# FACETS OF THE NODE-WEIGHTED STEINER ARBORESCENCE POLYTOPE

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BASED ON THE PAPER

"FACETS OF TWO STEINER ARBORESCENCE POLYHEDRA"

TO APPEAR IN <u>MATH. PROG.</u>

# THE PROBLEM

## GIVEN

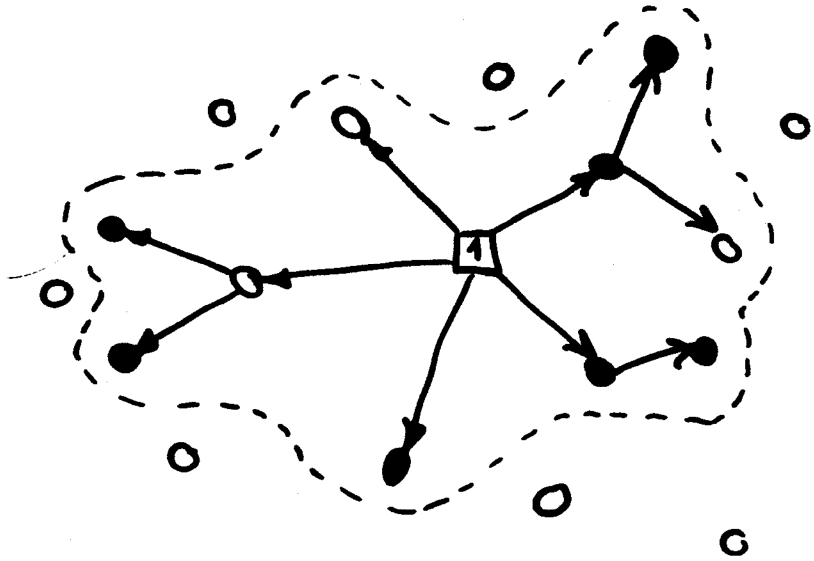
- A COMPLETE DIGRAPH $G = (V := \{1, \dots, n\}, A)$
- A DISTINGUISHED ROOT NODE, SAY NODE 1
- A TARGET NODE SET $T \subseteq V \setminus \{1\}$, $T \supseteq \emptyset$

## DEFINE

- STEINER NODE SET $\overline{T} := (V \setminus \{1\}) \setminus T$
- STEINER ARBORESCENCE (SA) = ARBORESCENCE (DIRECTED TREE) ROOTED AT NODE 1 AND SPANNING A NODE SET $Q \supseteq T \cup \{1\}$

● TARGET NODE

○ STEINER NODE



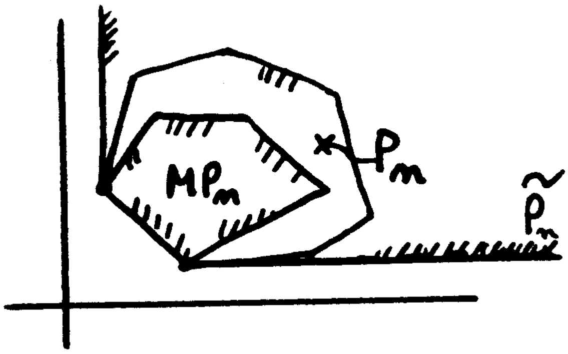

- MINIMAL SA (MSA): SA HAVING NO STEINER NODE AS A LEAF

# THREE SA POLYHEDRA

$P_n :=$ conv. hull incidence vect.s of all SA's

$MP_n :=$ " " " " " " MSA's

$\tilde{P}_n := P_n + \mathbb{R}^A_+ = MP_n + \mathbb{R}^A_+$ DOMINANT OF $P_n$ and $MP_n$




$$MP_n \subset P_n \subset \tilde{P}_n$$

$$\min(cx : x \in MP_n) \geq$$
$$\min(cx : x \in P_n) \geq$$
$$\min(cx : x \in \tilde{P}_n)$$

where $\min(*) = \min(*) = \min(*)$ in case $c \geq 0$

LITERATURE ON SA POLYHEDRA :

→ $\tilde{P}_n$ : BALL, LIU & PULLEYBLANK (1987)
CHOPRA & RAO (1988)
LIU (1988)

→ $MP_n$ : BALL, LIU & PULLEYBLANK (1987)
LIU (1988)

→ $P_n$ : ?

