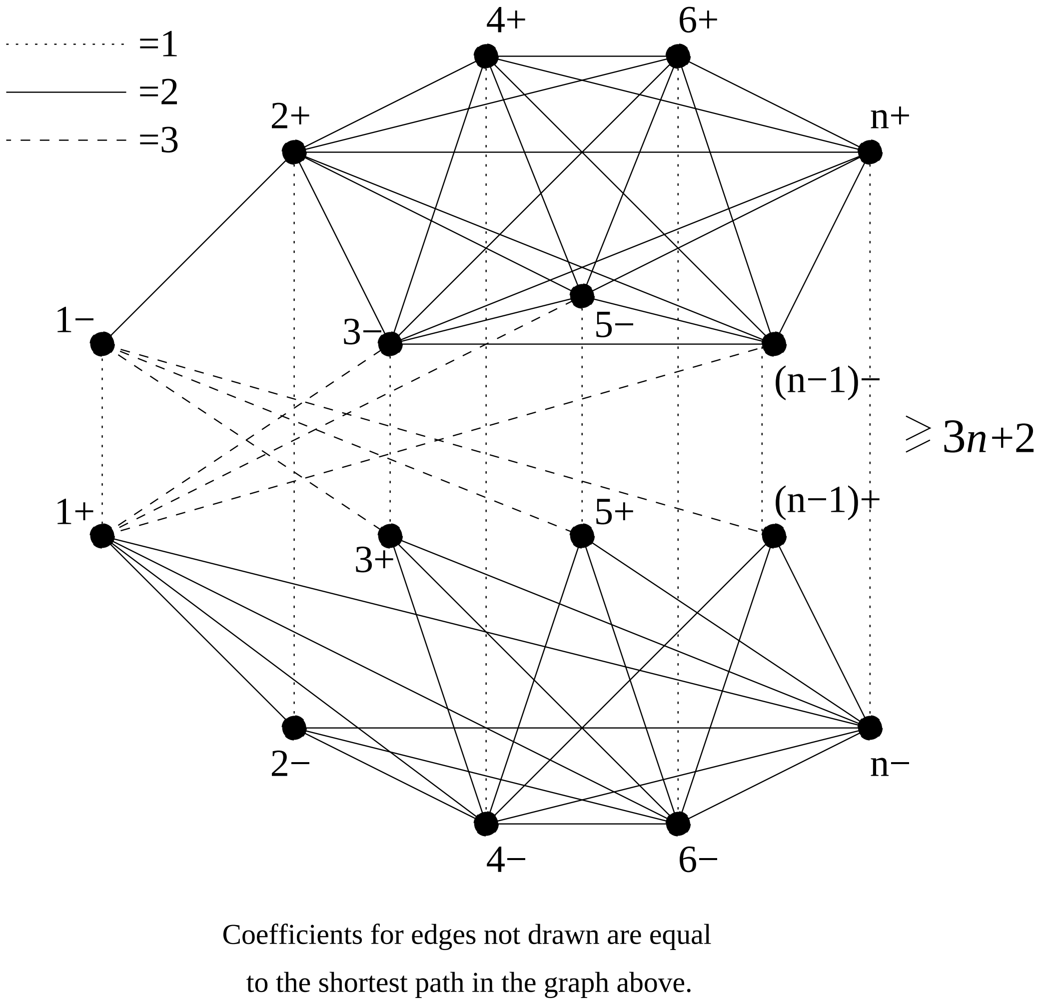


Figure 8: The $2n$-node SymCAT inequality in tight-triangular form

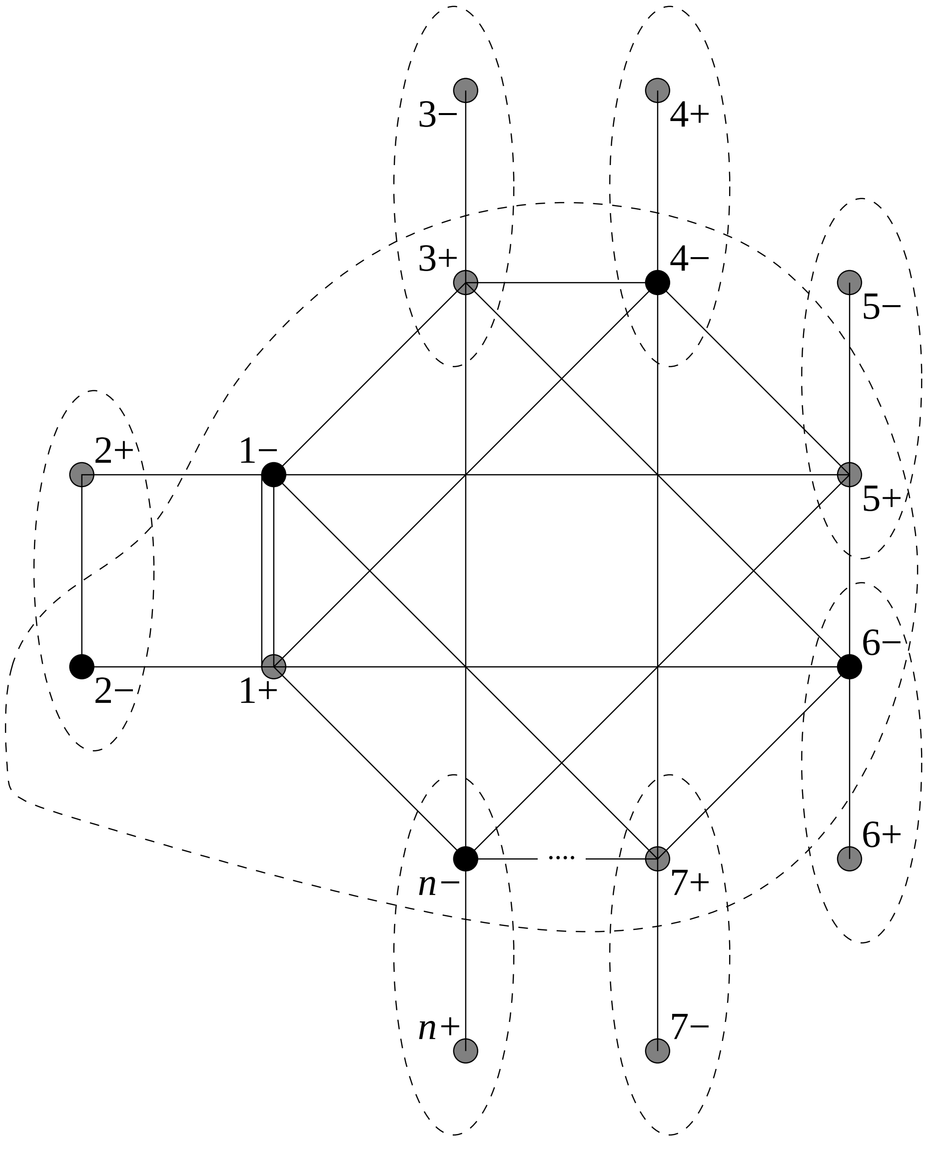


Figure 9: The comb used in Chvátal derivation.

