Mixed-Integer Cuts from Cyclic Groups

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Abstract

We analyze a separation procedure for Mixed-Integer Programs related to the work of Gomory and Johnson on interpolated subadditive functions. This approach has its roots in the Gomory-Johnson characterization on the master cyclic group polyhedron. To our knowledge, the practical benefit that can be obtained by embedding interpolated subadditive cuts in a cutting plane algorithm was not investigated computationally by previous authors. In this paper we compute, for the first time, the lower bound value obtained when adding (implicitly) *all* the interpolated subadditive cuts that can be derived from the individual rows of an optimal LP tableau, thus approximating the optimization over the Gomory's corner polyhedron. The computed bound is compared with that obtained when only Gomory mixed-integer cuts are used, on a very large test-bed of MIP instances.

Key words: Mixed-Integer Programming, Subadditive cuts, Gomory cuts, Gyclic Group and Corner polyhedra.

1 Introduction

In this paper we study the Integer Linear Program (ILP)

$$\min\{c^T x : Ax = b, x \geq 0 \text{ integer}\} \tag{1}$$

where A is a rational $m \times n$ matrix, b is a rational m-dimensional vector, and $c \in \mathcal{R}^n$ is a cost vector, and we address the two associated polyhedra:

$$P := \{x \in \mathcal{R}^n_+ : Ax = b\} \tag{2}$$

$$P_I := conv\{x \in \mathcal{Z}^n_+ : Ax = b\} = conv(P \cap \mathcal{Z}^n) \ . \tag{3}$$

The mixed-integer case where some variables are not restricted to be integer, will be addressed in Section 3.

We propose an exact separation procedure for the class of so-called *interpolated* (or *template*) *subadditive cuts*, based on the characterization of Gomory and Johnson [12, 13, 14] of the *master cyclic group polyhedron* defined as

$$T(k,r) = conv\{t \in \mathcal{Z}_+^{k-1} : \sum_{i=1}^{k-1}(i/k) \cdot t_i \equiv r/k \pmod 1\} \tag{4}$$

where $k \geq 2$ (group order) and $r \in \{1, \cdots, k-1\}$ are given integers. The space $\mathcal{R}^{k-1}$ of the t variables is called the T*-space* in [15]. It is known that the mapping the original x-variable space into the T-space allows one to use polyhedral information on $T(k,r)$ to derive valid inequalities for P_I. To our knowledge, however, the practical benefit that can be obtained by embedding the whole family of cyclic-group cuts in a cutting plane algorithm was not investigated computationally by previous authors. As a matter of fact, a number of recent papers [15, 16, 6, 7, 8, 3] deals only implicitly with cyclic-group separation, as they address the so-called Gomory's *shooting experiment*. Roughly speaking, in this experiment the point $t^* \in \mathcal{R}^{k-1}$ to be separated is generated at random (hence corresponding to a random "shooting direction" in the T-space), and statistics on the frequency of the most-violated facets of $T(k,r)$ are collected. A very recent paper presenting some computational results is the one by Koppe, Louveaux, Weismantel and Wolsey [20], where a compact formulation of the cyclic-group separation problem is embedded into the original ILP model—this however produces a huge extended formulation with limited practical applications. Also related to our work are the papers by Cornuejols, Li and Vandenbussche [5], where a subfamily of cyclic-group cuts (called k-cuts) is investigated both theoretically and computationally, and by Letchford and Lodi [19], where a different subfamily is addressed.

This paper is organized as follows. In Section 2 we present the theory of Gomory and Johnson [12, 13, 14] on interpolated subadditive functions (called *template functions* in [7]) and their role in generating valid inequalities for P_I. We also introduce an exact separation procedure for interpolated subadditive cuts based on an LP model taken from the Gomory-Johnson characterization of the master cyclic group polyhedron. This separation procedure allows us to exploit effectively the whole family of interpolated subadditive cuts to improve the LP relaxation quality. In Section 3 we consider the mixed-integer case, where model (1) becomes a MIP involving continuous variables, whereas in Section 4 we address some implementation issues related to the presence of bounded variables. In Section 5, the quality of the generated cuts is analyzed computationally. In particular we compute, for the first time, the lower bound value obtained when adding (implicitly) *all* the interpolated subadditive cuts that can be derived from the individual rows of an optimal LP tableau. This leads to an approximation[1] of the optimization over of the Gomory's *corner polyhedron* [14], thus giving a partial answer to the question posed in [16] (and also addressed in [20]) on the quality of this relaxation. The bound we compute is compared with that obtained when only Gomory mixed-integer cuts are used, on a very large test-bed of MIP instances [21] library. The outcome is that Gomory mixed-integer cuts play a very special role among subadditive cuts, in that they typically produce, alone, a lower bound increase which is comparable to that obtained when the whole family of cuts is considered. This result confirms the theoretical findings of Dash and Gunluk [7], who showed that interpolated subadditive cuts are dominated by Gomory mixed-integer cuts in a probabilistic sense, as well as the computational experience of Cornuejols, Li and Vandenbussche [5] on the subfamily of k-cuts. Some interesting directions of work are finally addressed in Section 6.

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2 Cuts from Subadditive Functions

The *fractional part* $\phi(a)$ of a real value a is defined as

$$\phi(a) := a - \lfloor a \rfloor \ ,$$

where $\lfloor a \rfloor$ denotes the largest integer not greater than a. Given a positive integer k and two real values $a, b \in \mathcal{R}$, we write $a \equiv b \pmod{k}$ if $a - b$ is an integer multiple of k. In this paper we are interested in deriving valid inequalities for P_I that are not implied by the system $Ax = b$, $x \geq 0$. To this end, given any equation

$$\alpha^T x = \beta \tag{5}$$

valid for P_I, where $(\alpha, \beta) \in \mathcal{Q}^{n+1}$ and $\phi(\beta) > 0$, we consider the group polyhedron

$$G(\alpha, \beta) := conv\{x \in \mathcal{Z}^n_+ : \sum_{j=1}^{n} \alpha_j x_j \equiv \beta \pmod{1}\} \supseteq P_I \ . \tag{6}$$

For example, the equation $\alpha^T x = \beta$ can be obtained by setting $(\alpha, \beta)^T := u^T(A, b)$ for any $u \in \mathcal{Q}^m$ such that $\phi(u^T b) > 0$. This is the case, e.g., when the equation is read from the tableau associated with a fractional optimal solution of the LP relaxation of (1).

In particular, we address the following separation problem:

[1] Besides getting rid of the effects of interpolation, optimizing *exactly* over the corner polyhedron would require to take into account *all* the tableau rows simultaneously.

Definition (g-SEP) *Given any point $x^* \geq 0$ and the equation $\alpha^T x = \beta$ with rational coefficients and such that $\phi(\beta) > 0$, find (if any) a valid inequality for $G(\alpha, \beta)$ that is violated by x^*.*

Notice that, as far as $G(\alpha, \beta)$ is concerned, one can replace each entry in (α, β) by its fractional part, hence one can assume without loss of generality $0 \leq \alpha_j < 1$ for all j, and $0 < \beta < 1$.

A function $g : \mathcal{R} \to \mathcal{R}$ is called subadditive if $g(a + b) \leq g(a) + g(b)$ for any $a, b \in \mathcal{R}$. We call a subadditive function $g(\cdot)$ periodic in $[0, 1)$ if $g(a + 1) = g(a)$ for all $a \in \mathcal{R}$. As in this paper we are only interested in subadditive functions $g(\cdot)$ that are periodic in $[0, 1)$ and such that $g(0) = 0$, in the sequel we will name this kind of functions just *subadditive*.

Given a valid equation $\alpha^T x = \beta$ for P_I, it is easy to show that the inequality

$$\sum_{j=1}^{n} g(\alpha_j) x_j \geq g(\beta) \tag{7}$$

is valid for $G(\alpha, \beta)$ (and hence for P_I) whenever $g(\cdot)$ is subadditive. For example, taking $g(\cdot) = \phi(\cdot)$ one obtains the well-know *Gomory fractional cut* [10]

$$\sum_{j=1}^{n} \phi(\alpha_j) x_j \geq \phi(\beta) \ ,$$

whereas taking the subadditive *GMI function* $\gamma^\beta(\cdot)$ defined as

$$\gamma^\beta(a) = \begin{cases} \phi(a) & \text{if } \phi(a) \leq \phi(\beta) \\ \phi(\beta) \frac{1-\phi(a)}{1-\phi(\beta)} & \text{otherwise} \end{cases} \quad \text{for all } a \in \mathcal{R} \tag{8}$$

one obtains the stronger *Gomory Mixed-Integer* (GMI) cut [11]

$$\sum_{j=1}^{n} \min\{\phi(\alpha_j),\, \phi(\beta) \frac{1 - \phi(\alpha_j)}{1 - \phi(\beta)}\}\, x_j \geq \phi(\beta) \ . \tag{9}$$

see Figure 1 for an illustration.

A basic result, due to Gomory and Johnson [13, 14], is that *all* the nontrivial facets[2] of $G(\alpha, \beta)$ are defined by inequalities of this type.[3] As a consequence, our separation problem (*g-SEP*) can be rephrased as the problem of defining a suitable subadditive function $g(\cdot)$ that produces a cut violated by the given point x^*, i.e., such that $\sum_{j=1}^n g(\alpha_j) x_j^* < g(\beta)$.

[2] A facet if called *nontrivial* if it is not defined by a nonnegativity constraint

[3] In addition, Gomory and Johnson have shown that non-dominated inequalities only arise when the complementarity condition $g(a) + g(b) = g(a + b)(= g(\beta))$ holds whenever $a + b \equiv \beta \pmod 1$.

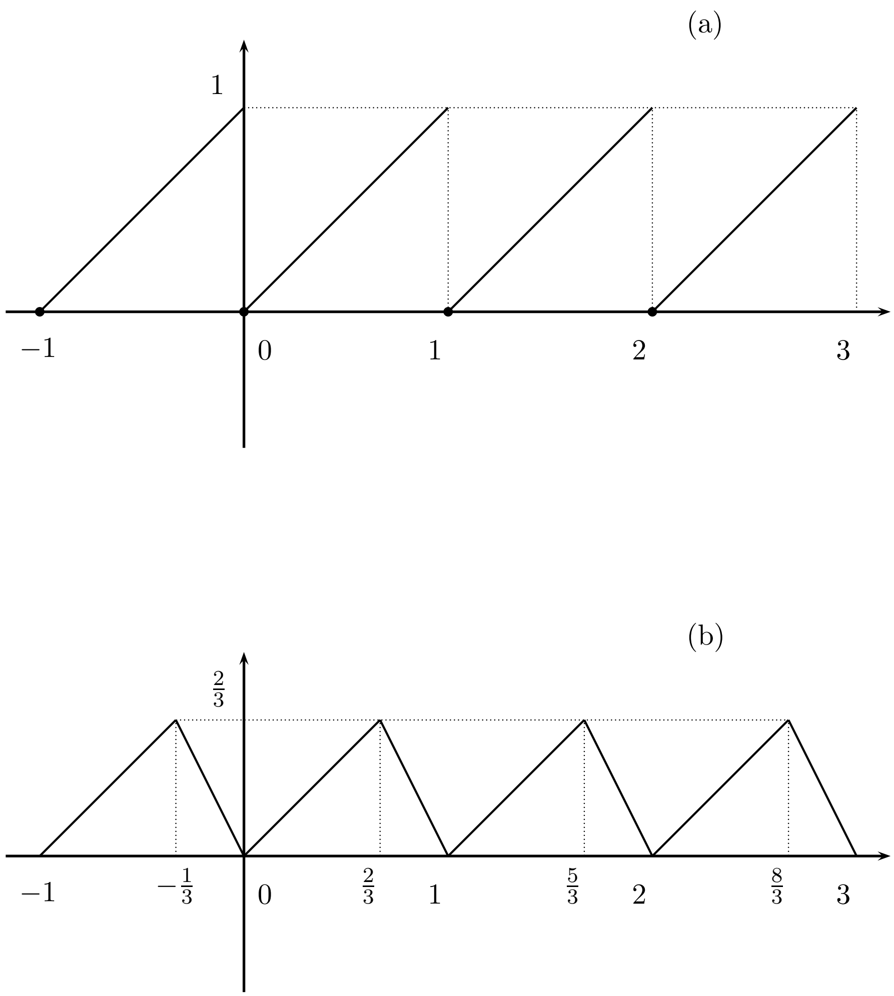

Figure 1: Two subadditive functions: the fractional part $\phi(\cdot)$ (top) and the GMI function $\gamma^{2/3}(\cdot)$ (bottom)