THE STUDY OF $P_n$ HAS PRACTICAL RELEVANCE ONLY IF THERE EXIST APPLICATIONS IN WHICH $c \not\geq 0$

# THE NODE-WEIGHTED SA PROBLEM

A. Segev, Networks 1987

$c : A \to \mathbb{R}_+$ ARC COSTS
$w : \overline{T} \to \mathbb{R}_+$ STEINER-NODE PENALTIES

COST OF A SA :=

$\sum ( c_{ij} :$ arc $(i,j)$ is used $) +$
$\sum ( w_j :$ node $j$ is unvisited $)$

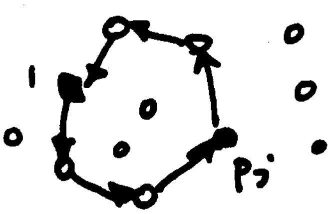
$\Rightarrow$ EQUIVALENT TO $\min ( c' x : x \in P_n ) + \sum_{j \in \overline{T}} w_j$

$$c'_{ij} := \begin{cases} c_{ij} & j \in T \\ c_{ij} - w_j & j \in \overline{T} \end{cases}$$


- RELATED PROBLEMS

SPANNING VERSION $\longrightarrow$ NON-SPANNING VERSION
MIN-COST ARBORESCENCE $\longrightarrow$ STEINER ARBORESCENCE
TSP 

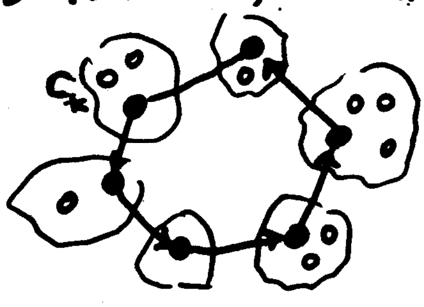
$\longrightarrow$ PRIZE-COLLECTING TSP




Balas, Networks 1989

$\sum ( p_j : j$ VISITED $) \geq$ GOAL

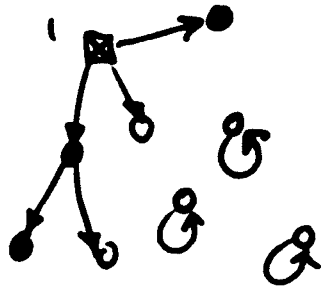
$\longrightarrow$ GENERALIZED TSP F., SALAZAR, TOTH in prep.




CLUSTERS $C_1, \dots, C_m$

$|\{ j \in C_k : j \text{ VISITED} \}| \geq 1$

# THE SA POLYTOPE, $P_n$



→ $x_{ij} = \begin{cases} 1 & \text{arc } (i,j) \text{ used} \\ 0 & \text{otherwise} \end{cases}$

→ $x_{jj} = \begin{cases} 1 & \text{node } j \text{ UNVISITED} \\ 0 & \text{otherwise} \end{cases}$

→ ARC SET $A := (V \times V) \setminus \{(i,1) : i \in V\} \setminus \{(j,j) : j \in T\}$

(so $|A| = n^2 - n - |T|$ )

→ NOTATION : GIVEN $x \in \mathbb{R}^A$ ; $S_1, S_2 \subseteq V$

$$x(S_1, S_2) := \sum (x_{ij} : i \in S_1, j \in S_2, (i,j) \in A)$$

* EQUALITY SYSTEM FOR $P_n$ :

$$x(V, j) = 1 \quad \text{for all } j \in V \setminus \{1\}$$

$$\Rightarrow \dim(P_n) = |A| - n + 1 = (n-1)^2 - |T|$$

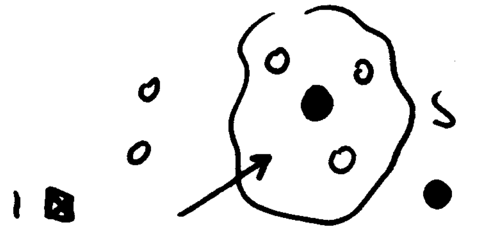
* TRIVIAL FACETS OF $P_n$ (ASSUMING $n \geq 4$) :