3.3 Proof that SymCATs are Facets for TTSP

This section shows a general method that can be used to show that inequalities obtained from the lifting procedure are facet-defining for the STSP. It is based on the idea of creating a tree structure outlining the order the coefficients can be maximally lifted. The root of the tree can be chosen freely because the lifting process has one degree of freedom (see below). As a specific example, the process will be shown on the symCAT inequality on 12 nodes, which is derived from the CAT inequality on a 6-node ATSP from Figure 4, and then we will generalize this for higher n. If only the coefficients whose variables are fixed to zero are lifted, one gets an inequality valid for the twin traveling salesman polytope (TTSP). If this can be shown to be facet-defining, most of the work will be done, as it is fairly easy to show that lifting the remaining variables (those fixed to one), creates a facet-defining inequality for the STSP (see Section 4).

The odd CAT inequality of the ATSP that we use here arises from a closed alternating trail where node 1 is both a source and sink, node 2 is neither a source nor sink, and the cycle visits in order (ignoring directions) $1, 3, 4, 5, 6$, and back to 1. (See Figure 4) The odd nodes greater than 1 are only sources and the even nodes greater than 2 are only sinks. Denote this inequality on the expanded undirected graph as $ax \leq 3$. Lifting this to the TTSP yields:

$$\begin{array}{lll} a_{i^+j^+} := 0 & \text{for all } i, j, & \\ a_{i^-j^-} := 1 & \text{for all } i, j \in \{1, 2, 4, 6\} & \quad (5) \\ a_{i^-j^-} := 0 & \text{otherwise .} & \end{array}$$

In lifting the coefficients for the missing E^+ and E^- variables, notice that one could choose any variable and assign its coefficient an arbitrary value and still have a valid TTSP inequality, for the following reason. Let $ax \leq a_0$ be a valid TTSP inequality, and let $\epsilon > 0$. Define a' by:

$$\begin{array}{lll} a'_e := a_e + \epsilon & \text{for all } e \in E^+ & \\ a'_e := a_e - \epsilon & \text{for all } e \in E^- & \quad (6) \\ a'_e := a_e & \text{otherwise.} & \end{array}$$

Then $a'x \leq a_0$ is a valid TTSP inequality that defines the same face as $ax \leq a_0$ since the equation $x(E^+) = x(E^-)$ is valid for the TTSP and the STSP.

Hence, we may assign any single coefficient to any value (we choose to set $a_{1^+2^+} := 0$) to begin our rooted tree. This choice is arbitrary as the proof could start at any node. Once one coefficient is assigned, other coefficients can be assigned by the following operation:

(1) Choose a tour using all six edges in E_0 and six of the edges in E^{+-} whose values for x make the inequality $ax \leq 3$ tight. Note that for the TTSP, $a_e = 0$ for any $e \in E_0$, and the values for a_e for $e \in E^{+-}$ are taken from the odd CAT inequality on the ATSP.

(2) Alter the tour with a 2-interchange move, swapping out two edges from E^{+-} and adding two edges, one from E^{+} and one from E^{-}. One of these two new edges should be an edge whose coefficient is already assigned, and the other should be an edge whose coefficient is not yet assigned. Since the new tour must satisfy the inequality $ax \leq 3$, the unassigned coefficient, a_e, has a maximum allowable value, and that value will make $ax = 3$. The objective of the proof is to find a sequence of these operations where the maximum allowable values match the lifted values in (5). A tree structure is used in place of the sequence, as there are often several good choices for the next edge to be assigned in the sequence, and the tree structure shows off the patterns in the generalization more easily.

Using the tour $1^-1^+2^-2^+4^-4^+5^-5^+6^-6^+3^-3^+1^-$ (which is tight since $a_{1^+2^-}$, $a_{5^+6^-}$, and $a_{3^+1^-}$ are equal to 1), choose a 2-interchange move that removes edges $\{1^+, 2^-\}$ and $\{2^+, 4^-\}$, and adds edges $\{1^+, 2^+\}$ and $\{2^-, 4^-\}$. Note that $a_{1^+2^+}$ is already assigned to zero, and the new tour still uses variables $a_{5^+6^-}$, and $a_{3^+1^-}$ which are 1. Thus, to make $ax = 3$, the variable $a_{2^-4^-}$ must be set to 1. This matches the value in (5), so we can assign $a_{2^-4^-} := 1$ after we assign $a_{1^+2^+} := 0$.

Not every possible 2-interchange move will create a useful assignment. For example, if we start with the tour $1^-1^+2^-2^+3^-3^+4^-4^+5^-5^+6^-6^+1^-$ (this is tight since $a_{1^+2^-}$, $a_{3^+4^-}$, and $a_{5^+6^-}$ are equal to 1), and choose a 2-interchange move that replaces edges $\{1^+, 2^-\}$ and $\{2^+, 3^-\}$ with edges $\{1^+, 2^+\}$ and $\{2^-, 3^-\}$, the maximum value allowed for $a_{2^-3^-}$ would be 1, but our target for this variable is 0. A different tour and different 2-interchange move later in the process will create the upper bound of 0 we are looking for.

