Mixed-Integer Cuts from Cyclic Groups

Matteo Fischetti
University of Padova, Italy
matteo.fischetti@unipd.it

Cristiano Saturni
University of Padova, Italy
cristiano.saturni@unipd.it

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Motivation

- Gomory cuts play a very important in modern MIP solvers
- Gomory cuts are easily read from the optimal tableau rows associated with fractional components (almost inexpensive to generate)
- **Question**:

Is it worth to invest more computing time in the attempt of improving Gomory cuts?

- Two possible answers:

1. Derive Gomory cuts from a more clever combination of the initial tableau rows
   $\longrightarrow$ M.F. and A. Lodi "Optimizing over the first Chvàtal closure"
2. **Given a fractional row of the optimal tableau, look for a most-violated cut within a wide family (including Gomory cuts)**
   $\longrightarrow$ **this talk.**

The Master Cyclig Group Polyhedron

- We study the Integer Linear Program (ILP):

$$\min\{c^T x : Ax = b, x \geq 0 \text{ integer}\} \tag{1}$$

where A is a rational $m \times n$ matrix, and the two associated polyhedra:

$$P := \{x \in \mathbb{R}^n_+ : Ax = b\} \tag{2}$$

$$P_I := conv\{x \in \mathbb{Z}^n_+ : Ax = b\} = conv(P \cap \mathbb{Z}^n) \ . \tag{3}$$

- We propose an exact separation procedure for the class of **interpolated** (or *template*) **subadditive cuts** based on the characterization of Gomory and Johnson (1972) of the following **master cyclic group polyhedron**:

$$T(k,r) = conv\{t \in \mathbb{Z}^{k-1}_+ : \sum_{i=1}^{k-1} (i/k) \cdot t_i \equiv r/k \text{ (mod 1)}\} \tag{4}$$

where $k \geq 2$ (group order) and $r \in \{1, \cdots, k-1\}$ are given integers

- The space $\mathbb{R}^{k-1}$ of the t variables is called the T*-space*

Previous work

- It is known that the mapping the original x-variable space into the T-space allows one to use polyhedral information on $T(k,r)$ to derive valid inequalities for P_I (**Gomory and Johnson**, 1972)

- Recent papers by **Gomory, Johnson, Araoz, and Evans** and by **Dash and Gunluk** deal with the Gomory's **shooting experiment**: the point $t^* \in \mathbb{R}^{k-1}$ to be separated is generated at random (hence it corresponds to a random "shooting direction" in the T-space), and statistics on the frequency of the most-violated facets of $T(k,r)$ are collected

- **Koppe, Louveaux, Weismantel and Wolsey** (2004) study a compact formulation of the cyclic-group separation problem is embedded into the original ILP model—huge formulation with limited practical applications

- **Letchford and Lodi** (2002) and **Cornuejols, Li and Vandenbussche** (2003) address specific subfamilies of cyclic-group cuts

- To our knowledge, the **practical** benefit that can be obtained by **implementing** these cuts in a cutting plane algorithm was not investigated computationally by previous authors

Separation over the Group Polyhedron

- Given any equation

$$\alpha^T x = \beta \tag{5}$$

valid for P_I, where $(\alpha, \beta) \in \mathbb{R}^{n+1}$ and β fractional, we consider the **group polyhedron** (in the x-space)

$$G(\alpha, \beta) := conv\{x \in \mathbb{Z}^n_+ : \sum_{j=1}^{n} \alpha_j x_j \equiv \beta \text{ (mod 1)}\} \supseteq P_I \ . \tag{6}$$

- E.g., the equation $\alpha^T x = \beta$ can be obtained by setting $(\alpha, \beta)^T := u^T(A, b)$ for any $u \in \mathbb{R}^m$ such that $u^T b$ is fractional $\Rightarrow$ e.g., an equation read from the tableau associated with a fractional optimal solution of the LP relaxation

- **Separation problem (g-SEP): Given any point $x^* \geq 0$ and the equation $\alpha^T x = \beta$ with rational coefficients and fractional β, find (if any) a valid inequality for $G(\alpha, \beta)$ that is violated by x^***

Cuts from Subadditive Functions

- We call a function $g : \mathbb{R} \rightarrow \mathbb{R}_+$ **subadditive** if

  1. $g(a + b) \leq g(a) + g(b)$ for any $a, b \in \mathbb{R}$

  and, in addition,

  2. $g(\cdot)$ is periodic in $[0, 1)$, i.e., $g(a + 1) = g(a)$ for all $a \in \mathbb{R}$
  3. $g(0) = 0$

- Gomory and Johnson (1970) showed that, given the equation $\alpha^T x = \beta$, **all** the nontrivial facets of $G(\alpha, \beta)$ are defined by inequalities of the type

$$\sum_{j=1}^{n} g(\alpha_j) x_j \geq g(\beta) \tag{7}$$

  with $g(\cdot)$ subadditive $\Rightarrow$ g-SEP can be rephrased as follows:

- Separation problem (g-SEP): Given any point $x^* \geq 0$ and the equation $\alpha^T x = \beta$ with rational coefficients and fractional β, **find a subadditive function** $g(\cdot)$ such that $\sum_{j=1}^{n} g(\alpha_j) x_j^* < g(\beta)$

Examples

- Taking $g(\cdot) = \phi(\cdot)$ (fractional part) one obtains the well-know **Gomory fractional cut** (1958):

$$\sum_{j=1}^{n} \phi(\alpha_j) x_j \geq \phi(\beta) \; ,$$

- Taking the subadditive **GMI function** $\gamma^{\beta}(\cdot)$ defined as

$$\gamma^{\beta}(a) = \begin{cases} \phi(a) & \text{if } \phi(a) \leq \phi(\beta) \\ \phi(\beta) \frac{1-\phi(a)}{1-\phi(\beta)} & \text{otherwise} \end{cases} \quad \text{for all } a \in \mathbb{R} \tag{8}$$

one obtains the stronger **Gomory Mixed-Integer (GMI)** cut:

$$\sum_{j=1}^{n} \gamma^{\beta}(\alpha_j)\, x_j \geq \gamma^{\beta}(\beta) = \phi(\beta) \; . \tag{9}$$

Illustration

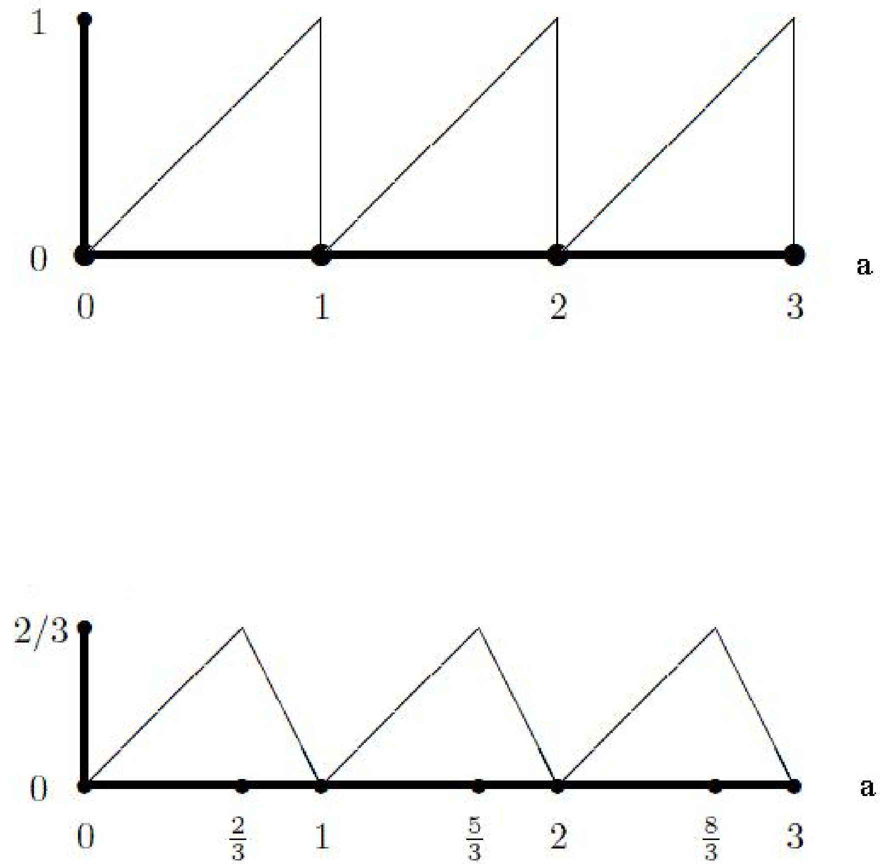

Figure 1: Two subadditive functions: the fractional part $\phi(\cdot)$ (top) and the GMI function $\gamma^{2/3}(\cdot)$ (bottom).

A separation algorithm for subadditive cuts

- Given the equation $\alpha^T x = \beta$, let $k \geq 2$ be the smallest integer such that $k(\alpha, \beta)$ is integer (called **ideal** k)

- The subadditivity of $g(\cdot)$ implies that the same property holds over the discrete set $\{0, 1/k, 2/k, \cdots, (k-1)/k\}$ $\Rightarrow$ a **necessary** condition for subadditivity is that the "sampled" values $g_i := g(i/k)$ satisfy the following g-**system**:

$$\begin{cases} g_h \leq g_i + g_j, & 1 \leq i, j, h \leq k-1 \text{ and } i + j \equiv h \pmod{k} \\ g_0 = 0, & \\ 0 \leq g_i \leq 1, & i = 1, \cdots, k-1 \end{cases} \tag{10}$$

  where bounds $0 \leq g_i \leq 1$ play a normalization role.