$$x_{ij} \geq 0 \quad \text{for all } (i,j) \in A$$

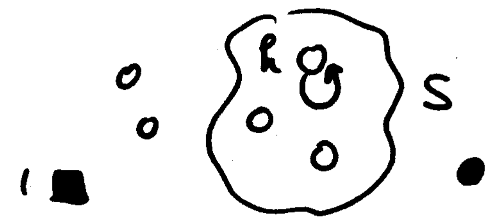
# CONNECTIVITY CONSTRAINTS (→ FACETS)

$x(V\setminus S, S) \ge 1$ , $S \subseteq V\setminus\{1\}$ s.t. $T \cap S \neq \emptyset$

$x(V\setminus S, S) \ge 1 - x_{hh}$ , $S \subseteq \overline{T}$ , $h \in S$



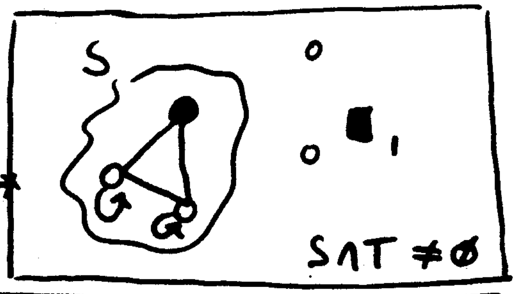

case $T \cap S \neq \emptyset$




case $T \cap S = \emptyset$

## EQUIVALENT FORMULATION (SEC's)

$x(S,S) \le |S| - 1$




$x(S,S) - x_{hh} \le |S| - 1$

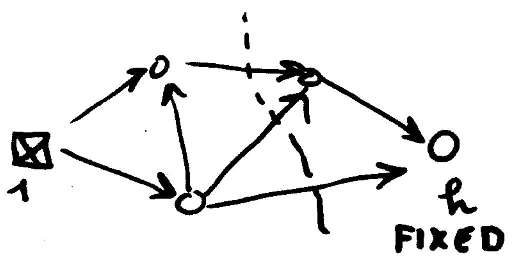

ALL LOOPS $(j,j)$ , $j \in S$, BUT ONE HAVE COEFFICIENT 1

## POLYNOMIAL-TIME SEPARATION : $O(n^4)$ TIME

- GIVEN $y \in \mathbb{R}^A$ FIND , IF ANY, $h \in V\setminus\{1\}$ AND $S \subseteq V\setminus\{1\}$ s.t. $h \in S$ AND

  $y(V\setminus S, S) < 1 - y_{hh}$ (where $y_{hh} := 0$ if $h \in T$)

- CAPACITATED NETWORK ( $y_{ij}$ = CAPACITY ARC $(i,j)$ )