Figure 10 shows one possible tree diagram that can lead to the appropriate assignments for each of the variables. Each dependency in the tree is associated with a tour. Given the tour, there is only one possible 2-interchange move in the tour that adds the two edges associated with the parent and child in the dependency. Therefore, the tree and list of tours constitute the proof that, given the lifting from $F(V^+ \cup V^-)$ to $\text{TTS}(V^+ \cup V^-)$ is valid, it is also maximal. The labels on the arcs of the tree in Figure 10 refer to the tours in the following list:

$1a : 1^-1^+2^-2^+4^-4^+5^-5^+6^-6^+3^-3^+1^-$ $\quad 1b : 1^-1^+2^-2^+6^-6^+5^-5^+4^-4^+3^-3^+1^-$
$1c : 1^-1^+2^-2^+4^-4^+3^-3^+6^-6^+5^-5^+1^-$ $\quad 1d : 1^-1^+2^-2^+6^-6^+3^-3^+4^-4^+5^-5^+1^-$
$2a : 1^-1^+4^-4^+5^-5^+6^-6^+3^-3^+2^-2^+1^-$ $\quad 2b : 1^-1^+4^-4^+3^-3^+6^-6^+5^-5^+2^-2^+1^-$
$2c : 1^-1^+6^-6^+5^-5^+4^-4^+3^-3^+2^-2^+1^-$
$3a : 1^-1^+4^-4^+3^-3^+5^-5^+6^-6^+2^-2^+1^-$ $\quad 3b : 1^-1^+6^-6^+3^-3^+5^-5^+4^-4^+2^-2^+1^-$
$4a : 1^-1^+3^-3^+4^-4^+5^-5^+6^-6^+2^-2^+1^-$ $\quad 4b : 1^-1^+5^-5^+4^-4^+3^-3^+6^-6^+2^-2^+1^-$
$4c : 1^-1^+3^-3^+6^-6^+5^-5^+4^-4^+2^-2^+1^-$ $\quad 4d : 1^-1^+5^-5^+6^-6^+3^-3^+4^-4^+2^-2^+1^-$
$5a : 1^-1^+2^-2^+3^-3^+4^-4^+5^-5^+6^-6^+1^-$ $\quad 5b : 1^-1^+2^-2^+5^-5^+4^-4^+3^-3^+6^-6^+1^-$

The above list of tours and the tree in Figure 10 prove that the 12-node symCAT inequality is facet-defining on the TTSP.

For the general case, notice that even nodes greater than 2 are indistinguishable in the odd CAT and symCAT inequalities. The same is true for odd nodes greater than 1. For this reason, the tour (5a above)

$$1^-1^+2^-2^+3^-3^+4^-4^+5^-5^+6^-6^+1^-$$

could be represented by

$$1^-1^+2^-2^+odd^-odd^+even^-even^+odd^-odd^+even^-even^+1^-.$$

Tour 5b would become the same generic tour. Also, note that tours 1a through 1d would be the same, as would 2a through 2c, 3a and 3b, and finally 4a through 4d. To generalize to a larger odd CAT, additional even-odd pairs can be inserted into each general tour, giving a tree structure that can be used for any size odd CAT (see Figure 11). Notice that when an even node is used in both the parent node and child node of an arc in the tree, they will always be referencing different even nodes in this tree. In the previous example, one can note that the assignment of $a_{3^-6^-}$ was a child of the assignment of $a_{2^+4^+}$, while the assignment of $a_{3^-4^-}$ was a child of the assignment of $a_{2^+6^+}$. This is not necessary, since there does exist a tour that could be used to assign $a_{3^-4^-}$ after $a_{2^+4^+}$, but avoiding these tours makes the generalization simpler.

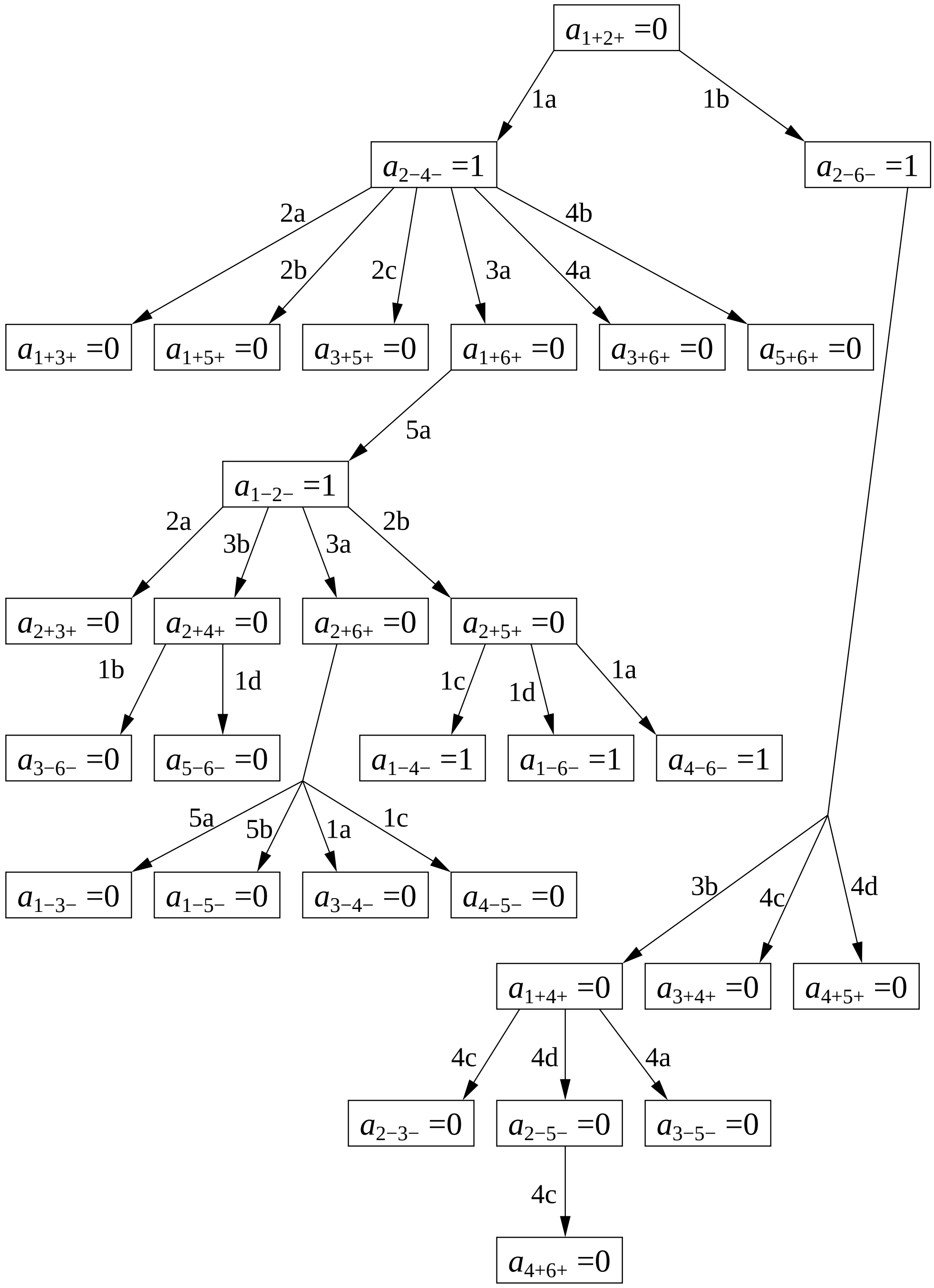


Figure 10: A tree showing the order of lifted coefficients for $n = 6$.