Now let $k \geq 2$ be the smallest integer such that $k(\alpha, \beta)$ is integer, whose existence follows from the assumption that (α, β) is rational. This value of k will be called *ideal* with respect to equation (5). Of course, the subadditivity (plus periodicity) of $g(\cdot)$ implies that the same property holds over the discrete set $\{0, 1/k, 2/k, \cdots, (k-1)/k\}$. In other words, a necessary condition for subadditivity is that the "sampled" values $g_i := g(i/k)$, $i = 0, \cdots, k-1$ satisfy the following set of linear constraints, called *g-system* in the sequel:

$$g_h \leq g_i + g_j\ , \quad 1 \leq i, j, h \leq k-1 \text{ and } i + j \equiv h \pmod{k} \tag{10}$$

$$g_0 = 0 \tag{11}$$

$$0 \leq g_i \leq 1\ , \quad i = 1, \cdots, k-1\ , \tag{12}$$

where bounds (12) will play a normalization role in the sequel.

Any solution $(g_0, \cdots, g_{k-1})$ of the g-system above can be completed so as to define a subadditive function $g : \mathcal{R} \to \mathcal{R}$ through a simple *interpolation procedure* due to Gomory and Johnson [13]. This procedure simply takes a linear interpolation of the values $g_0, \cdots, g_{k-1}$ over $[0, 1)$, and then extends the resulting piecewise-linear function to $\mathcal{R}$, in the obvious periodic way. More formally, for any $a \in \mathcal{R}$ the interpolated value $g(a)$ is defined as $g(a) = (1-\theta) g_i + \theta g_{i+1}$, where $\theta \in [0, 1)$ and $i \in \{0, \cdots, k-1\}$ are such that $\phi(a) = (1-\theta)\, i/k + \theta\, (i+1)/k$, and $g_k := g_0$ because of periodicity.

A key observation at this point is that, being k ideal, the actual value of $g(\cdot)$ outside the sample points i/k is immaterial, since $g(\cdot)$ only needs to be evaluated on these sample points when computing the coefficients in (7). Therefore, the interpolation procedure does not actually restrict the space of the possible subadditive functions—as it would be the case for a different choice of k. As a consequence, we can *exactly* rephrase *g-SEP* as the following LP:

$$g - SEP_k : \quad \min\{ \sum_{i=1}^{k-1} t_i^* g_i : (10) - (12)\ \}\ , \tag{13}$$

where $r := k\, \phi(\beta)$, $t_i^* := \sum_j \{x_j^* : \phi(\alpha_j) = i/k\}$ for $i \in \{1, \cdots, k-1\} \setminus \{r\}$, and $t_r^* := \sum_j \{x_j^* : \phi(\alpha_j) = r/k\} - 1$ so as to take into account the role of the right hand side $g(\beta) = g_r$ in (7). With these definitions, the objective function $\sum_{i=1}^{k-1} t_i^* g_i$ is precisely the opposite of the violation of a cut of the form (7), hence a violated such cut exists if and only if the optimal value of $g - SEP_k$ is strictly negative.

Unfortunately, the ideal k is very often too large to be used in practice, so one has to choose a smaller value in order to produce a manageable g-system. In this case, the interpolation procedure does restrict (often considerably) the range of subadditive functions that can be captured by $g - SEP_k$. Moreover, for a non-ideal k the definition of the weights t_i^* becomes slightly more involved, due to the need of taking interpolation into account. More specifically, for any integer $k \geq 2$ (not necessarily ideal) the weights t_i^* in (13) are

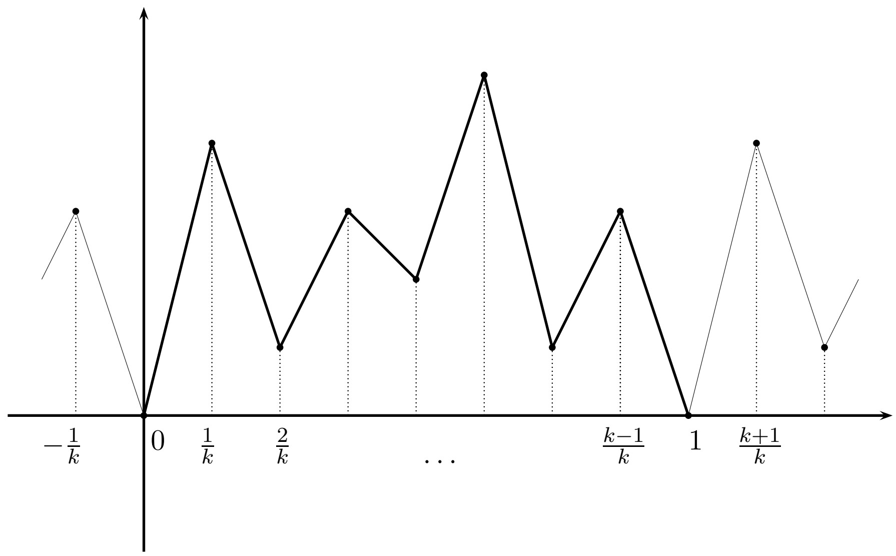

Figure 2: The Gomory-Johnson interpolation procedure: the sample values $g(0)$, $g(1/k)\cdots, g((k-1)/k)$ are connected by straight lines so to get the subadditive function $g(\cdot)$ over $[0,1)$, which is then extended periodically over $\mathcal{R}$.

defined through the algorithm outlined in Figure 3, that works as follows. At step 1, we define two fictitious values α_0 and x_0^* so as to re-write the (opposite of the) cut violation $\sum_{j=1}^n g(\alpha_j)x_j^* - g(\beta)$ in the more convenient form $\sum_{j=0}^n g(\alpha_j)x_j^*$. At step 2, all weights t_i^* are initially set to zero. At step 3, for each $j = 0, \cdots, n$ we locate the interval $[i/k, (i+1)/k)$ that contains $\phi(\alpha_j)$, where $i+1$ is replaced by 0 in case $i+1 = k$ so as to take periodicity into account. At step 4 we define a "displacement" parameter $\theta \in [0,1)$ giving the exact position of $\phi(\alpha_j)$ within this interval; by definition, we have $\theta = 0$ if $\phi(\alpha_j) = i/k$, whereas θ approaches its limit 1 as $\phi(\alpha_j)$ approaches $(i+1)/k$. We then split, at step 5, the contribution $g(\alpha_j)\,x_j^*$ between t_i^* and t_{i+1}^*, in a way proportional to θ. Note that the procedure also works in case of ideal k, where we always have $\theta = 0$ at step 4.

We finally observe that, for the interpolated function $g(\cdot)$, we sometimes have $g(a) > g(\beta)$ as, e.g., in the case illustrated in Figure 4. In this figure we compare a GMI function and the corresponding interpolated version for $k = 3$ (top), and scale them so as to get the same right-hand-side value (bottom) so as to make clear the fact that the interpolated function produces a dominated inequality. Therefore, an interpolated subadditive cut

define the fictitious values $\alpha_0 := \beta$ and $x_0^* := -1$;
initialize $t_0^* := t_1^* := \cdots := t_{k-1}^* := 0$;
for $j = 0, 1, \cdots, n$ such that $x_j^* > 0$ and $\phi(\alpha_j) > 0$ do
   let $i := \lfloor k\,\phi(\alpha_j) \rfloor$ and $h = i + 1 \bmod k$;
   let $\theta := k\phi(\alpha_j) - i$;
   update $t_i^* := t_i^* + (1-\theta)x_j^*$ and $t_h^* := t_h^* + \theta x_j^*$
enddo

Figure 3: Defining the weights t_i^* in $g - SEP_k$ for any given k and x^*

$\sum_{j=1}^n g(\alpha_j)x_j \ge g(\beta)$ can easily be improved to its *clipped* form:

$$\sum_{j=1}^{n} \min\{g(\alpha_j), g(\beta)\} x_j \ge g(\beta) \tag{14}$$

whose validity follows trivially from the integrality of x.

3 Dealing with Continuous Variables

We next address the case where some variables x_j with $j \in \mathcal{C}$ (say) are not restricted to be integer valued. In this case, Gomory and Johnson [13, 14] showed that, for any subadditive function $g(\cdot)$, it is enough to modify cut (7) into

$$\sum_{j \in \mathcal{I}}^{n} g(\alpha_j)x_j + \sum_{j \in \mathcal{C}: \alpha_j > 0} slope_+\, \alpha_j x_j + \sum_{j \in \mathcal{C}: \alpha_j < 0} slope_-\, \alpha_j x_j \ge g(\beta)\ , \tag{15}$$

where $\mathcal{I} := \{1, \cdots, n\} \setminus \mathcal{C}$ is the index set of the integer-valued variables,

$$slope_+ := \lim_{\delta \to 0^+} g(\delta)/\delta$$

is the slope of $g(\cdot)$ in 0^+, and

$$slope_- := \lim_{\delta \to 0^-} g(\delta)/\delta \ = - \lim_{\delta \to 0^+} g(1-\delta)/\delta$$

is the slope of $g(\cdot)$ in 0^- (or, equivalently, in 1^-). Notice that, by definition, $slope_+ > 0$ and $slope_- < 0$, hence all coefficients in (15) are nonnegative.