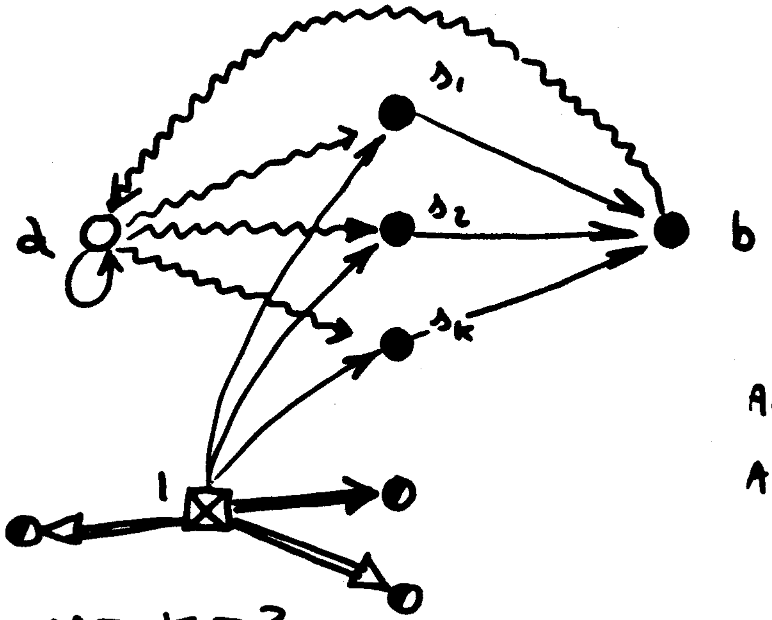

FIND THE MIN-CUT FROM NODE 1 TO NODE $h$

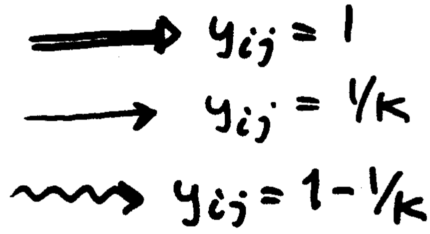
$\Rightarrow$ MAX-FLOW ALGORITHM

# A NEW CLASS OF FACETS

→ Let $a \in \overline{T}$ , $b, s_1, s_2, \ldots, s_k \in T$ $(k \geq 2)$.

→ Consider the following fractional point $y$ :




ALL IN-DEGREE EQ.S SATISFIED

ALL CONN. CONSTR.S SATISFIED

CASE $k = 3$

→ CHVÁTAL RANK ONE INEQUALITY :

$1/k$ TIMES $x(\{a,b,s_1,\ldots,s_k\},\{a,b,s_1,\ldots,s_k\}) \leq k+1$

$1-1/k$ TIMES $x_{a,s_j} + x_{s_j,a} + x_{aa} \leq 1$ , $j=1,\ldots,k$

$1-1/k$ TIMES $x(V, a) = 1$

$1-1/k$ TIMES $x(V, b) = 1$

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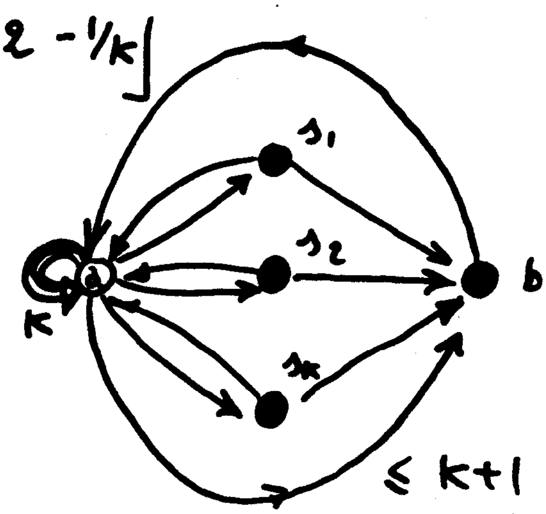
$$= \sum_{i,j} \lfloor \alpha_{ij} \rfloor x_{ij} \leq \lfloor k+2-1/k \rfloor$$

$$k\, x_{aa} + \sum_{j=1}^{k} (x_{as_j} + x_{s_j a}) + x_{ab} +$$

$$+ x_{ba} + \sum_{j=1}^{k} x_{s_j b} \leq k+1$$




SIMPLE $F_k$ - INEQUALITY

# SEQUENTIAL LIFTING ON LOOPS

- GIVEN $h \in T$, DEFINE

$$P'_n := \text{SA polytope with } T' := T \setminus \{h\}$$

($\rightarrow$ ADD THE LOOP $(h,h)$ TO THE ARC SET)