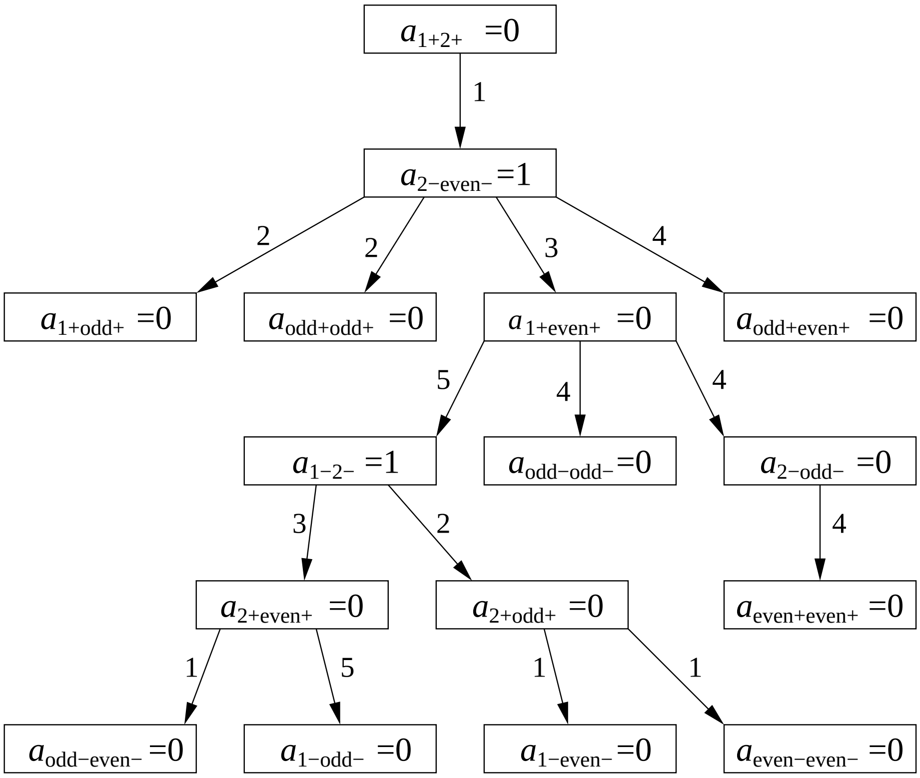


Figure 11: A tree showing the order of lifted coefficients in the general case

$1 : 1^-1^+2^-2^+ even^- even^+ odd^- odd^+ (\ldots) even^- even^+ odd^- odd^+ 1^-$
$2 : 1^-1^+ even^- even^+ odd^- odd^+ (\ldots) even^- even^+ odd^- odd^+ 2^-2^+1^-$
$3 : 1^-1^+ even^- even^+ odd^- odd^+ odd^- odd^+ (\ldots) even^- even^+ 2^-2^+1^-$
$4 : 1^-1^+ odd^- odd^+ even^- even^+ odd^- odd^+ (\ldots) even^- even^+ 2^-2^+1^-$
$5 : 1^-1^+2^-2^+ odd^- odd^+ even^- even^+ odd^- odd^+ (\ldots) even^- even^+ 1^-$

Parentheses indicate where an arbitrary number of even-odd pairs may be inserted.

This list of tours and the tree in Figure 11 prove that our class of symCAT inequalities is facet-defining for $\text{TTS}(V^+ \cup V^-)$.

We will show that the class of symCAT inequalities is also facet-defining for $\text{STS}(V^+ \cup V^-)$ using methods developed in the next section.

4 The Cloning Coefficient in A2S Lifting

In this section we analyze an important property of A2S lifting, with the aim of establishing useful bounds on some of the lifting coefficients. We deal with a generic facet-defining $\text{ATS}(V)$ inequality

$$ax \le a_0,$$

and denote by

$$\overline{a}y \le \overline{a}_0$$

the corresponding inequality for $\text{STS}(V^+ \cup V^-)$. To simplify notation, we define

$$\begin{array}{lcl} E^+ & := & \{\{i^+, j^+\} : i < j \in V\} \\ E^- & := & \{\{i^-, j^-\} : i < j \in V\} \\ E^0 & := & \{\{i^+, i^-\} : i \in V\} \end{array}$$

The variables in E^0 are initially set to 1 and the variables in $E^+ \cup E^-$ are initially set to 0 in $F(V^+ \cup V^-)$. Moreover, we assume without loss of generality that $\overline{a}_{i^+i^-} = 0$ holds for each $\{i^+, i^-\} \in E^0$ before lifting, which implies $\overline{a}_0 = a_0$ at the starting point where $\overline{a}y \le \overline{a}_0$ is facet-defining for $F(V^+ \cup V^-)$. Finally, we concentrate on the situation where one of the variables fixed to 1, namely $y_{i^+i^-}$ for an $\{i^+, i^-\} \in E^0$, is lifted first. This is motivated by the fact that the lifting coefficient of such a variable can then be computed easily.

Given a (facet-defining) inequality $ax \leq a_0$ for ATS(V), let the *cloning coefficient* a_{kk} for each vertex $k \in V$ be computed as

$$a_{kk} := max\{a_{ik} + a_{kj} - a_{ij} \; : \; i \neq j \in V \setminus \{k\}\}.$$

Balas and Fischetti [3] show that if we add a vertex i' with a new coefficient vector a' that satisfies

$$\begin{array}{rcllcl} a'_{i'v} & = & a_{iv}, & a'_{vi'} & = & a_{vi} \quad \forall v \in V \setminus \{i\} \\ a'_{ii'}, a'_{i'i} & = & a_{ii} & & & \\ a'_{uv} & = & a_{uv} & & & \text{otherwise} \end{array}$$

then $a'x \leq a_o + a_{ii}$ is valid and facet-defining for the ATSP. We now give the main theorem of this section.