- **However ... we also need to compute the value of $g(\cdot)$ outside the sample points $1/k, 2/k, \cdots, (k-1)/k$ so as to get the required subadditive function $g : \mathbb{R} \rightarrow \mathbb{R}_+$**

Interpolation

- Any solution $(g_0, \cdots, g_{k-1})$ of the g-system above can be completed so as to define a subadditive function $g : \mathbb{R} \longrightarrow \mathbb{R}_+$ through a simple **interpolation procedure** due to Gomory and Johnson (1972):
  1. take a linear interpolation of the values $g_0, \cdots, g_{k-1}$ over $[0, 1)$,
  2. extend the resulting piecewise-linear function to $\mathbb{R}$, in the obvious periodic way

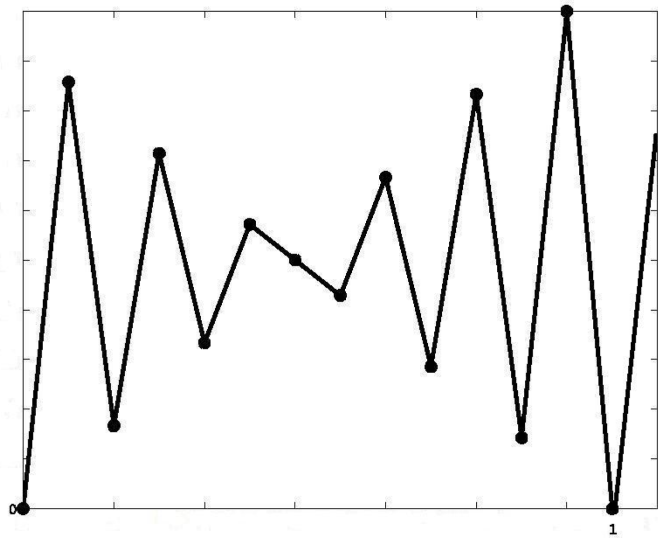

Figure 2: The Gomory-Johnson interpolation procedure

T-space separation

- A given x^* violates a cut of the form

$$\sum_{j=1}^{n} g(\alpha_j) x_j \geq g(\beta)$$

iff

$$\sum_{j=0}^{n} g(\alpha_j) x_j^* < 0$$

where $\alpha_0 := \beta$ and $x_0^* := -1$ to simplify notation

- Observation: k **ideal** $\Rightarrow$ the value of $g(\cdot)$ outside the sample points i/k is immaterial

$$\sum_{j=0}^{n} g(\alpha_j) x_j^* = \sum_{i=1}^{k-1} g(i/k) \, [\sum_{j:\phi(\alpha_j)=i/k} x_j^*] =: \sum_{i=1}^{k-1} g(i/k) \; t_i^*$$

- Hence we can model *g-SEP* **exactly** as the following LP (in the T-space):

$$g - SEP_k : \quad \min\{ \sum_{i=1}^{k-1} t_i^* g_i : \text{ ``}g\text{-system'' } \} \;, \tag{11}$$

Dealing with a nonideal k

- Unfortunately, the ideal k is very often too large to be used in practice $\Rightarrow$ choose a smaller value in order to produce a manageable g-system

- In this case, the interpolation procedure **does restrict** (often considerably) the range of subadditive functions that can be captured by $g - SEP_k$

- **Modified definition of the weights t_i^* needed to take interpolation into account**

- For any given integer $k \geq 2$ (not necessarily ideal), the separation weights t_i^* are defined through the following "splitting" algorithm:

```
1. define the fictitious values α_0 := β and x*_0 := −1;
2. initialize t*_0 := t*_1 := ··· := t*_{k−1} := 0;
2. for j = 0, 1, ···, n such that x*_j > 0 and φ(α_j) > 0 do
3.    let i := ⌊k φ(α_j)⌋ and h = i + 1 mod k;
4.    let θ := kφ(α_j) − i;
5.    update t*_i := t*_i + (1 − θ)x*_j and t*_h := t*_h + θx*_j
6. enddo
```