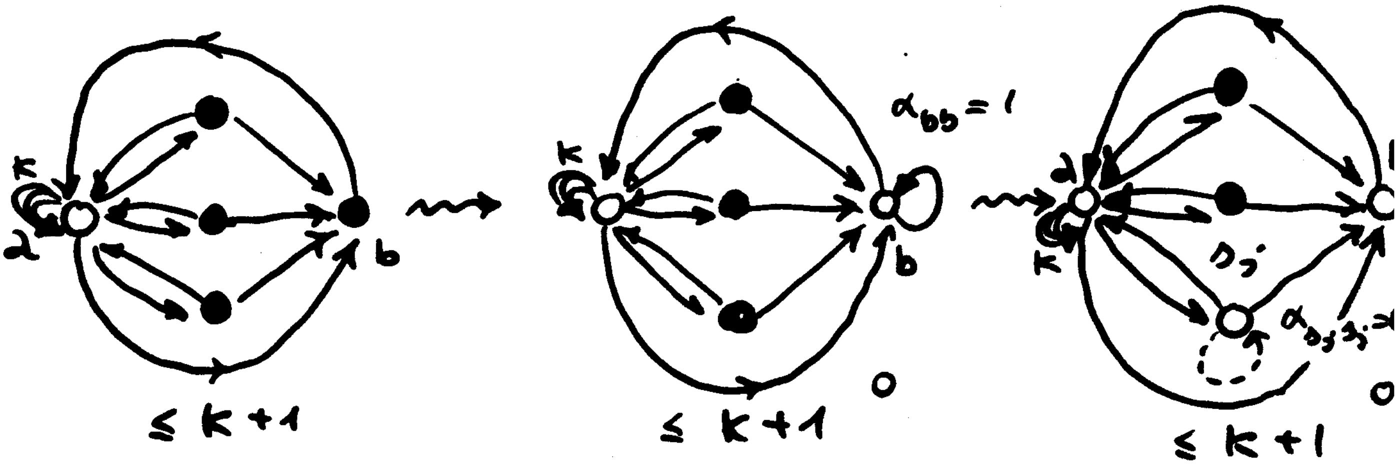
- GIVEN ANY (FACET-INDUCING) $\alpha x \leq \alpha_0$ FOR $P_n$, OBTAIN A (FACET-INDUCING) INEQUALITY FOR $P'_n$

$$\alpha x + \alpha_{hh} x_{hh} \leq \alpha_0 ,$$

WHERE $\alpha_{hh}$ MAX-VALUE s.t. $\alpha x + \alpha_{hh} x_{hh} \leq \alpha_0$ VALID FOR $P'_n$.

- FACETS SPANNING VERSION $\rightarrow$ FACETS NON-SPANNING VER
  - " Arborescence $\rightarrow$ Steiner Arborescence
  - " TSP $\rightarrow$ PC-TSP, G-TSP, ...

- APPLICATION TO SIMPLE $F_K$ - INEQUALITIES




$\rightsquigarrow \ldots$ $\alpha_{22} = K$ $\alpha_{ii} = 0 \quad \forall i \in V \setminus \{1,2,b\}$

$\alpha_{bb} = 1$

# NODE CLONING

* Let $h \in V$


$P_n :=$ SA polytope $n$ nodes, target set $T$

$P_{n+1} :=$ ... $n+1$ nodes, target set $T'$ where

$$T' := \begin{cases} T & \text{if } h \in \overline{T} \\ T \text{ or } T \cup \{n+1\} & \text{otherwise} \end{cases}$$

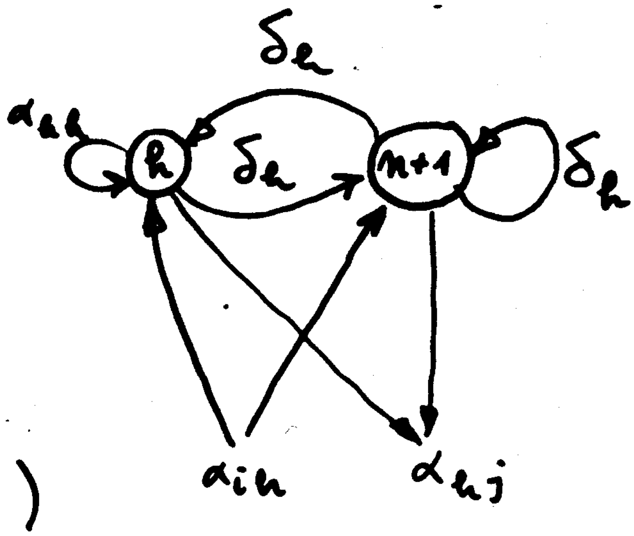
( $h$ Steiner node $\Rightarrow$ $n+1$ Steiner node as well )