Theorem 2 *Let $ax \leq a_0$ define a facet of ATS(V) and let $\overline{a}y \leq \overline{a}_0$ be its A2S counterpart, and thus facet-defining for $F(V^+ \cup V^-)$. Let $i \in V$. Then*

$$\overline{a}y + a_{ii}y_{i^+i^-} \leq a_0 + a_{ii} \tag{7}$$

represents a maximal lifting of the coefficient of the variable $y_{i^+i^-}$ if this coefficient is lifted first in the lifting sequence from $F(V^+ \cup V^-)$ to STS($V^+ \cup V^-$).

Proof: We must establish that (7) is valid and that for any $\epsilon > 0$, the inequality

$$\overline{a}y + (a_{ii} - \epsilon)y_{i^+i^-} \leq a_0 + (a_{ii} - \epsilon) \tag{8}$$

is not valid.

We first establish validity. Let $\hat{y}$ be a feasible tour for the current polytope. If $\hat{y}_{i^+i^-} = 1$, clearly (7) holds, so suppose $\hat{y}_{i^+i^-} = 0$. With a two-interchange on $\hat{y}$, create a tour $\overline{y}$ where edges $\{i^+, j^-\}$ and $\{k^+, i^-\}$ are replaced by edges $\{i^+, i^-\}$ and $\{k^+, j^-\}$. From the definition of the cloning coefficient, we have

$$\overline{a}\hat{y} \leq \overline{ay} + a_{ii}\overline{y}_{i^+i^-}.$$

Since (7) holds for $\overline{y}$ and $\hat{y}_{i^+i^-} = 0$, (7) holds for $\hat{y}$. Since $\hat{y}$ was arbitrary, (7) is valid.

To show that (8) is invalid, we first choose an arc (k, j) from the ATSP, such that $a_{ii} = a_{ij} + a_{ki} - a_{kj}$. Since the inequality is facet-defining for the ATSP, there exists a tight tour on the ATSP using the arc (k, j), which corresponds to a tight tour on $F(V^+ \cup V^-)$, which we will denote $\hat{y}$. With a two-interchange on $\hat{y}$, create a tour $\overline{y}$ where edges $\{i^+, i^-\}$ and $\{k^+, j^-\}$ are replaced by edges $\{i^+, j^-\}$ and $\{k^+, i^-\}$. (This is the reverse of the two-interchange done in the first part of the proof.) Because of the choice of (k, j), $\overline{a}\overline{y} = \overline{a}\hat{y} = a_0 + a_{ii}$. Since $\{i^+, i^-\}$ is not an edge of $\overline{y}$, we have

$$\overline{a}\overline{y} + (a_{ii} - \epsilon)\overline{y}_{i^+i^-} = \overline{a}\overline{y} = a_0 + a_{ii} > a_0 + (a_{ii} - \epsilon)$$

which proves that (8) is invalid. □

Theorem 3 *The class of symCAT inequalities is facet-defining for STSP($V^+ \cup V^-$).*

Proof: The class of symCAT inequalities were shown in the previous section to be facet-defining for TTSP($V^+ \cup V^-$). If the lifting of the E^0 coefficients were maximal, our theorem would follow. If these E^0 edges were lifted first, the maximal lifting would be given by the cloning coefficients of the corresponding nodes. Maximally lifting these E^0 edges whose variables are fixed to 1 later in the sequence can only result in larger values than the cloning coefficients (if they change at all). Note that this relationship is larger, not smaller, because variables fixed to 1 must be complemented before lifting and restored after lifting. This also changes the right-hand side of the inequality.

Since using the cloning coefficients for the E^0 edges does not make the symCAT inequalities invalid, but using smaller values clearly would make our inequalities invalid, the E^0 edges have been maximally lifted as required.

□

5 Curtain Inequalities

The ATS class of curtain inequalities has a definition depending on how many nodes are in the cycle of the *cycle inequality* that the curtain inequality is lifted from. We will treat only the case where the number of nodes in this cycle is 4κ for some integer $\kappa \geq 2$. Let C be the cycle visiting in sequence the nodes $i_1, i_2, \ldots, i_{4\kappa}$, where $4\kappa < n$. For notational ease, we will relabel the nodes in our graph so that the nodes on this cycle are $1, 2, \ldots, 4\kappa$. Define $S_1 := \{1, 3, 5, \ldots, 4\kappa - 1\}$ (the set of odd cycle nodes), $C_1 := \{(1, 3), (3, 5), \ldots, (4\kappa - 1, 1)\}$, and $L_1 := \{(3, 4\kappa - 1), (5, 4\kappa - 3), \ldots, (4\kappa - 1, 3)\} \setminus \{(2\kappa + 1, 2\kappa + 1)\}$. Then the curtain inequality is as follows:

$$ax := x(C) + x(E(S_1)) + x(C_1) + x(L_1) \leq 4\kappa - 1. \tag{9}$$

The curtain inequalities are facet-defining for ATS [1].

5.1 Deriving a new STS inequality class

We tried our lifting methods on an asymmetric curtain inequality whose cycle has 12 nodes. We used 12 nodes because we believed it would be more likely to reveal any generalities since the eight node case has only one pair of anti-parallel arcs in L_1. From this experiment, we were led to hypothesize the following facet-defining STS class of inequalities which we will call *symCurtain inequalities.* This class appears to be a new class of STSP inequalities, similar to that of the inequality derived from the curtain in [1]. The inequalities on 16 and 24 nodes are pictured in Figures 12 and 13, respectively.

$$\overline{a}x \leq 12\kappa - 1, \tag{10}$$

where

$$\begin{array}{ll} \overline{a}_{i^+j^+} = \overline{a}_{i^-j^-} = 1 & i, j \in \{1, 3, 5, \ldots, 4\kappa - 1\}, \\ \overline{a}_{i^+(i+1)^+} = \overline{a}_{i^-(i-1)^-} = 1 & i \in \{1, 3, 5, \ldots, 4\kappa - 1\}, \\ \overline{a}_{i^+i^-} = 1 & i \in \{2, 4, 6, \ldots, 4\kappa\}, \\ \overline{a}_{i^+i^-} = 3 & i \in \{1, 3, 5, \ldots, 4\kappa - 1\}, \\ \overline{a}_e = 0 & \text{otherwise.} \end{array}$$

In the definition above, and for the remainder of this section, nodes in the cycle should be considered modulo 4κ. For example, when $i = 1$ node $i - 1$ refers to node 4κ and not the non-existent node 0. Note that $\overline{a}_{i+i^-}$ is defined to be the cloning coefficient for node i in the curtain inequality.