Weakness of interpolation

- Observe that, for the interpolated function $g(\cdot)$, we sometimes have $g(a) > g(\beta) \;\Rightarrow$ an interpolated subadditive cut $\sum_{j=1}^{n} g(\alpha_j) x_j \geq g(\beta)$ can easily be improved to its **clipped** form:

$$\sum_{j=1}^{n} \min\{g(\alpha_j), g(\beta)\} x_j \geq g(\beta) \tag{12}$$

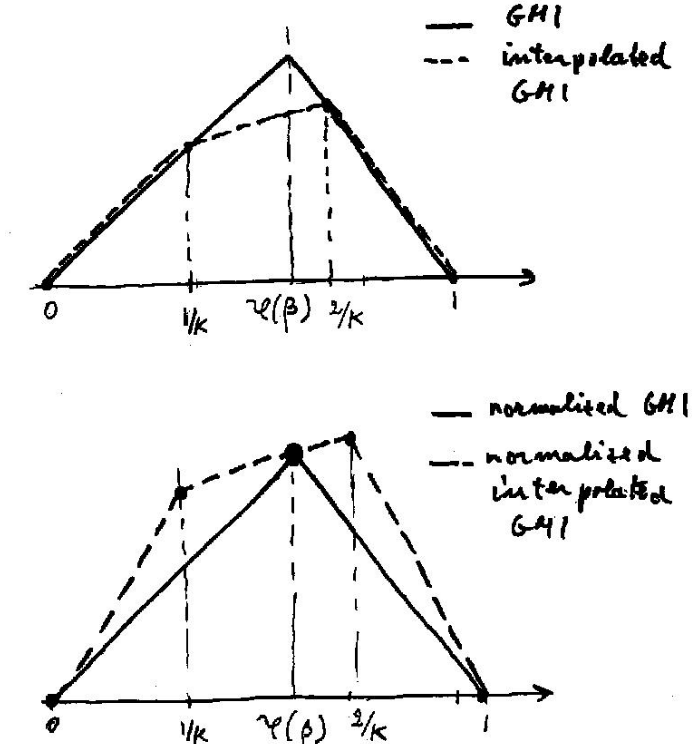

Figure 3: GMI and interpolated GMI functions (normalization of the rhs value)

Dealing with continuous variables

- Mixed-integer case: some variables x_j with $j \in \mathcal{C}$ (say) are not restricted to be integer valued

- Gomory and Johnson (1972) showed that, for any subadditive function $g(\cdot)$, it is enough to modify cut

$$\sum_{j=1}^{n} g(\alpha_j) x_j \geq g(\beta)$$

into

$$\sum_{j \in \mathcal{I}}^{n} g(\alpha_j) x_j + \sum_{j \in \mathcal{C}: \alpha_j > 0} slope_{+}\, \alpha_j x_j + \sum_{j \in \mathcal{C}: \alpha_j < 0} slope_{-}\, \alpha_j x_j \geq g(\beta) \ , \qquad (13)$$

where

$\mathcal{I} := \{1, \cdots, n\} \setminus \mathcal{C}$ is the index set of the integer variables,
$slope_{+} := \lim_{\delta \to 0^{+}} g(\delta)/\delta$ is the slope of $g(\cdot)$ in 0^{+}, and
$slope_{-} := \lim_{\delta \to 0^{-}} g(\delta)/\delta$ is the slope of $g(\cdot)$ in 0^{-} (or, equivalently, in 1^{-})

- Intuitive explanation based on a simple **scaling argument** $\Rightarrow$ one can deal with continuous variables without any modification of the separation procedure (used as a black box)

Computational experiments

- Preliminary computational analysis aimed at comparing the quality of Gomory mixed-integer cuts with that of the interpolated sudadditive cuts, when embedded in a pure cutting plane method

- Test-bed includes MIPLIB 3.0 instances (reformulated is standard form)

- After the solution of first LP relaxation of our model, we store in our equation pool all the tableau rows $\alpha^T x = \beta$ with fractional right-hand side β.

- This pool is never updated during the run, i.e., we deliberately avoid generating subadditive cuts of rank greater than 1

- At each round of separation, at most 200 cuts are generated

- Each run is aborted at the root node, i.e., no branching is allowed.

Lessons learned

- As reported by other authors, **GMI cuts are hard to beat**

- For a given equation $\alpha^T x = \beta$, a GMI cut often captures (alone) the power of the whole family of subadditive cuts based on that equation $\Rightarrow$ **a single GMI cut is often sufficient to bring x^* inside the corresponding group polyhedron $G(\alpha, \beta)$**

- Interpolated subadditive cuts typically become competitive with (or better than) GMI cuts for $k \geq 20$, though their separation requires a substantial computing-time overhead

- Large number of subadditive cuts generated and the small improvement obtained in some cases $\Rightarrow$ a more conservative policy that generates GMI cuts first, and only afterward resorts to *g-SEP* to generate new violated subadditive cuts

- Better compromise between lower bound quality and computing time: use a clever set of non-interpolated subadditive functions (GMI, k-cuts or other template functions) first, and apply *g-SEP* separation only afterwards

- This goes into the direction suggested by Andreello, Caprara and Fischetti (2003) for an effective use of easy-to-compute cuts such as GMI and k-cuts