* Given any nontrivial facet-inducing $\alpha x \le \alpha_0$ for $P_n$, compute

$$\delta_h := \begin{cases} 0 & \text{, if } h = 1 \\ \max\{\alpha_{ih} : (i,h) \in A\} & \text{, otherwise} \end{cases}$$

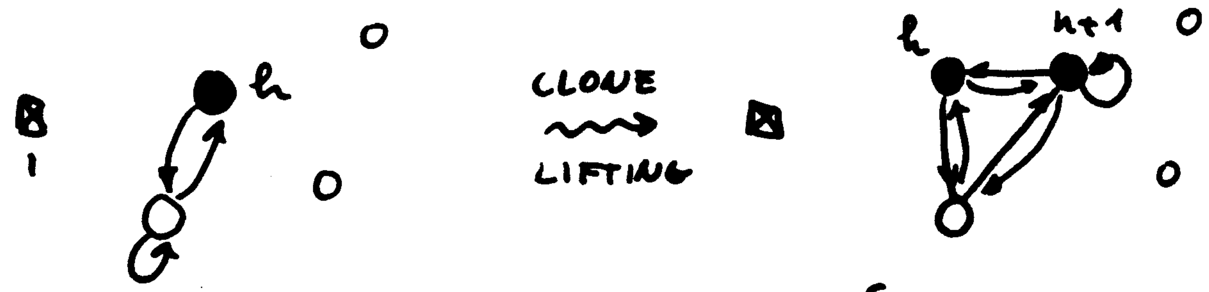
* Then the inequality :

$$\alpha x + \sum_{i \in V} \alpha_{ih} x_{i,n+1} + \sum_{j \in V} \alpha_{hj} x_{n+1,j} + \delta_h \left( x_{h,n+1} + x_{n+1,h} + x_{n+1,n+1} \right) \le \alpha_0 + \delta_h$$