The above result has an intuitive explanation based on the following simple scaling argument. Let $j \in \mathcal{C}$ be the index of any continuous variable. We introduce a scaled copy $\tilde{x}_j = Mx_j$ of x_j, where $M > 0$ is a suitable scaling factor, and impose that $\tilde{x}_j$ can only assume integer values. This is of course correct only if M is chosen so as not to cut any feasible point of the

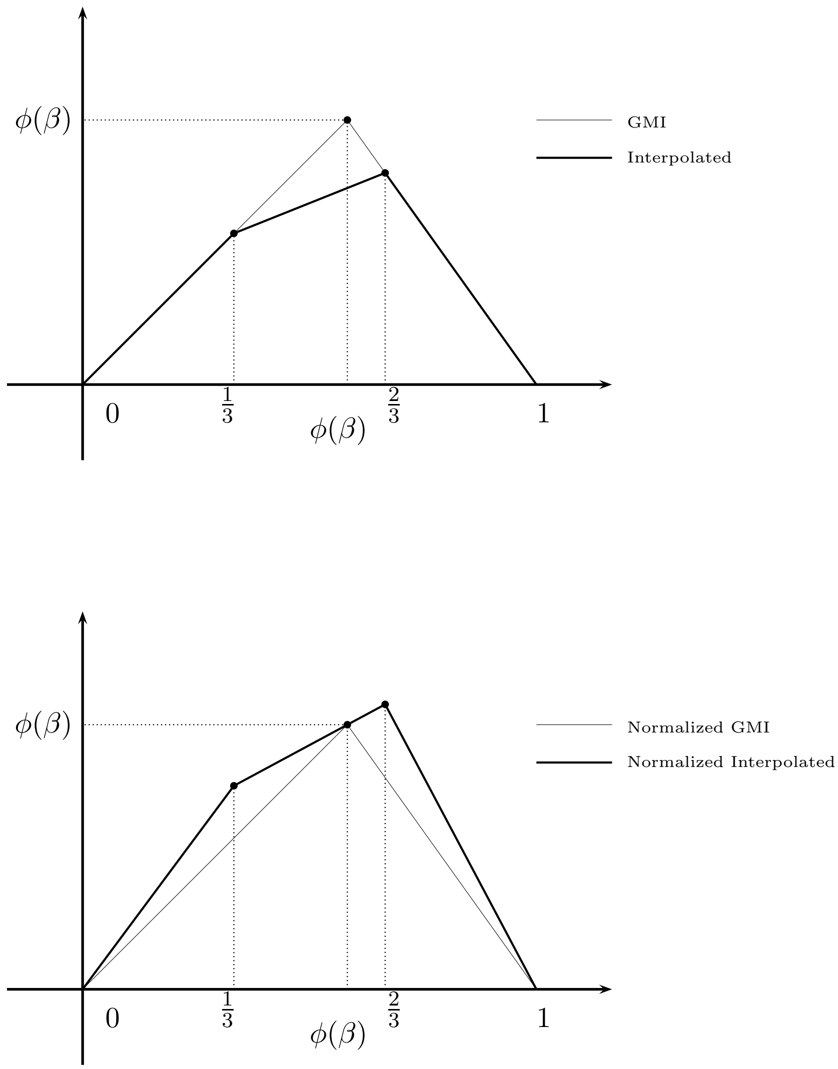

Figure 4: GMI and interpolated GMI functions

original MIP set, which is always possible due the rational data assumption. Notice that M can be assumed to be arbitrarily large, since multiplying a valid M by a positive integer yields another valid M. Now, replacing x_j by $\tilde{x}_j/M$ changes the j-th coefficient in equation $\alpha^T x = \beta$ from α_j to $\tilde{\alpha}_j := \alpha_j/M$, while increasing the j-th component of x^* from x_j^* to $\tilde{x}_j^* := Mx_j^*$. Being $\tilde{x}_j$ constrained to be integer, we can compute its coefficient in (7) as $g(\tilde{\alpha}_j) = g(\alpha_j/M)$. If $\alpha_j > 0$, for $M \to +\infty$ we have that α_j/M tends to 0^+, hence for sufficiently large M we have $g(\tilde{\alpha}_j) = g(\alpha_j/M) = \mathit{slope}_+ \alpha_j/M$. Analogously, if $\alpha_j < 0$ we have $\alpha_j/M \to 0^-$ when $M \to +\infty$, hence for sufficiently large M we have $g(\tilde{\alpha}_j) = g(\alpha_j/M) = \mathit{slope}_- \alpha_j/M$. This shows that the coefficient of the scaled variable $\tilde{x}_j$ in the subadditive cut is $\mathit{slope}_* \, \alpha_j/M$, where $\mathit{slope}_* = \mathit{slope}_-$ or slope_- depending on the sign of α_j. The back substitution $x_j = \tilde{x}_j/M$ then yields the coefficient $\mathit{slope}_* \, \alpha_j$ for the original (continuous) variable x_j, as in (15).

As a consequence of the above scaling argument, we can deal with continuous variables without any modification of our separation procedure, that can used as a black box. To this end, it is enough to implement a pre-processing scaling phase for the continuous variables, and then a post-processing phase where the separated cut returned by the black box is expressed in terms of the original (non-scaled) variables. In case of interpolated subadditive functions, a suitable scaling factor for each continuous variable x_j is $M_j := k|\alpha_j|$, that maps the original coefficient α_j into $\tilde{\alpha}_j = \alpha_j/M_j = \pm 1/k$.

Alternatively, one can modify slightly the separation procedure of Figure 3 so as to take into account continuous variables in an explicit way. To this end, we observe that for interpolated subadditive functions with interpolation values $g_i := g(i/k)$ for $i = 0, \cdots, k-1$ and $g_0 = 0$, one has $\mathit{slope}_+ = g_1/(1/k)$ and $\mathit{slope}_- = -g_{k-1}/(1/k)$. Hence, in the definition of the weights t_i^* used in the separation problem (13) the value x_j^* of a continuous variable x_j ($j \in \mathcal{C}$) contributes to t_1^* or t_{k-1}^*, depending on the sign of α_j. To be more specific, for each $j \in \mathcal{C}$, $j \neq 0$, one has to skip steps 3-4 in Figure 3, and update $t_1^* := t_1^* + k\,|\alpha_j|\,x_j^*$ in case $\alpha_j > 0$, and $t_{k-1}^* := t_{k-1}^* + k\,|\alpha_j|\,x_j^*$ otherwise.

The above considerations show that, in presence of several continuous variables with nonzero $\alpha_j x_j^*$, values g_1 and g_{k-1} play a crucial role in the separation, the lower these values the better. Hence GMI cuts qualify as the strongest subadditive cuts when continuous variables are present, since they have the property of being associated with a subadditive function $\gamma^\beta(\cdot)$ where slope_+ and slope_- are as small as possible; see [14].

Finally, we observe that the clipping of coefficient $g(\alpha_j)$ to $\min\{g(\alpha_j), g(\beta)\}$ in (14) is guaranteed to be valid only for integer-constrained variables x_j, so it cannot be applied for $j \in \mathcal{C}$.

4 Dealing with bounded variables

It is often the case that the original MIP model involves variable bounds of the form $x_j \leq UB_j$ for some variables x_j. Commercial LP solvers typically deal with variable upper bounds in an implicit way, hence the tableau rows are not exactly the same as in the textbook theory. (E.g., due to the presence of nonbasic variables at their upper bound, a basic fractional variable is sometimes associated with a tableau row whose right-hand-side value is integer.) We next outline a possible way to handle bounded variables.

Given any equation $\alpha^T x = \beta$ we address the possibility of complementing some bounded variables x_j (i.e., replacing x_j by $UB_j - x_j$) before actually invoking the separator. Of course, an optimal choice of the variables to be complemented can improve the chances of finding a violated cut, but this choice does not appear easy.

To illustrate this point, let us consider a pure 0-1 problem where all variables are bounded by 1, and let $\sum_{j=1}^{n} \alpha_j x_j = \beta$ by any "knaspack" equation valid for P_I. Without loss of generality one can assume $0 \leq \alpha_j \leq \beta \leq 1$ for all j. A set $Q \subseteq \{1, \cdots, n\}$ with $\sum_{j \in Q} \alpha_j > \beta$ is called a *cover*. Given any minimal (with respect to set inclusion) cover Q, we complement the corresponding variables by replacing x_j by $1 - \overline{x}_j$, and obtain

$$\sum_{j \in Q} \alpha_j \overline{x}_j + \sum_{j \notin Q} (-\alpha_j) x_j = \sum_{j \in Q} \alpha_j - \beta =: \theta > 0 \tag{16}$$

where $\theta < 1$ and $\alpha_j > 0$ for all $j \in Q$ because of the minimality of the cover. Now take the subadditive function $g(a) := \lceil \phi(a) \rceil - \phi(a)$, and observe that $0 < \alpha_j < 1$ implies $g(\alpha_j) = 1 - \alpha_j$ and $g(-\alpha_j) = g(1 - \alpha_j) = 1 - \phi(1 - \alpha_j) = 1 - (1 - \alpha_j) = \alpha_j$. Applying $g(\cdot)$ to (16) we then obtain the cut

$$\sum_{j \in Q} (1 - \alpha_j) \overline{x}_j + \sum_{j \notin Q} \alpha_j x_j \geq g(\theta) = 1 - \theta, \tag{17}$$

and adding together (16) and (17) we finally get the so-called *cover inequality* [22]

$$\sum_{j \in Q} \overline{x}_j = \sum_{j \in Q} (1 - x_j) \geq 1 \tag{18}$$

Note that one can have exponentially many different cover inequalities, each associated with a different subset Q of complemented variables.

The above example suggests that there is no easy way to determine the best set of variables to be complemented, just as there is no easy way to locate the set Q that produces a most-violated cover inequality when dealing with knapsack constraints. Hence some heuristics have to be applied. A natural choice (also used by other authors) is to complement only the variables that assume the LP status "nonbasic at its upper bound" in the corresponding optimal solution. This guarantees that the equations inserted in our pool have the familiar "textbook form" where all nonbasic variables x_j have value $x_j^* = 0$ with respect to the basic LP solution x^* of the same tableau.

5 Computational Results

We next report a computational analysis aimed at comparing the quality of Gomory mixed-integer cuts with that of the interpolated sudadditive cuts, when embedded in a pure cutting plane method. We also report a comparison with an important class of (non-interpolated) subadditive cuts, namely, the k-cuts described in [5].

Our test-bed includes all MIPLIB 3.0 and 2003 instances taken form [21], except those with unknown optimal solution or having some variables with negative lower bound (we also excluded from our test-bed some very large instances, namely, all those having an LP file larger than than 1.7 MB). In addition, we addressed the hard ILP instances available at the Alper Atamtürk's home page [2], associated with multiple-knapsack problems involving both binary and general-integer (either bounded or unbounded) variables. Finally, our test-bed includes a set of random (both bounded and binary) single-knapsack problems generated as in [5], namely:

$$\begin{array}{l} \min \sum_{i=1}^{n} p_i x_i \\ \text{s.t.} \sum_{i=1}^{n} w_i x_i \le c \\ 0 \le x_i \le b_i \text{ and integer for all } i = 1 \dots n \end{array} \tag{19}$$

with p_i and w_i uniformly random integers in $[1, 1000]$, b_i uniformly random integers in $[5, 10]$ and $c = \lfloor 0.5 \sum_{i=1}^{n} w_i b_i \rfloor$. Binary knapsack problems were generated in the same way, by setting $b_i = 1$ for all i.

For the problems involving "$\le$" or "$\ge$" constraints we built an equivalent formulation in standard form, that includes slack variables in an explicit way. The bounds on the variables, instead, were dealt with in an implicit way, as outlined in Section 4.

All LP's were solved through the commercial software ILOG-Cplex 9.0 [17, 18]. Computing times are expressed in CPU seconds and refer to a notebook with a 512MB RAM and a 1.6Mhz AMD Processor.