5.2 Proof that SymCurtains are valid

Theorem 4 *The inequality (10) is valid for STS($V^+ \cup V^-$).*

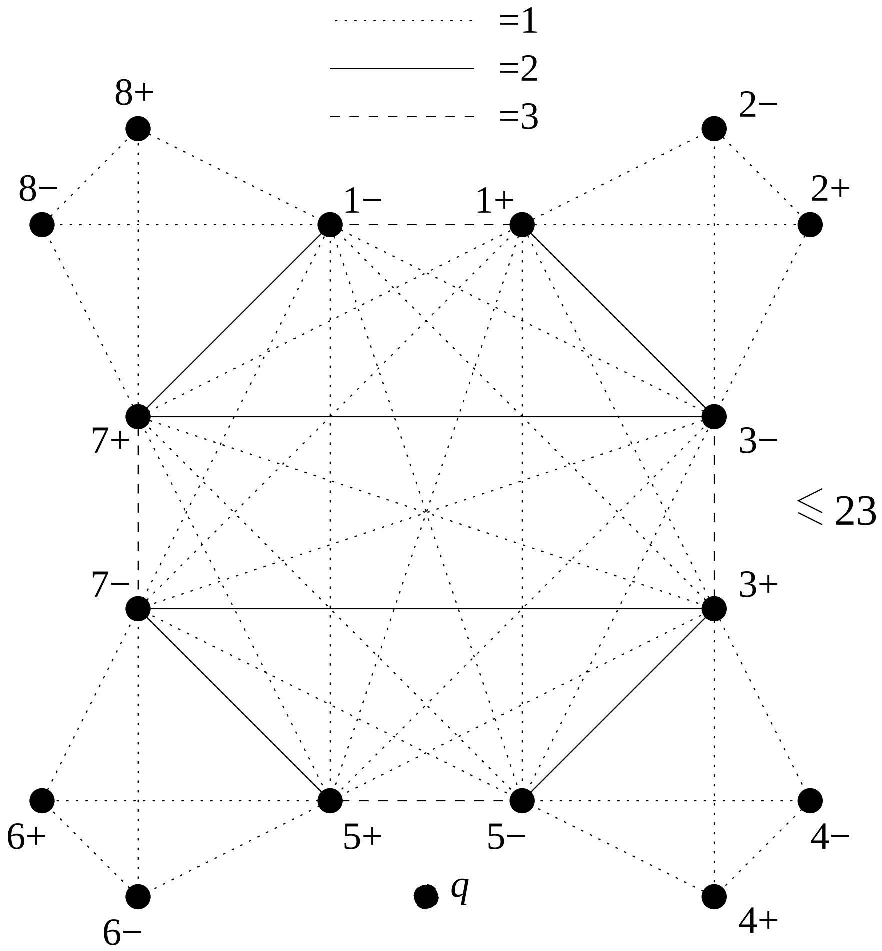


Figure 12: The symCurtain inequality on 16 nodes.

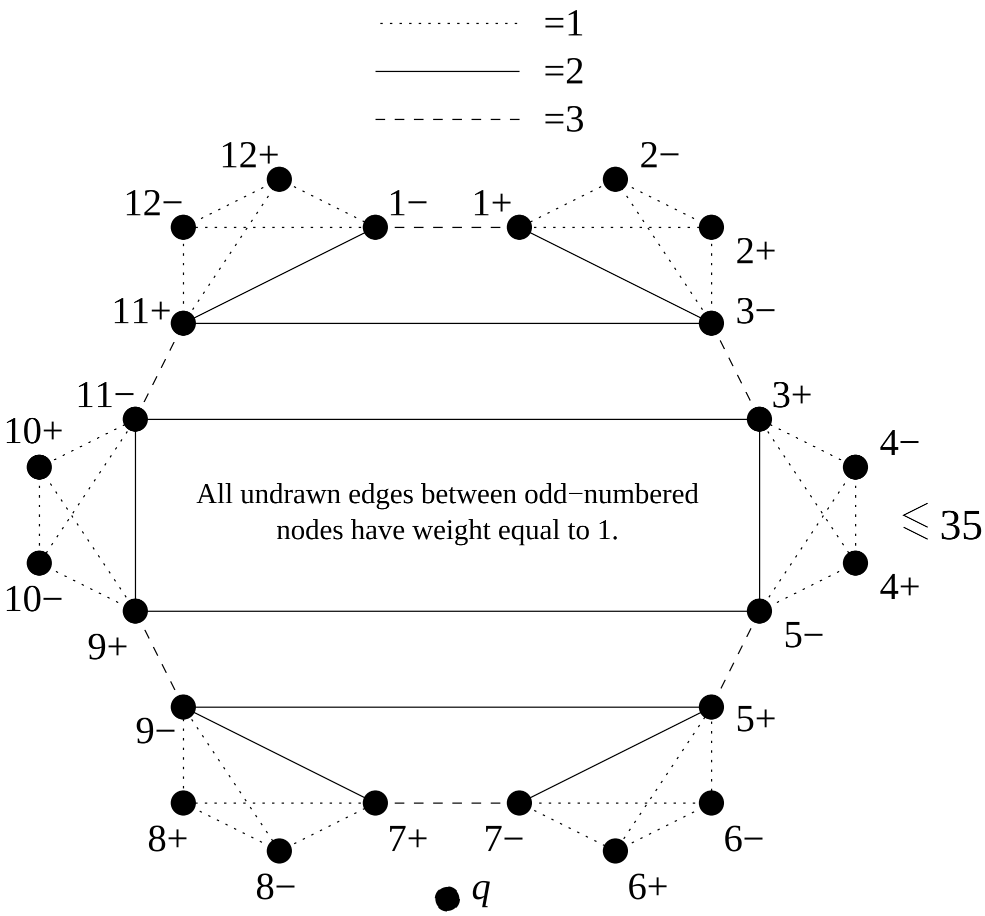


Figure 13: The symCurtain inequality on 24 nodes.