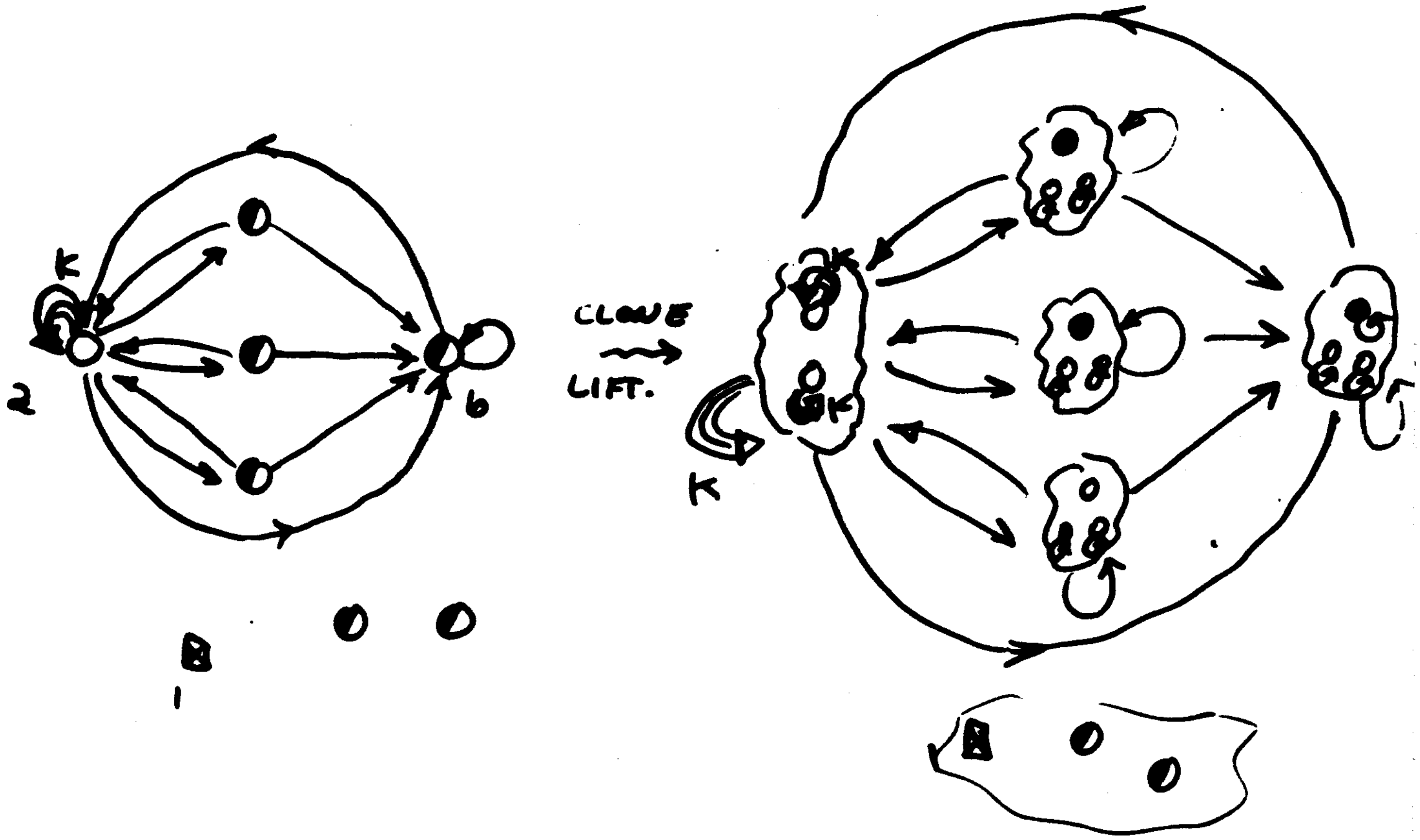



defines a FACET of $P_{n+1}$. Note that the new node $n+1$ acts as a "CLONE" of node $h$ (although $\alpha_{hh} \neq \alpha_{n+1,n+1}$ is allowed

## Applications




SEC : $\alpha x \leq \alpha_0$ $\qquad\qquad$ $\delta_h = 1$

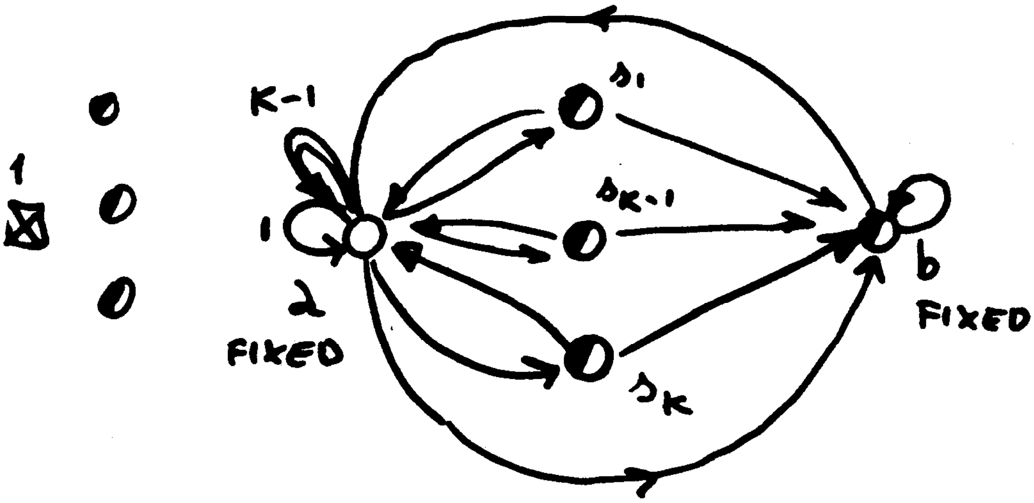



SIMPLE $F_K$ - INEQ.'S ⇝ GENERAL $F_K$ - INEQ.'S

# A POLYNOMIAL-TIME SEPARATION ALGORITHM FOR SIMPLE $F_K$ - INEQUALITIES

Let $y \in \mathbb{R}^A$ satisfy IN-DEGREE eq.s & SEC's.

Assume $a, b$ FIXED.