Our order of business was to approximate the optimization over the Gomory's corner polyhedron associated with the optimal solution of the LP relaxation of our MIP model (without any MIP preprocessing). To this end, after the solution of the first LP relaxation of our model, we stored in an equation pool all the tableau rows $\alpha^T x = \beta$ with fractional right-hand side β, along with the list of the variables that are at their upper bound in the optimal LP solution (these variables were always complemented before invoking our separation procedures, no matter their value in the current point

x^* to be separated). Neither the pool nor the list of complemented variables was updated during the run, i.e., we deliberately avoided generating subadditive cuts of rank greater than 1 (or, to be more precise, we avoid cuts derived from equations different from those associated with the single rows of the first LP tableau). The same applies to Gomory mixed-integer cuts, that were derived from the equations in the pool and added (at once) to the LP. At each round of separation, at most 200 cuts were generated. Each run was aborted at the root node, i.e., no branching was allowed.

The outcome of our experiments on the MIPLIB istances is shown in Tables 1-4 (mixed-integer problems) and 5-6 (pure-integer problems). In these tables, the first numerical value under the name of the problem gives the optimal value, the second the optimal value of the LP relaxation, and the third the computing time (in CPU seconds) needed to solve the LP relaxation. Atamtürk's instances are instead addressed in Tables 7 and 8. Column "Type of Cuts" gives the type of cuts generated: Gomory mixed-integer cuts (GMI), k-cuts [5] for all $k = 1, 2, \cdots, 50$ (1:50-cuts), and interpolated subadditive cuts when fixing $k = 10, 20, 30, 60$. A star indicates an improved bound with respect to GMI. All other columns are self-explanatory. As to knapsack problems (KP), for both the binary and the bounded cases we generated 7 sets of problems, each set consisting of 30 instances with n variables ($n = 10, 50, 100, 500, 1000, 5000, 10000$).[4] The corresponding average results are reported in Table 9.

According to the tables, interpolated subadditive cuts do improve the quality of the LP relaxation, but for MIPLIB instances they seldom beat GMI cuts. This negative result confirms the theoretical findings of Dash and Gunluk [7], who showed that interpolated subadditive cuts are dominated by Gomory mixed-integer cuts in a probabilistic sense, as well as the computational experience reported by Cornuejols, Li and Vandenbussche [5] for the subfamily of k-cuts. On the other hand, for the hard Atamtürk's multiple-knapsack instances the interpolated cuts allow for a considerable improvement over the GMI bound. As expected, larger values of k produce better bounds, but the separation procedure becomes computationally quite expensive for $k > 20$.

For the random single-knapsack problems, the results show that a few interpolated subadditive cuts are able to improve the GMI bound considerably. It should be observed however that, for KP instances, the integrality gap is extremely narrow, so even a very small improvement of the lower bound produces a significant difference in the percentage of gap closed. Comparison with 1:50-cuts shows clearly the negative effect of the interpolation—without interpolation, the family of subadditive cuts would contain all k-cuts, hence

[4]These problems are very sensitive to the *mipgap* tolerance parameter used by ILOG CPLEX: in these experiments we used the value 10^{-9} for this parameter.

our procedure would guarantee a bound never worse than the 1:50-cut one. This is confirmed by the additional results reported in Table 10, where we addressed KP instances generated as explained above, but with p_i and w_i uniformly random integers in $[1, 100]$ instead of $[1, 1000]$ (this guarantees that the ideal k is not larger than 100, hence it can be handled effectively by our separation procedure; see row "ideal k" in the table). As expected, working with the ideal k produces significantly better results–at least, for our KP instances. A plot of the percentage of closed gap vs. the group-order k is given in Figure 5 for a sample binary KP instance with 50 variables; note that, as expected, the bound growth is not monotone (though for other instances the curve is much more regular, with a saturation starting well before the ideal k).

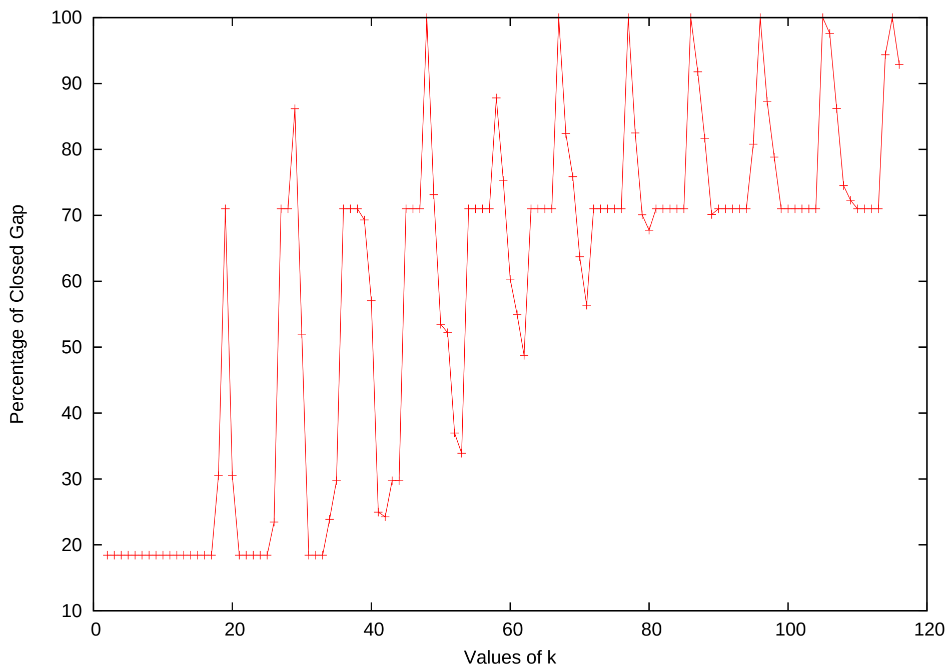

Figure 5: Percentage of closed gap vs. the group-order k for a sample binary KP instance with 50 variables; the ideal k is 77 in this example

It is unclear at this point whether working with a large k so as to get rid of the interpolation effect, is likely to produce a significant improvement over GMI cuts for a wide class of MIPs. We conjecture that, for problems with a large number of constraints as those in the MIPLIB, the considerable number of GMI cuts read from the optimal tableau is likely to bring the fractional point inside all the group polyhedra associated with the single tableau rows—hence the new fractional point cannot be cut anymore by any subadditive cut (interpolated or not), meaning that the root-node lower bound cannot be improved any further by our approach. If this explanation is true, then the research on improved bounds based on mod-1 considerations should concentrate on finding row combinations different from those in the optimal tableau (a topic investigated in a recent paper by Fischetti and Lodi [9]), or has to take into account two or more tableau rows at a same time so as to better approximate the corner polyhedron.

6 Future directions of work

Future work on cyclic-group cuts should investigate the following issues.

Integration within a branch-and-cut method

As already mentioned, an explanation of the good performance of GMI cuts in that these cuts are often able to bring the fractional point inside all the group polyhedra associated with the single tableau rows. However, at a later time, the cutting plane (and branching) process is likely to introduce other constraints that may bring the fractional point outside these group polyhedra, thus triggering our separation procedure to produce new violated subadditive cuts to "move back" the fractional point inside the same group polyhedra–thus hopefully improving the lower bound.

Moreover, the large number of subadditive cuts generated and the small improvement obtained in some cases, would suggest a more conservative policy to better exploit subadditive cuts within a branch-and-cut solution scheme. To be more specific, we believe that a better compromise between lower bound quality and computing time could be reached if one uses first a clever set of non-interpolated subadditive functions to derive quickly an initial set of violated inequalities (including GMI and k-cuts), and applies *g-SEP* separation only afterwards. This goes into the direction suggested by Andreello, Caprara and Fischetti [1] for an effective use of easy-to-compute cuts such as GMI and k-cuts.

Using higher-rank subadditive cuts

In our computational experiments we deliberately avoided exploiting the rows of the optimal tableaux obtained after the addition of new cuts, as we were only interested in the rank-1 corner polyhedron associated with the first tableau. In practice, however, one could derive subadditive cuts from the equations of *any* tableau, just as one can generate several rounds GMI cuts. Practical experience shows that the quality of GMI cuts tends to deteriorate rapidly as new cuts are added, hence one typically avoids their generation after a while. It would be interesting to investigate whether subadditive cuts are also affected by a similar tailing-off phenomenon, the hope being that choosing a cut in the whole family of sudadditive functions (rather than choosing only the GMI one) can lead to some improvement.

Replacing a single GMI cut by two subadditive cuts

In an early stage of our study, we conjectured the effectiveness of GMI cut be due to the fact that it is the deepest[5] one in the family of subadditive cuts, with respect to the fractional LP solution x^* associated with the initial optimal tableau (the one whose rows are stored in our equation pool). Indeed, it is known [4] (and geometrically intuitive) that deep cuts are likely to be the most effective to be used in cutting plane algorithms.

Being the amount of violation for the LP solution x^* with respect to any subadditive cut read from the associated tableau a constant, the cut depth actually depends only on the coefficient norm $\|g\|^2$, the smaller the norm the deeper the cut.

In order to avoid any dependency on the actual value of the coefficients in the tableau rows, we decided to work on the T-space, and computed the deepest cut by solving the quadratic problem $\min\{\sum_{i=1}^{k-1} g_i^2 : (10) - (12)\}$. Surprisingly, we found that the GMI cut (tough quite deep) is not the deepest one, the latter arising when setting $g_i = 0$ for $i = 0$, $g_i = 1$ for $i = r$, and $g_i = 0.5$ otherwise. This produces the cut $\sum_{i=1}^{k-1} t_i + t_r \geq 2$ saying that any integer $t \in T(k,r)$ has to satisfy the disjunction $(t_r \geq 1) \bigvee (\sum_{i \neq r} t_i \geq 2)$. The counterpart of the above cut in the x-space reads

$$\sum_{j:\phi(\alpha_j)>0} x_j + \sum_{j:|\phi(\alpha_j)-\phi(\beta)|\leq\epsilon} x_j \geq 2 \tag{20}$$

where ϵ is a very small positive value.

According to our computational experience, however, cut (20) has a poor practical performance. As a matter of fact, its associated subadditive function does not correspond to an extreme point of the g-system, hence the corresponding cut does not define a facet of $T(k,r)$. This implies that we can obtain two (and possibly more) facet-defining inequalities for $T(k,r)$ whose convex combination gives the deepest cut (20), a situation illustrated in Figure 6. E.g., for $T(10,5)$ the two extreme solutions $(g_0, \cdots, g_9)$ of the g-model are given in the rows of the following matrix:

$$\begin{pmatrix} 0, & 1, & 0, & 1, & 0, & 1, & 0, & 1, & 0, & 1 \\ 0, & 1/3, & 2/3, & 1/3, & 2/3, & 1, & 2/3, & 1/3, & 2/3, & 1/3 \end{pmatrix}$$

whose combination with weights 1/4 (first row) and 3/4 (second row) produce precisely the deepest-cut function, namely:

$$\begin{pmatrix} 0, & 1/2, & 1/2, & 1/2, & 1/2, & 1, & 1/2, & 1/2, & 1/2, & 1/2 \end{pmatrix}$$

It is then natural to investigate the possibility of using the above *pair* of subadditive cuts instead of (or together with) the usual GMI one, in the

[5]The *depth* of a cut $g^T x \geq g_0$ with respect to a point $\bar{x}$ is computed as $|g^T \bar{x} - g_0|/\|g\|^2$

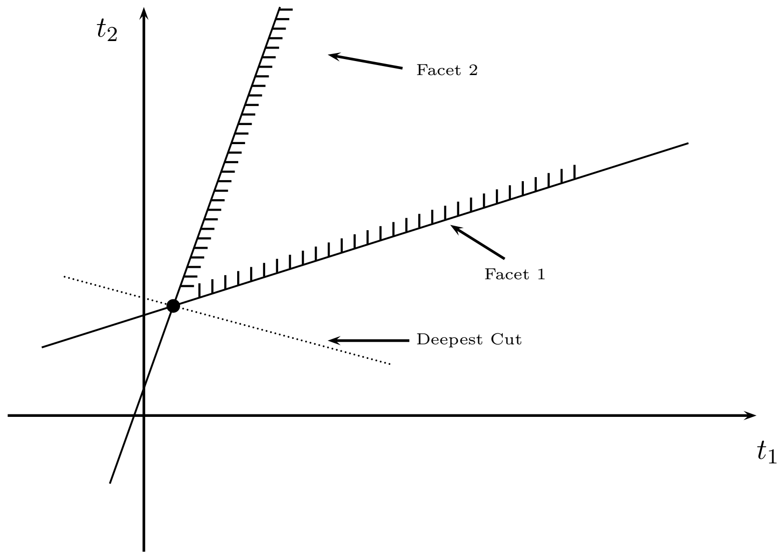

Figure 6: The deepest cut (w.r.t. to the origin) is not a facet of $T(k, r)$, hence it can expressed as the sum of two (or more) such facets

hope that they can "cut from different angles" the factional vertex, hence producing an improved performance without the overhead involved in the solution of the cyclic-group separation problem.