Proof: By contradiction, consider a tour $\hat{x}$ that violates our curtain inequality, i.e.,

$$\overline{a}\hat{x} \geq \overline{a}_0 + 1 =: b_0.$$

Define V_1 to be the set of odd labeled vertices and V_2 to be the set of even labeled vertices. Define V_1^+ to be the subset of V_1 with a superscripted plus. Similarly define V_1^-, V_2^+, and V_2^-. Recall that q is outside the cycle of the curtain inequality. Define the edge sets

$$\begin{array}{rcl} E_0 & = & E(V_2^+) \cup E(V_2^-), \\ E_1 & = & \{\{i^+, (i+1)^+\} \, : \, i \in V_1\} \cup \{\{i^-, (i-1)^-\} \, : \, i \in V_1\}, \\ Q & = & E(V^+) \cup E(V^-) \setminus (E_0 \cup E_1) \cup_{i \in V_1} ((\delta(i^+) \cup \delta(i^-)) \cap \delta(q)). \end{array}$$

Consider a valid STS$(V^+ \cup V^-)$ inequality $bx \leq b_0 := \overline{a}_0 + 1$ derived by adding up the following inequalities:

$$\begin{array}{lcll} x(\delta(i^+)) & \leq & 2, & \\ x(\delta(i^-)) & \leq & 2 & \forall i \in V_1, \\ x_{i^+i^-} & \leq & 1 & \forall i \in V, \\ -x(Q) & \leq & 0. & \end{array}$$

One can verify that $\overline{a} \leq b$. Hence,

$$b_0 \leq \overline{a}\hat{x} \leq b\hat{x} \leq b_0,$$

and so the inequalities in the derivation of $bx \leq b_0$ are all tight at $\hat{x}$.

We will now transform $\hat{x}$ into a tour $\overline{x}$ by a sequence

$$\hat{x} = x^0 \rightarrow x^1 \ldots \rightarrow x^m \ldots \rightarrow x^l = \overline{x}$$

of 2-interchanges, for which

$$\begin{array}{l} \overline{a}\overline{x} = \overline{a}\hat{x}, \\ \overline{x}(E_0 \cup E_1) = 0. \end{array}$$

We first eliminate the m edges of E_1 that are used in the $\hat{x}$ tour. If x^k uses, say, the edges $\{i^+, (i+1)^+\}$ and $\{v, (i+1)^-\}$ ($v \neq (i+1)^+$) for $i \in V_1$, then form x^{k+1} by replacing these two edges with $\{v, (i+1)^+\}$ and $\{i^+, (i+1)^-\}$. A similar operation is performed if x^k uses the edge $\{(i-1)^-, i^-\}$ for $i \in V_1$. We now eliminate the E_0 edges. If x^{m+k} uses an edge $\{i^+, j^+\}$ in E_0, with $j > i$ (or $\{i^+, q\}, \{q, j^+\}$) then there is a corresponding edge $\{i_2^-, j_2^-\}$ in E_0

also used in x^{m+k}. Form x^{m+k+1} by replacing these two edges with the 2 edges in $\delta(V^+) \cap \delta(V^-)$ which keep x^{m+k+1} connected, and inserting q into one of these entering edges if necessary.

Now we satisfy

$$\begin{aligned} \overline{x}(E(V^+) \cup E(V^-)) &= 0, \\ \overline{x}_{i^+i^-} &= 1 \quad \forall i \in V. \end{aligned}$$

Because of our transformation from ATS(V) to STS($V^+ \cup V^-$), and the fact that the curtain inequality is valid for ATS(V), it follows that

$$\overline{ax} \leq \overline{a}_0,$$

a contradiction. □

5.3 Proof that SymCurtains are Facets

Using the method introduced in section 3.3, we can arbitrarily choose to set one coefficient, and show by a tree relationship how the remaining coefficients for the TTS polytope can be assigned (see Figure 14). The tours used to show the relationships in the tree are given in Figure 15. The tree, tours, and the following theorem prove that our class of symCurtain inequalities are facet-defining on the STSP.

Theorem 5 *Inequality (10) is facet-defining for $STS(V^+ \cup V^-)$.*

Proof: From our last theorem, inequality (10) is valid for $STS(V^+ \cup V^-)$. Note that (10) is uniquely determined by the bounds shown in the tree of Figure 14, and (10) is also valid for $TTS(V^+ \cup V^-)$. Thus, when the E^0 edges are ignored, we have that (10) is facet-defining for $TTS(V^+ \cup V^-)$. Since the E^0 coefficients can not be any smaller than the corresponding ATS cloning coefficients, and these values are obtained tightly, the lifting to the polytope $STS(V^+ \cup V^-)$ is maximal, from which our theorem follows. □

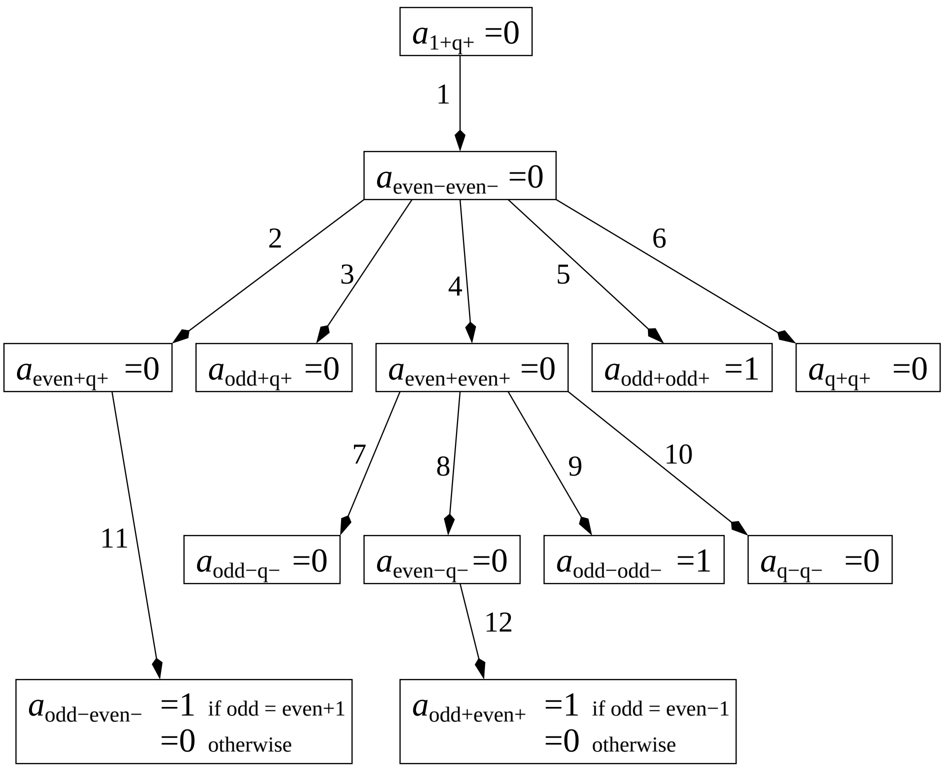


Figure 14: A tree showing the order of lifted coefficients for the symCurtain inequality.

x refers to the pair of nodes x^-, x^+.
e_i = an even node in the cycle. o_i = an odd node in the cycle.
q_i = a node outside the cycle.
ee = all unspecified even nodes in the cycle, in any order.
qq = all unspecified nodes outside the cycle, in any order.