→ CONTRIBUTION of $s_K$ to the $F_K$ - ineq. :

Left-hand side: $y_{aa} + y_{a s_K} + y_{s_K a} + y_{s_K b}$

Right-hand side : 1

GREEDY ALGORITHM:

<u>FOR ALL</u> $a \in V \setminus \{1\}$ <u>DO</u>

  <u>FOR ALL</u> $b \in V \setminus \{1, a\}$ <u>DO</u>

    <u>BEGIN</u>

      $K := 0$;

      <u>FOR</u> <u>ALL</u> $j \in V \setminus \{1, a, b\}$ <u>DO</u>

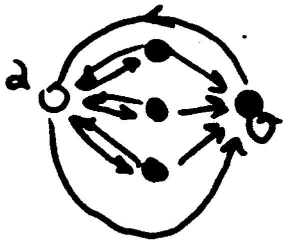
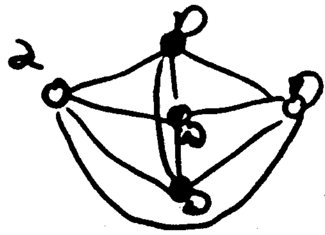
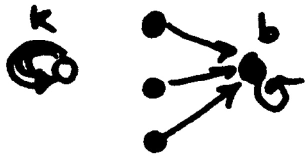
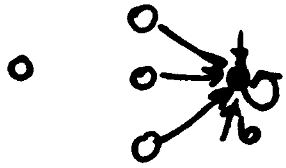
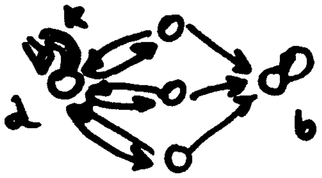
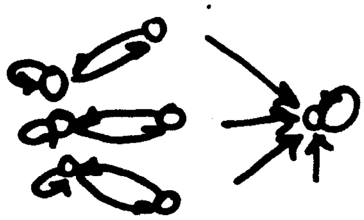
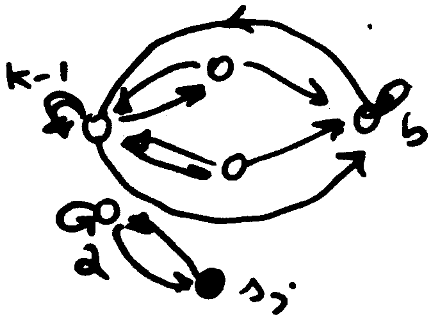
        <u>IF</u> $y_{ja} + y_{aj} + y_{aa} + y_{jb} > 1$ <u>THEN</u>

          <u>BEGIN</u> $K := K + 1$ ; $s_K := j$ <u>END</u> ;

      <u>IF</u> $K > 0$ <u>THEN</u> "CHECK THE SIMPLE $F_K$- INEQ. FOUND"

    <u>END</u>

MOREOVER :

| CASE | EQUIV. INEQ. | | DOMINATED BY | |
|---|---|---|---|---|
| $y_{aa} = 0$ |  | $\leq k+1$ |  | SEC<br>$\leq k+1$ |
| $y_{aa} = 1$ |  | |  | IN-DEGREE |
| $y_{ab} + y_{ba} = 0$ |  | |  | SEC's<br>IN-DEG |
| $y_{s_j b} = 0$ |  | |  | $F_{k-1}$-INEQ<br>SEC |

# IMPROVED IDENTIFICATION PROCEDURE:

**FOR ALL** $a \in V \setminus \{1\}$ s.t. $0 < y_{aa} < 1$ **DO**
  **FOR ALL** $b \in V \setminus \{1, a\}$ s.t. $y_{ab} + y_{ba} > 0$ **DO**
    **BEGIN**
      $k := 0$;
      **FOR ALL** $j \in V \setminus \{1, a, b\}$ s.t. $y_{jb} > 0$ **DO**
        **IF** $y_{ja} + y_{aj} + y_{aa} + y_{jb} > 1$ **THEN**
          **BEGIN** $k := k+1$ ; $s_k := j$ **END** ;
      **IF** $k > 0$ **THEN** "CHECK THE SIMPLE $F_k$- INEQ. FOUND"
    **END**