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Table 1: MIPLIB mixed integer problems (part 1).

Problem	Type of Cuts	Final LB	Closed Gap (%)	Separation Time (seconds)	Total Time (seconds)	Number of Cuts
	1:50-cuts	924.00	100.00	10.04	14.85	275
10teams	GMI	924.00	100.00	0.20	2.95	162
924.00	K=10	924.00	100.00	0.33	2.73	162
917.00	K=20	924.00	100.00	0.53	3.04	162
0.16	K=30	924.00	100.00	1.06	3.69	162
	K=60	924.00	100.00	12.07	14.66	162
	1:50-cuts	1002.66	11.15	0.96	1.48	500
aflow30a	GMI	1002.66	11.15	0.01	0.05	31
1158.00	K=10	1001.55	10.51	0.07	0.17	56
983.17	K=20	1002.21	10.89	0.22	0.38	64
0.01	K=30	1002.35	10.97	0.83	0.99	72
	K=60	1002.54	11.08	12.30	12.45	72
	1:50-cuts⋆	1014.56	5.48	3.91	5.75	379
aflow40b	GMI	1014.56	5.48	0.06	0.39	38
1168.00	K=10⋆	1014.27	5.30	0.19	0.78	57
1005.66	K=20⋆	1014.46	5.42	0.43	1.29	64
0.11	K=30⋆	1014.50	5.44	1.05	1.84	63
	K=60⋆	1014.51	5.45	17.81	18.60	66
	1:50-cuts	872157.51	60.43	0.32	0.42	600
bell3a	GMI	872157.51	60.43	0.00	0.00	50
878430.32	K=10	872015.80	59.53	0.04	0.04	83
862578.64	K=20	872092.20	60.02	0.22	0.23	93
0.00	K=30	872099.44	60.06	0.59	0.62	96
	K=60	872140.08	60.32	11.27	11.29	93
	1:50-cuts	8660422.46	14.53	0.21	0.27	464
bell5	GMI	8660422.46	14.53	0.01	0.01	40
8966406.49	K=10	8654669.96	12.92	0.03	0.04	59
8608417.95	K=20	8657274.83	13.65	0.13	0.14	60
0.00	K=30	8658662.54	14.04	0.40	0.41	59
	K=60⋆	8661152.43	14.73	12.73	12.76	77
	1:50-cuts	6.92	0.00	0.06	0.06	0
blend2	GMI	6.92	0.00	0.00	0.00	0
7.60	K=10	6.92	0.00	0.01	0.01	0
6.92	K=20	6.92	0.00	0.01	0.01	0
0.01	K=30	6.92	0.00	0.02	0.02	0
	K=60	6.92	0.00	0.88	0.88	0
	1:50-cuts	62.69	1.74	5.52	8.17	224
danoint	GMI	62.69	1.74	0.04	0.67	52
65.67	K=10	62.65	0.39	0.25	1.16	74
62.64	K=20	62.65	0.55	1.00	2.01	81
0.21	K=30	62.66	0.68	3.55	4.55	80
	K=60	62.67	0.96	71.12	72.26	82
	1:50-cuts⋆	291699.72	54.28	2.34	2.79	646
fiber	GMI	279391.25	49.35	0.03	0.05	46
405935.18	K=10⋆	268704.27	45.08	0.12	0.19	75
156082.52	K=20⋆	277434.30	48.57	0.39	0.49	103
0.01	K=30⋆	282035.42	50.41	1.43	1.58	133
	K=60⋆	285955.48	51.98	33.19	33.29	119
	1:50-cuts	1171213.72	11.74	0.00	0.03	204
flugpl	GMI	1171213.72	11.74	0.00	0.00	10
1201500.00	K=10	1171004.98	11.13	0.01	0.01	19
1167185.73	K=20	1171142.46	11.53	0.04	0.04	20
0.00	K=30	1171193.55	11.68	0.14	0.14	18
	K=60	1171197.64	11.69	3.85	3.86	19
	1:50-cuts	112239.39	59.77	3.04	3.77	800
gen	GMI	112239.39	59.77	0.03	0.14	45
112313.00	K=10	112216.67	47.35	0.17	0.33	71
112130.04	K=20	112222.13	50.34	0.50	0.68	84
0.05	K=30	112228.97	54.07	2.96	3.21	76
	K=60	112239.37	59.75	55.87	56.20	94
	1:50-cuts	25568014.40	30.17	21.37	24.11	1816
gesa2	GMI	25568014.40	30.17	0.18	0.35	91
25779856.37	K=10	25565828.17	29.45	0.56	0.86	178
25476489.68	K=20	25566860.50	29.79	1.63	2.04	206
0.09	K=30	25567378.76	29.96	5.32	5.83	206
	K=60	25567569.09	30.02	100.04	100.60	204
	1:50-cuts⋆	25568387.00	30.29	24.88	27.32	2041
gesa2_o	GMI	25567400.54	29.97	0.26	0.41	121
25779856.37	K=10⋆	25565930.89	29.48	0.74	1.04	218
25476489.68	K=20⋆	25567122.07	29.88	1.72	2.09	251
0.05	K=30⋆	25567712.56	30.07	5.50	5.94	243
	K=60⋆	25568060.92	30.19	88.02	88.53	258

Table 2: MIPLIB mixed integer problems (part 2).

Problem	Type of Cuts	Final LB	Closed Gap (%)	Separation Time (seconds)	Total Time (seconds)	Number of Cuts
	1:50-cuts⋆	19092.03	73.09	0.11	0.18	410
gt2	GMI	18998.90	71.88	0.00	0.00	12
21166.00	K=10⋆	15241.29	23.11	0.01	0.01	16
13460.23	K=20⋆	15241.29	23.11	0.03	0.04	18
0.00	K=30⋆	15463.52	26.00	0.17	0.19	23
	K=60⋆	17986.99	58.74	4.80	4.81	23
	1:50-cuts	0.00	0.00	0.00	0.01	200
markshare1	GMI	0.00	0.00	0.00	0.00	6
1.00	K=10	0.00	0.00	0.00	0.00	6
0.00	K=20	0.00	0.00	0.02	0.02	7
0.00	K=30	0.00	0.00	0.04	0.04	6
	K=60	0.00	0.00	0.60	0.60	6
	1:50-cuts	0.00	0.00	0.01	0.02	200
markshare2	GMI	0.00	0.00	0.00	0.00	7
1.00	K=10	0.00	0.00	0.00	0.00	7
0.00	K=20	0.00	0.00	0.01	0.01	7
0.00	K=30	0.00	0.00	0.05	0.05	7
	K=60	0.00	0.00	0.55	0.55	7
	1:50-cuts⋆	10578.59	7.27	0.13	0.44	230
mas74	GMI	10570.72	6.67	0.00	0.01	12
11801.20	K=10⋆	10570.94	6.69	0.01	0.04	33
10482.80	K=20⋆	10576.54	7.11	0.12	0.15	44
0.01	K=30⋆	10581.80	7.51	0.83	0.90	71
	K=60⋆	10585.87	7.82	27.15	27.23	79
	1:50-cuts⋆	38971.87	7.02	0.07	0.30	221
mas76	GMI	38965.29	6.42	0.00	0.01	11
40005.10	K=10⋆	38968.36	6.70	0.02	0.04	25
38893.90	K=20⋆	38972.76	7.10	0.12	0.14	43
0.00	K=30⋆	38975.64	7.36	0.49	0.53	34
	K=60⋆	38977.76	7.55	19.21	19.26	52
	1:50-cuts⋆	−608.67	6.62	82.83	88.59	950
mkc	GMI	−609.41	5.09	0.69	1.16	142
−563.85	K=10⋆	−609.32	5.27	4.98	6.91	367
−611.85	K=20⋆	−609.32	5.27	8.56	11.19	463
0.09	K=30⋆	−609.08	5.76	16.49	20.09	600
	K=60⋆	−608.92	6.11	408.84	416.52	958
	1:50-cuts⋆	294.29	20.92	0.05	0.15	200
mod008	GMI	294.29	20.88	0.00	0.01	5
307.00	K=10⋆	293.95	18.80	0.00	0.00	8
290.93	K=20⋆	294.17	20.18	0.02	0.03	8
0.00	K=30⋆	294.28	20.81	0.06	0.06	8
	K=60⋆	294.28	20.83	2.29	2.30	9
	1:50-cuts	20484452.96	17.28	0.92	1.23	800
modglob	GMI	20484452.96	17.28	0.01	0.03	30
20740508.00	K=10	20472029.03	13.27	0.06	0.09	52
20430947.62	K=20	20478791.33	15.46	0.23	0.29	50
0.01	K=30	20481755.99	16.41	0.85	0.92	54
	K=60	20482796.42	16.75	24.97	25.03	55
	1:50-cuts	31.16	7.07	2041.36	2439.95	819
net12	GMI	31.16	7.07	33.49	66.35	296
214.00	K=10	30.31	6.64	29.38	75.80	457
17.25	K=20	30.69	6.83	39.63	86.34	462
14.53	K=30	31.02	7.00	47.41	92.83	433
	K=60	31.10	7.04	268.56	312.26	477
	1:50-cuts⋆	−19.23	19.74	0.70	1.34	349
opt1217	GMI	−19.23	19.72	0.01	0.04	28
−16.00	K=10⋆	−19.26	18.85	0.09	0.15	53
−20.02	K=20⋆	−19.25	19.11	0.16	0.23	55
0.01	K=30⋆	−19.23	19.67	0.57	0.71	68
	K=60⋆	−19.23	19.68	12.43	12.57	75
	1:50-cuts	0.00	0.00	0.02	0.04	200
pk1	GMI	0.00	0.00	0.00	0.01	15
11.00	K=10	0.00	0.00	0.01	0.01	15
0.00	K=20	0.00	0.00	0.02	0.03	15
0.01	K=30	0.00	0.00	0.07	0.07	15
	K=60	0.00	0.00	1.25	1.25	15
	1:50-cuts	5151.26	52.22	0.41	0.75	1846
pp08a	GMI	5151.26	52.22	0.00	0.01	53
7350.00	K=10	5073.19	50.52	0.06	0.08	92
2748.35	K=20	5115.98	51.45	0.16	0.18	102
0.00	K=30	5124.13	51.63	0.57	0.61	111
	K=60	5144.84	52.08	10.61	10.63	110

Table 3: MIPLIB mixed integer problems (part 3).