$1 : a_{1^+q_1^+}$ to $a_{e_1^-e_2^-}$ $(e_1, e_2 \neq 2)$
$1, e_1, ee, qq, q_1, e_2, 2, 3, 5, 7, \ldots, n-1, 1$
(if $e_2 = 2$ use tour $1, e_1, ee, qq, q_1, 2, 3, 5, 7, \ldots, n-1, 1$)

$2 : a_{e_1^-e_2^-}$ to $a_{e_3^+q_1^+}$
$e_3, e_1, qq, q_1, e_2, e_2+1, e_2+3, \ldots, e_2+n-1, ee, e_3$

$3 : a_{e_1^-e_2^-}$ to $a_{o_1^+q_1^+}$ $(e_1, e_2 \neq o_1+1)$
$o_1, e_1, qq, q_1, e_2, ee, o_1+1, o_1+2, o_1+4, \ldots, o_1+n-2, o_1$

$4 : a_{e_1^-e_2^-}$ to $a_{e_3^+e_4^+}$
$e_3, e_1, e_4, e_2, e_2+1, e_2+3, \ldots, e_2+n-1, qq, ee, e_3$

$5 : a_{e_1^-e_2^-}$ to $a_{o_1^+o_2^+}$ $(o_1 = e_1-1, o_2 = e_2+1, o_1 \neq o_2)$
$1, 2, 3, \ldots, e_2-1, e_2+1, e_2, qq, e_2+2, e_2+3, \ldots, n, 1$

$6 : a_{e_1^-e_2^-}$ to $a_{q_1^+q_2^+}$
$q_1, e_1, q_2, e_2, e_2+1, e_2+3, \ldots, e_2+n-1, ee, qq, q_1$

$7 : a_{e_1^+e_2^+}$ to $a_{o_1^-q_1^-}$ $(e_1, e_2 \neq o_1-1)$
$e_1, o_1, o_1+2, o_1+4, \ldots, o_1+n-2, o_1-1, e_2, q_1, qq, ee, e_1$

$8 : a_{e_1^+e_2^+}$ to $a_{e_3^-q_1^-}$
$e_1, e_3, e_3+1, e_3+3, \ldots, e_3+n-1, e_2, q_1, qq, ee, e_1$

$9 : a_{e_1^+e_2^+}$ to $a_{o_1^-o_2^-}$ $(o_1 = e_1-1, o_2 = e_2+1)$
$1, 2, 3, \ldots, e_1-2, qq, e_1, o_1, o_1+2, e_1+2, e_1+3, \ldots, n, 1$

$10 : a_{e_1^+e_2^+}$ to $a_{q_1^-q_2^-}$
$e_1, q_1, e_2, q_2, qq, ee, e_1+1, e_1+3, e_1+5, \ldots, e_1+n-1, e_1$

$11a : a_{e_1^+q_1^+}$ to $a_{o_2^-e_2^-}$ $(e_1, e_2 \neq o_2-1)$
$e_1, e_2, ee, qq, q_1, o_2, o_2+2, o_2+4, \ldots, o_2+n-2, o_2-1, e_1$

$11b : a_{e_1^+q_1^+}$ to $a_{o_1^-e_1^-}$ $(e_1 = o_1-1)$
$1, 2, 3, \ldots, e_1-1, qq, q_1, e_1, e_1+1, e_1+2, \ldots, n, 1$

$12a : a_{e_1^-q_1^-}$ to $a_{o_2^+e_2^+}$ $(e_1, e_2 \neq o_2+1)$
$e_2, e_1, ee, o_2+1, o_2+2, o_2+4, \ldots, o_2+n-2, o_2, q_1, qq, e_2$

$12b : a_{e_1^-q_1^-}$ to $a_{o_1^+e_1^+}$ $(e_1 = o_1+1)$
$1, 2, 3, \ldots, e_1, q_1, qq, e_1+1, e_1+2, \ldots, n, 1$

Figure 15: The tours used to show the dependencies in the tree of Figure 14.

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References

[1] Balas, E. and M. Fischetti. "Lifted Cycle Inequalities for the Asymmetric Traveling Salesman Problem." *Mathematics of Operations Research, Vol. 24, No. 2*, May 1999, 273-292.

[2] Balas, E. "The Asymmetric Assignment Problem and Some New Facets of the Traveling Salesman Polytope" *SIAM Journal on Discrete Math, 2*, 1989, 425-451.

[3] Balas, E. and M. Fischetti. "A lifting procedure for the asymmetric traveling salesman polytope and a large new class of facets." *Math. Programming 58*, 1993, 325-352.

[4] Balas, E. and M. Fischetti. "Polyhedral Theory for the Asymmetric Traveling Salesman Problem." *The Traveling Salesman Problem and its Variations* (G. Gutin and A. Punnen ed.s), Kluwer, 117–168, 2002.

[5] Christof, T., M. Jünger and G. Reinelt (1991). A complete description of the traveling salesman polytope on 8 nodes, *Operations Research Letters 10, 497-500.*

[6] Jünger, M.,G. Reinelt and G. Rinaldi. "The Traveling Salesman Problem." In M.O. Ball, T.L. Magnanti, C.L. Monma and G.L. Nemhauser (editors), *Network Models.* North-Holland, 1995, 225-330.

[7] Karp, R., "Reducibility Among Combinatorial Problems." In R.E. Miller and J.W. Thatcher (editors), *Complexity of Computer Computations.* Plenum Press, 1972, 85-103.

[8] Naddef, D. and G. Rinaldi "The Graphical Relaxation: A New Framework for the Symmetric Travelling Salesman Polytope", *Math. Programming 58*, 1993, 53-88.

[9] Padberg, M.W. "On the Facial Structure of Set Packing Polyhedra" *Mathematical Programming 5*, 1973, 199-215.