Problem	Type of Cuts	Final LB	Closed Gap (%)	Separation Time (seconds)	Total Time (seconds)	Number of Cuts
	1:50-cuts⋆	14447.39	9.87	3.84	6.72	374
qnet1	GMI	14445.72	9.78	0.09	0.26	55
16029.69	K=10⋆	14446.24	9.80	0.22	0.49	59
14274.10	K=20⋆	14446.24	9.80	0.38	0.64	63
0.05	K=30⋆	14446.24	9.80	1.01	1.32	69
	K=60⋆	14447.10	9.85	23.74	24.09	75
	1:50-cuts⋆	13748.85	42.02	1.07	1.99	400
qnet1_o	GMI	13739.62	41.79	0.03	0.07	11
16029.69	K=10⋆	13686.09	40.43	0.09	0.18	22
12095.57	K=20⋆	13714.81	41.16	0.21	0.33	29
0.02	K=30⋆	13724.55	41.41	0.49	0.61	31
	K=60⋆	13738.61	41.76	14.90	15.03	30
	1:50-cuts	29046932.15	15.53	46.90	64.34	550
rentacar	GMI	29046932.15	15.53	0.78	6.82	22
30356761.00	K=10	28806137.64	0.00	1.51	7.47	32
28806137.64	K=20	28806137.64	0.00	3.01	7.72	31
7.19	K=30	28806137.64	0.00	9.91	15.91	35
	K=60	28877221.50	4.58	150.90	159.31	39
	1:50-cuts	40830.12	39.16	6.35	8.42	4266
set1ch	GMI	40830.12	39.16	0.02	0.08	138
54537.80	K=10	38975.16	30.93	0.12	0.22	258
32007.73	K=20	40274.48	36.69	0.38	0.50	263
0.00	K=30	40622.20	38.24	1.40	1.53	287
	K=60	40776.80	38.92	37.04	37.17	294
	1:50-cuts	201721.28	23.51	1.53	1.86	1188
timtab1	GMI	201721.28	23.51	0.01	0.03	136
764772.00	K=10	199777.70	23.24	0.13	0.19	194
28694.00	K=20	200979.95	23.41	0.47	0.54	218
0.00	K=30	201158.22	23.43	1.76	1.82	244
	K=60	201721.28	23.51	25.59	25.64	254
	1:50-cuts	84359.79	60.27	76.98	90.45	13262
tr12-30	GMI	84359.79	60.27	0.13	0.36	348
130596.00	K=10	73768.19	51.17	0.36	0.67	550
14210.43	K=20	80578.11	57.02	0.87	1.42	806
0.01	K=30	83103.48	59.19	3.45	4.02	733
	K=60	83817.48	59.81	72.54	73.25	771
	1:50-cuts	15.86	9.70	0.27	0.30	118
vpm1	GMI	15.86	9.70	0.00	0.00	7
20.00	K=10	15.81	8.66	0.04	0.06	18
15.42	K=20	15.86	9.70	0.09	0.09	11
0.00	K=30	15.85	9.37	0.37	0.38	15
	K=60	15.86	9.70	3.90	3.92	11
	1:50-cuts	10.36	12.15	0.47	0.60	390
vpm2	GMI	10.36	12.15	0.01	0.02	20
13.75	K=10	10.35	11.82	0.05	0.07	27
9.89	K=20	10.35	12.05	0.23	0.25	33
0.01	K=30	10.35	12.06	0.85	0.88	36
	K=60	10.36	12.11	19.28	19.33	42
	1:50-cuts	390.10	57.47	0.25	0.46	1588
egout	GMI	390.10	57.47	0.01	0.02	40
568.10	K=10	242.52	22.20	0.03	0.04	60
149.59	K=20	292.92	34.25	0.13	0.15	69
0.00	K=30	304.87	37.10	0.59	0.60	75
	K=60	319.41	40.58	14.32	14.33	71
	1:50-cuts	1497.17	10.65	2.76	3.89	2349
fixnet6	GMI	1497.17	10.65	0.01	0.07	60
3983.00	K=10	1470.33	9.69	0.07	0.14	91
1200.88	K=20	1478.40	9.97	0.09	0.14	92
0.00	K=30	1483.45	10.16	0.28	0.32	76
	K=60	1493.53	10.52	7.89	7.96	91
	1:50-cuts	31.16	7.07	2041.36	2439.95	819
net12	GMI	31.16	7.07	33.49	66.35	296
214.00	K=10	30.31	6.64	29.38	75.80	457
17.25	K=20	30.69	6.83	39.63	86.34	462
14.53	K=30	31.02	7.00	47.41	92.83	433
	K=60	31.10	7.04	268.56	312.26	477
	1:50-cuts	−43.00	0.00	0.56	0.68	666
noswot	GMI	−43.00	0.00	0.00	0.02	48
−41.00	K=10	−43.00	0.00	0.08	0.12	63
−43.00	K=20	−43.00	0.00	0.18	0.20	54
0.00	K=30	−43.00	0.00	1.24	1.27	58
	K=60	−43.00	0.00	16.27	16.28	56

Table 4: MIPLIB mixed integer problems (part 4).

Problem	Type of Cuts	Final LB	Closed Gap (%)	Separation Time (seconds)	Total Time (seconds)	Number of Cuts
	1:50-cuts	52.26	10.37	0.06	0.10	327
rgn	GMI	52.26	10.37	0.00	0.00	16
82.20	K=10	51.66	8.56	0.03	0.05	27
48.80	K=20	52.00	9.59	0.05	0.06	28
0.00	K=30	52.17	10.08	0.24	0.27	32
	K=60	52.17	10.08	4.75	4.77	28
	1:50-cuts	6066.01	31.32	0.66	0.98	884
pp08aCUTS	GMI	6066.01	31.32	0.00	0.03	46
7350.00	K=10	6043.49	30.11	0.06	0.11	72
5480.61	K=20	6059.97	30.99	0.28	0.31	76
0.01	K=30	6059.67	30.98	1.12	1.19	91
	K=60	6063.23	31.17	28.15	28.20	92
	1:50-cuts	−924.23	0.93	6.35	11.15	410
qiu	GMI	−924.23	0.93	0.07	0.73	36
−132.87	K=10	−924.97	0.84	0.24	1.16	59
−931.64	K=20	−924.57	0.89	0.63	1.52	59
0.21	K=30	−924.44	0.90	2.33	3.43	60
	K=60	−924.44	0.90	55.77	56.95	71
	1:50-cuts	27908501.37	47.56	14.99	16.99	1550
gesa3	GMI	27908501.37	47.56	0.19	0.41	100
27991042.65	K=10	27906378.34	46.21	0.62	0.93	151
27833632.45	K=20	27907466.03	46.91	1.40	1.76	183
0.10	K=30	27908302.96	47.44	5.61	6.07	203
	K=60	27908406.33	47.50	108.63	109.18	200
	1:50-cuts	27928910.08	60.53	19.59	21.30	1580
gesa3_o	GMI	27928910.08	60.53	0.30	0.50	145
27991042.65	K=10	27925661.11	58.46	0.82	1.14	216
27833632.45	K=20	27927682.03	59.75	1.86	2.28	256
0.08	K=30	27928467.67	60.25	5.18	5.75	303
	K=60	27928634.90	60.35	95.07	95.67	282

Table 5: MIPLIB pure integer problems (part 1).

Problem	Type of Cuts	Final LB	Closed Gap (%)	Separation Time (seconds)	Total Time (seconds)	Number of Cuts
	1:50-cuts	340160.00	100.00	0.90	5.03	200
air03	GMI	340160.00	100.00	0.12	1.03	36
340160.00	K=10	340160.00	100.00	0.24	1.14	36
338864.25	K=20	340160.00	100.00	0.26	1.23	36
0.30	K=30	340160.00	100.00	0.29	1.27	36
	K=60	340160.00	100.00	1.19	2.09	36
	1:50-cuts⋆	55590.96	9.23	94.33	880.95	427
air04	GMI	55583.78	8.04	2.02	335.94	202
56137.00	K=10⋆	55580.69	7.52	3.16	514.19	283
55535.44	K=20⋆	55583.86	8.05	5.93	510.79	300
34.82	K=30⋆	55585.19	8.27	10.13	581.08	370
	K=60⋆	55586.21	8.44	197.07	723.21	389
	1:50-cuts⋆	25903.63	5.24	82.52	172.38	423
air05	GMI	25899.70	4.45	1.39	34.44	201
26374.00	K=10⋆	25898.87	4.28	2.62	44.05	229
25877.61	K=20⋆	25900.87	4.69	3.54	58.50	260
3.19	K=30⋆	25900.96	4.70	9.10	95.19	298
	K=60⋆	25902.02	4.92	177.51	307.77	386
	1:50-cuts	−2451470.55	41.65	2.65	40.17	398
cap6000	GMI	−2451470.55	41.65	0.05	0.59	9
−2451377.00	K=10	−2451474.58	39.14	0.08	0.73	11
−2451537.33	K=20	−2451474.58	39.14	0.12	0.78	11
0.33	K=30	−2451472.58	40.38	0.17	0.82	11
	K=60	−2451472.49	40.44	1.59	2.26	11
	1:50-cuts⋆	4665.16	13.40	1.62	3.13	200
l152lav	GMI	4664.41	12.25	0.05	0.40	51
4722.00	K=10⋆	4664.03	11.67	0.22	0.79	88
4656.36	K=20⋆	4664.60	12.54	0.52	1.17	88
0.09	K=30⋆	4665.26	13.55	2.23	3.59	237
	K=60⋆	4665.87	14.48	83.55	86.58	349
	1:50-cuts⋆	996.97	56.88	0.07	0.09	300
lseu	GMI	991.87	55.09	0.00	0.01	13
1120.00	K=10⋆	996.29	56.64	0.02	0.03	22
834.68	K=20⋆	997.34	57.01	0.07	0.08	25
0.00	K=30⋆	998.64	57.47	0.41	0.42	42
	K=60⋆	1000.29	58.04	8.17	8.19	31
	1:50-cuts	−13164.00	100.00	4798.42	4884.79	10275
manna81	GMI	−13164.00	100.00	13.58	16.55	812
−13164.00	K=10	−13164.00	100.00	21.28	24.32	812
−13297.00	K=20	−13164.00	100.00	21.95	25.06	812
0.13	K=30	−13164.00	100.00	23.74	27.13	812
	K=60	−13164.00	100.00	33.91	38.17	812
	1:50-cuts	115081.21	82.20	4057.42	4244.84	14200
mitre	GMI	115081.21	82.20	12.71	15.95	552
115155.00	K=10	115067.88	78.98	21.18	25.85	662
114740.52	K=20	115081.21	82.20	60.69	67.74	725
0.34	K=30	115081.21	82.20	72.65	80.40	701
	K=60	115081.21	82.20	573.16	583.40	738
	1:50-cuts	6535.50	21.47	1.54	3.49	200
mod010	GMI	6535.50	21.47	0.05	0.42	34
6548.00	K=10	6535.46	21.24	0.14	0.67	38
6532.08	K=20	6535.46	21.24	0.17	0.66	36
0.09	K=30⋆	6535.75	23.04	0.26	0.70	36
	K=60⋆	6536.00	24.61	2.88	3.40	40
	1:50-cuts⋆	2843.74	56.85	0.00	0.02	219
p0033	GMI	2830.95	54.60	0.00	0.00	8
3089.00	K=10⋆	2629.39	19.14	0.02	0.03	15
2520.57	K=20⋆	2690.91	29.97	0.07	0.07	18
0.00	K=30⋆	2749.28	40.23	0.33	0.34	22
	K=60⋆	2842.25	56.59	4.73	4.73	19
	1:50-cuts⋆	7007.91	17.96	0.14	0.22	200
p0201	GMI	7000.56	16.97	0.00	0.02	22
7615.00	K=10⋆	7002.42	17.22	0.03	0.08	61
6875.00	K=20⋆	7002.43	17.22	0.05	0.13	71
0.01	K=30⋆	7004.88	17.55	0.24	0.30	76
	K=60⋆	7004.96	17.56	2.54	2.61	86
	1:50-cuts	179882.58	3.70	0.45	0.53	400
p0282	GMI	179882.58	3.70	0.00	0.02	32
258411.00	K=10	179711.49	3.49	0.04	0.05	38
176867.50	K=20	179830.00	3.63	0.10	0.12	43
0.00	K=30	179784.53	3.58	0.41	0.44	45
	K=60	179830.00	3.63	5.90	5.93	43

Table 6: MIPLIB pure integer problems (part 2).

Problem	Type of Cuts	Final LB	Closed Gap (%)	Separation Time (seconds)	Total Time (seconds)	Number of Cuts
	1:50-cuts⋆	3668.81	40.04	3.80	4.49	2113
p0548	GMI	3667.64	40.02	0.02	0.06	55
8691.00	K=10⋆	738.79	5.06	0.09	0.14	88
315.25	K=20⋆	3258.85	35.14	0.29	0.35	99
0.01	K=30⋆	3139.64	33.72	1.09	1.16	102
	K=60⋆	3493.37	37.94	23.29	23.40	113
	1:50-cuts⋆	2691.42	0.61	8.24	10.35	1300
p2756	GMI	2691.09	0.54	0.08	0.16	35
3124.00	K=10⋆	2689.80	0.24	0.24	0.44	57
2688.75	K=20⋆	2690.00	0.29	0.34	0.54	61
0.02	K=30⋆	2690.00	0.29	0.56	0.76	62
	K=60⋆	2690.00	0.29	8.65	8.81	65
	1:50-cuts	405.14	6.75	978.83	1019.77	676
seymour	GMI	405.14	6.75	16.09	30.95	276
423.00	K=10	405.13	6.69	26.64	45.43	410
403.85	K=20	405.13	6.72	27.83	45.03	427
24.01	K=30	405.14	6.75	51.41	69.78	401
	K=60	405.14	6.75	290.60	305.18	348
	1:50-cuts	−5000.00	(−)	43.27	72.82	200
disctom	GMI	−5000.00	(−)	1.45	27.86	200
−5000.00	K=10	−5000.00	(−)	2.16	28.47	200
−5000.00	K=20	−5000.00	(−)	2.48	28.97	200
1.25	K=30	−5000.00	(−)	3.11	29.69	200
	K=60	−5000.00	(−)	13.84	40.28	200
	1:50-cuts	0.00	(−)	0.02	0.05	200
enigma	GMI	0.00	(−)	0.00	0.00	8
0.00	K=10	0.00	(−)	0.00	0.00	8
0.00	K=20	0.00	(−)	0.01	0.01	8
0.00	K=30	0.00	(−)	0.04	0.04	8
	K=60	0.00	(−)	0.62	0.63	8
	1:50-cuts⋆	−74219846.41	29.43	5.45	27.53	653
harp2	GMI	−74251958.32	22.35	0.04	0.19	30
−73899798.00	K=10⋆	−74247224.08	23.40	0.23	0.69	58
−74353341.50	K=20⋆	−74236993.08	25.65	0.35	0.97	62
0.04	K=30⋆	−74236058.30	25.86	0.93	1.71	71
	K=60⋆	−74225928.01	28.09	26.86	27.79	75
	1:50-cuts	13.00	0.00	0.38	0.40	209
stein27	GMI	13.00	0.00	0.01	0.01	84
18.00	K=10	13.00	0.00	0.03	0.04	84
13.00	K=20	13.00	0.00	0.07	0.07	84
0.00	K=30	13.00	0.00	0.17	0.18	84
	K=60	13.00	0.00	2.31	2.34	84
	1:50-cuts	22.00	0.00	1.26	1.29	200
stein45	GMI	22.00	0.00	0.04	0.09	200
30.00	K=10	22.00	0.00	0.10	0.14	200
22.00	K=20	22.00	0.00	0.21	0.26	200
0.01	K=30	22.00	0.00	0.50	0.52	200
	K=60	22.00	0.00	5.03	5.06	200

Table 7: Atamtürk's bounded problems (averages over 5 instances).

Set	Type of Cuts	Closed Gap (%)	Separation Time (seconds)	Total Time (seconds)	Average Number of Cuts
mik.250-10-100	1:50-cuts⋆	74.69	3.501	8.294	2948.40
	GMI	67.41	0.012	0.034	100.00
	k=5	66.60	0.052	0.182	174.60
	k=10	66.78	0.098	0.266	204.60
	k=20⋆	67.52	0.405	0.623	227.40
	k=30⋆	70.10	2.311	2.622	257.00
	k=60⋆	72.02	64.913	65.314	289.20
mik.250-10-50	1:50-cuts⋆	76.34	1.975	23.976	2100.60
	GMI	50.87	0.012	0.022	50.00
	k=5	50.37	0.026	0.118	86.00
	k=10	50.37	0.066	0.180	103.40
	k=20⋆	51.14	0.216	0.363	114.00
	k=30⋆	55.62	0.993	1.206	132.00
	k=60⋆	66.04	35.559	35.898	163.40
mik.250-10-75	1:50-cuts⋆	72.54	3.179	14.369	2795.80
	GMI	56.19	0.016	0.034	75.00
	k=5	55.18	0.048	0.198	130.20
	k=10	55.33	0.070	0.268	153.00
	k=20⋆	56.20	0.318	0.593	174.40
	k=30⋆	60.78	1.741	2.155	200.20
	k=60⋆	66.22	56.732	57.366	232.60
mik.250-20-100	1:50-cuts⋆	75.27	3.145	7.713	2689.80
	GMI	70.46	0.022	0.044	100.00
	k=5	69.83	0.048	0.182	174.60
	k=10	69.99	0.096	0.248	204.00
	k=20	70.42	0.423	0.643	229.20
	k=30⋆	72.25	2.087	2.379	257.00
	k=60⋆	73.68	71.585	72.052	286.40
mik.250-20-50	1:50-cuts⋆	74.77	1.929	19.041	2049.60
	GMI	51.60	0.012	0.028	50.00
	k=5	51.09	0.034	0.118	86.00
	k=10	51.08	0.062	0.184	103.20
	k=20⋆	51.87	0.234	0.389	115.40
	k=30⋆	56.41	1.066	1.320	132.20
	k=60⋆	66.12	37.179	37.594	166.60
mik.250-20-75	1:50-cuts⋆	74.37	2.936	12.135	2586.60
	GMI	61.29	0.018	0.040	75.00
	k=5	60.21	0.046	0.206	130.20
	k=10	60.38	0.078	0.286	152.40
	k=20⋆	61.35	0.334	0.637	176.80
	k=30⋆	66.33	1.658	2.063	201.80
	k=60⋆	70.14	54.352	55.023	241.20
mik.250-5-100	1:50-cuts⋆	71.17	3.990	9.996	3196.00
	GMI	61.29	0.020	0.038	100.00
	k=5	60.43	0.054	0.154	174.60
	k=10	60.58	0.098	0.258	204.80
	k=20⋆	61.43	0.413	0.633	229.00
	k=30⋆	64.61	2.351	2.688	265.40
	k=60⋆	67.35	75.697	76.123	297.00

Table 8: Atamtürk's unbounded problems (averages over 5 instances).

Set	Type of Cuts	Closed Gap (%)	Separation Time (seconds)	Total Time (seconds)	Average Number of Cuts
mik.250-1-100	1:50-cuts⋆	74.55	3.898	9.474	3187.20
	GMI	63.26	0.014	0.032	100.00
	k=5	62.63	0.058	0.182	174.60
	k=10	62.78	0.100	0.232	203.80
	k=20⋆	63.28	0.411	0.619	225.80
	k=30⋆	66.27	1.891	2.129	262.20
	k=60⋆	70.83	64.801	65.294	303.20
mik.250-1-50	1:50-cuts⋆	78.52	2.117	23.440	2185.40
	GMI	50.07	0.008	0.024	50.00
	k=5	49.58	0.036	0.118	86.00
	k=10	49.58	0.062	0.184	103.40
	k=20⋆	50.34	0.232	0.379	114.00
	k=30⋆	54.74	0.993	1.234	132.60
	k=60⋆	65.52	35.902	36.212	157.20
mik.250-1-75	1:50-cuts⋆	76.14	3.419	19.087	3014.40
	GMI	54.83	0.018	0.034	75.00
	k=5	53.82	0.046	0.176	130.20
	k=10	53.97	0.086	0.282	153.00
	k=20	54.82	0.312	0.589	174.40
	k=30⋆	59.53	1.642	2.043	199.00
	k=60⋆	67.45	56.796	57.483	236.00
mik.500-1-100	1:50-cuts⋆	75.13	10.467	83.001	3293.20
	GMI	63.48	0.046	0.130	100.00
	k=5	62.42	0.124	2.175	174.60
	k=10	62.53	0.204	2.636	206.40
	k=20⋆	63.57	0.571	3.112	224.80
	k=30⋆	68.01	2.221	5.772	267.40
	k=60⋆	71.92	76.314	84.912	321.40
mik.500-1-50	1:50-cuts⋆	76.67	4.418	75.238	2117.20
	GMI	50.68	0.024	0.080	50.00
	k=5	50.19	0.062	0.401	86.00
	k=10	50.19	0.102	0.479	102.80
	k=20⋆	50.88	0.284	0.793	113.60
	k=30⋆	55.41	1.096	1.961	133.60
	k=60⋆	66.10	37.354	38.610	157.80
mik.500-1-75	1:50-cuts⋆	74.92	8.360	91.904	2932.40
	GMI	55.13	0.034	0.102	75.00
	k=5	54.12	0.098	1.248	130.20
	k=10	54.28	0.160	1.364	151.40
	k=20	55.08	0.431	2.135	174.80
	k=30⋆	59.91	1.600	4.094	197.20
	k=60⋆	67.48	50.757	54.979	240.40

Table 9: Random knapsack problems with coefficients in $\{1, \ldots, 1000\}$.

n	Type of Cuts	Binary				Bounded			
		Closed Gap (%)	Sep. Time (sec)	Tot. Time (sec)	Avg. Num. of Cuts	Closed Gap (%)	Sep. Time (sec)	Tot. Time (sec)	Avg. Num. of Cuts
10	1:50-cuts	90.46⋆	0.000	0.002	49.93	94.75⋆	0.000	0.000	50.00
	GMI	76.15	0.000	0.000	1.00	74.47	0.000	0.001	1.00
	k=5	65.92	0.001	0.001	1.97	71.49	0.000	0.001	2.00
	k=10	81.27⋆	0.001	0.002	2.83	82.10⋆	0.001	0.003	2.63
	k=20	89.30⋆	0.003	0.006	3.33	91.25⋆	0.004	0.007	3.23
	k=30	91.51⋆	0.014	0.016	3.57	94.08⋆	0.012	0.016	3.80
	k=60	92.90⋆	0.229	0.232	3.50	97.36⋆	0.249	0.252	4.13
50	1:50-cuts	79.94⋆	0.002	0.007	49.90	88.83⋆	0.001	0.003	49.93
	GMI	50.71	0.000	0.001	1.00	42.44	0.000	0.001	1.00
	k=5	52.00⋆	0.001	0.002	2.07	36.46	0.001	0.002	1.73
	k=10	62.63⋆	0.001	0.005	3.20	46.62⋆	0.002	0.003	2.67
	k=20	69.44⋆	0.005	0.008	3.90	68.21⋆	0.005	0.009	3.77
	k=30	70.34⋆	0.016	0.020	4.97	73.74⋆	0.020	0.023	4.63
	k=60	77.78⋆	0.294	0.300	5.63	83.88⋆	0.360	0.365	5.67
100	1:50-cuts	78.41⋆	0.002	0.009	49.97	82.84⋆	0.002	0.008	49.43
	GMI	39.80	0.000	0.000	1.00	39.31	0.000	0.001	1.00
	k=5	34.88	0.001	0.003	1.77	34.12	0.002	0.004	1.77
	k=10	49.98⋆	0.001	0.003	2.80	44.71⋆	0.001	0.002	2.60
	k=20	58.65⋆	0.007	0.011	4.27	59.39⋆	0.007	0.008	4.20
	k=30	66.57⋆	0.019	0.024	5.50	67.65⋆	0.016	0.022	4.80
	k=60	74.88⋆	0.347	0.351	6.73	75.64⋆	0.357	0.363	6.23
500	1:50-cuts	74.09⋆	0.014	0.062	49.60	75.99⋆	0.012	0.060	49.93
	GMI	26.20	0.000	0.003	1.00	22.44	0.000	0.004	1.00
	k=5	26.69⋆	0.003	0.007	1.93	19.90	0.001	0.006	1.80
	k=10	34.13⋆	0.005	0.009	3.27	25.08⋆	0.002	0.008	2.77
	k=20	43.08⋆	0.010	0.017	4.87	37.03⋆	0.008	0.015	3.90
	k=30	48.74⋆	0.024	0.032	5.57	41.72⋆	0.019	0.027	4.50
	k=60	60.09⋆	0.366	0.384	7.33	54.93⋆	0.329	0.341	6.20
1000	1:50-cuts	56.97⋆	0.027	0.153	50.00	70.16⋆	0.030	0.152	49.13
	GMI	24.68	0.000	0.008	1.00	23.97	0.001	0.008	1.00
	k=5	22.09	0.003	0.012	2.17	22.51	0.003	0.012	2.10
	k=10	29.17⋆	0.004	0.018	3.23	27.08⋆	0.004	0.015	2.90
	k=20	36.07⋆	0.011	0.026	4.70	35.10⋆	0.007	0.020	4.23
	k=30	40.03⋆	0.027	0.048	5.67	39.40⋆	0.031	0.045	5.30
	k=60	47.21⋆	0.375	0.395	7.47	44.74⋆	0.324	0.344	5.90
5000	1:50-cuts	47.07⋆	0.156	0.910	49.90	53.96⋆	0.156	0.904	49.97
	GMI	17.03	0.004	0.063	1.00	12.78	0.002	0.060	1.00
	k=5	15.69	0.008	0.082	2.03	10.74	0.008	0.081	2.00
	k=10	19.50⋆	0.013	0.102	3.07	19.49⋆	0.012	0.099	3.10
	k=20	25.86⋆	0.021	0.140	4.63	25.01⋆	0.022	0.127	4.20
	k=30	29.64⋆	0.036	0.179	5.53	28.31⋆	0.034	0.159	4.93
	k=60	36.53⋆	0.433	0.656	8.43	36.29⋆	0.381	0.543	6.43
10000	1:50-cuts	38.59⋆	0.313	2.249	49.97	48.81⋆	0.306	2.214	49.93
	GMI	10.82	0.006	0.246	1.00	18.34	0.004	0.235	1.00
	k=5	10.43	0.016	0.286	1.93	16.05	0.012	0.261	1.77
	k=10	15.40⋆	0.023	0.331	2.97	20.37⋆	0.019	0.304	2.83
	k=20	20.40⋆	0.031	0.380	4.03	28.24⋆	0.031	0.345	3.97
	k=30	23.55⋆	0.055	0.483	5.60	30.90⋆	0.050	0.418	4.87
	k=60	28.14⋆	0.425	0.927	6.77	37.03⋆	0.359	0.852	6.70

Table 10: Random knapsack problems with coefficients in $\{1, \ldots, 100\}$.

n	Type of Cuts	Binary				Bounded			
		Closed Gap (%)	Sep. Time (sec)	Tot. Time (sec)	Avg. Num. of Cuts	Closed Gap (%)	Sep. Time (sec)	Tot. Time (sec)	Avg. Num. of Cuts
10	1:50-cuts	91.96⋆	0.001	0.002	47.03	96.54⋆	0.000	0.002	48.37
	GMI	77.61	0.000	0.002	1.00	74.98	0.000	0.000	1.00
	k=5	71.12	0.001	0.002	1.93	74.15	0.001	0.003	1.90
	k=10	84.14⋆	0.001	0.005	2.73	85.52⋆	0.002	0.004	2.70
	k=20	92.12⋆	0.003	0.006	3.23	92.18⋆	0.005	0.008	3.37
	k=30	93.30⋆	0.011	0.014	3.33	94.11⋆	0.014	0.017	3.83
	k=60	95.91⋆	0.216	0.221	3.70	96.37⋆	0.255	0.260	3.87
	Ideal k	96.92⋆	1.304	1.310	2.97	98.12⋆	1.251	1.256	3.20
50	1:50-cuts	85.56⋆	0.002	0.005	49.60	91.38⋆	0.001	0.006	49.13
	GMI	53.89	0.000	0.001	1.00	43.37	0.000	0.001	1.00
	k=5	55.43⋆	0.000	0.001	2.10	37.62	0.000	0.001	1.73
	k=10	65.72⋆	0.002	0.005	2.80	49.61⋆	0.003	0.004	2.30
	k=20	73.96⋆	0.007	0.009	3.80	67.45⋆	0.006	0.011	4.00
	k=30	78.80⋆	0.014	0.022	4.70	73.92⋆	0.017	0.022	4.50
	k=60	85.47⋆	0.330	0.337	5.33	84.98⋆	0.327	0.331	5.17
	Ideal k	90.29⋆	1.235	1.239	4.07	97.80⋆	1.399	1.406	4.13
100	1:50-cuts	82.53⋆	0.003	0.008	48.48	89.38⋆	0.003	0.009	48.72
	GMI	39.52	0.000	0.002	1.00	43.97	0.000	0.002	1.00
	k=5	36.32	0.002	0.004	1.86	41.22	0.000	0.002	1.83
	k=10	50.66⋆	0.002	0.005	2.90	50.27⋆	0.002	0.005	2.76
	k=20	64.36⋆	0.005	0.011	4.83	67.06⋆	0.007	0.011	3.79
	k=30	71.79⋆	0.020	0.027	5.83	77.06⋆	0.020	0.027	4.59
	k=60	80.30⋆	0.387	0.398	6.55	85.03⋆	0.353	0.357	5.76
	Ideal k	92.37⋆	2.042	2.050	5.17	93.85⋆	1.615	1.624	4.93
500	1:50-cuts	67.69⋆	0.016	0.057	48.97	87.92⋆	0.012	0.055	49.30
	GMI	21.93	0.002	0.006	1.00	25.72	0.000	0.004	1.00
	k=5	20.79	0.003	0.008	1.90	26.68⋆	0.001	0.006	1.80
	k=10	28.33⋆	0.002	0.010	3.03	36.09⋆	0.002	0.009	3.00
	k=20	40.85⋆	0.009	0.020	4.45	49.99⋆	0.009	0.019	4.57
	k=30	46.50⋆	0.024	0.034	5.03	53.87⋆	0.023	0.036	5.17
	k=60	56.12⋆	0.357	0.372	6.45	69.85⋆	0.415	0.430	6.83
	Ideal k	73.68⋆	1.756	1.768	6.34	96.80⋆	2.636	2.652	7.13
1000	1:50-cuts	79.88⋆	0.030	0.130	48.73	93.05⋆	0.023	0.124	48.23
	GMI	24.16	0.000	0.008	1.00	26.48	0.001	0.008	1.00
	k=5	22.99	0.002	0.012	1.80	25.23	0.005	0.015	1.67
	k=10	38.57⋆	0.006	0.021	3.47	31.93⋆	0.003	0.017	2.33
	k=20	48.07⋆	0.010	0.032	5.03	41.26⋆	0.009	0.022	3.43
	k=30	55.72⋆	0.030	0.055	6.83	54.78⋆	0.021	0.036	4.17
	k=60	65.81⋆	0.363	0.390	8.03	73.97⋆	0.377	0.402	5.97
	Ideal k	89.54⋆	2.036	2.067	6.63	100.00⋆	1.727	1.749	5.40
5000	1:50-cuts	100.00⋆	0.149	0.823	47.93	100.00⋆	0.125	0.814	47.70
	GMI	19.67	0.004	0.068	1.00	10.38	0.004	0.067	1.00
	k=5	20.91⋆	0.010	0.088	1.77	11.10⋆	0.009	0.083	1.47
	k=10	28.81⋆	0.018	0.105	2.60	19.98⋆	0.015	0.101	2.40
	k=20	39.70⋆	0.026	0.126	3.47	25.30⋆	0.025	0.130	3.57
	k=30	42.99⋆	0.042	0.163	4.50	33.11⋆	0.039	0.165	4.53
	k=60	55.66⋆	0.340	0.485	5.40	47.19⋆	0.429	0.637	7.27
	Ideal k	100.00⋆	1.909	2.036	4.33	100.00⋆	1.834	1.968	4.93
10000	1:50-cuts	100.00⋆	0.322	2.052	49.13	100.00⋆	0.269	2.073	49.63
	GMI	21.61	0.009	0.262	1.00	19.65	0.012	0.263	1.00
	k=5	19.61	0.022	0.299	1.73	20.79⋆	0.019	0.284	1.57
	k=10	26.16⋆	0.032	0.336	2.70	32.15⋆	0.025	0.302	2.27
	k=20	30.73⋆	0.041	0.368	3.27	41.04⋆	0.048	0.373	3.63
	k=30	37.32⋆	0.060	0.416	4.00	45.52⋆	0.062	0.431	4.27
	k=60	48.89⋆	0.403	0.876	6.10	52.99⋆	0.381	0.804	5.10
	Ideal k	100.00⋆	2.134	2.505	4.73	100.00⋆	2.816	3.192	4.67