Mixed-Integer Cuts from Cyclic Groups

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Motivation

- Gomory cuts play a very important in modern MIP solvers
- Gomory cuts are easily read from the optimal tableau rows associated with fractional components (almost inexpensive to generate)
- **Question**:

  Is it worth to invest more computing time in the attempt of improving Gomory cuts?

- Three possible answers:

  1. Derive standard Gomory cuts from the optimal tableau, and improve them afterwards
     $\longrightarrow$ Balas and Perregaard (2003), Andersen, Cornuejols and Li (2004), etc.
  2. Derive Gomory cuts from a more clever combination of the initial tableau rows
     $\longrightarrow$ M.F. and A. Lodi "Optimizing over the first Chvàtal closure"
  3. **Given a fractional row of the optimal tableau, look for a most-violated cut within a wide family (including Gomory cuts)**
     $\longrightarrow$ **this talk.**

The Master Cyclig Group Polyhedron

- We study the Integer Linear Program (ILP):

$$\min\{c^T x : Ax = b, x \geq 0 \text{ integer}\} \tag{1}$$

where A is a rational $m \times n$ matrix, and the two associated polyhedra:

$$P := \{x \in \mathbb{R}^n_+ : Ax = b\} \tag{2}$$

$$P_I := conv\{x \in \mathbb{Z}^n_+ : Ax = b\} = conv(P \cap \mathbb{Z}^n) \; . \tag{3}$$

- We propose an exact separation procedure for the class of **interpolated** (or *template*) **subadditive cuts** based on the characterization of Gomory and Johnson (1972) of the following **master cyclic group polyhedron**:

$$T(k,r) = conv\{t \in \mathbb{Z}^{k-1}_+ : \sum_{i=1}^{k-1} (i/k) \cdot t_i \equiv r/k \text{ (mod 1)}\} \tag{4}$$

where $k \geq 2$ (group order) and $r \in \{1, \cdots, k-1\}$ are given integers

- The space $\mathbb{R}^{k-1}$ of the t variables is called the T-*space*

Previous work

- It is known that the mapping the original x-variable space into the T-space allows one to use polyhedral information on $T(k,r)$ to derive valid inequalities for P_I (**Gomory and Johnson**, 1972)

- Recent papers by **Gomory, Johnson, Araoz, and Evans** and by **Dash and Gunluk** deal with the Gomory's **shooting experiment**: the point $t^* \in \mathbb{R}^{k-1}$ to be separated is generated at random (hence it corresponds to a random "shooting direction" in the T-space), and statistics on the frequency of the most-violated facets of $T(k,r)$ are collected

- **Koppe, Louveaux, Weismantel and Wolsey** (2004) study a compact (but huge) formulation of the cyclic-group separation problem is embedded into the original ILP model

- **Letchford and Lodi** (2002) and **Cornuejols, Li and Vandenbussche** (2003) address specific subfamilies of cyclic-group cuts

- To our knowledge, the **practical** benefit that can be obtained by **implementing** these cuts in a cutting plane algorithm was not investigated computationally by previous authors

Separation over the Group Polyhedron

- Given any equation

$$\alpha^T x = \beta \tag{5}$$

valid for P_I, where $(\alpha, \beta) \in \mathbb{R}^{n+1}$ and β fractional, we consider the **group polyhedron** (in the x-space)

$$G(\alpha, \beta) := conv\{x \in \mathbb{Z}^n_+ : \sum_{j=1}^{n} \alpha_j x_j \equiv \beta \text{ (mod 1)}\} \supseteq P_I \ . \tag{6}$$

- E.g., the equation $\alpha^T x = \beta$ can be obtained by setting $(\alpha, \beta)^T := u^T(A, b)$ for any $u \in \mathbb{R}^m$ such that $u^T b$ is fractional $\Rightarrow$ the equation is read from the tableau associated with a fractional optimal solution of the LP relaxation

- **Separation problem (g-SEP): Given any point $x^* \geq 0$ and the equation $\alpha^T x = \beta$ with rational coefficients and fractional β, find (if any) a valid inequality for $G(\alpha, \beta)$ that is violated by x^***

Cuts from Subadditive Functions

- We call a function $g : \mathbb{R} \rightarrow \mathbb{R}_+$ **subadditive** if

  1. $g(a+b) \leq g(a) + g(b)$ for any $a, b \in \mathbb{R}$

  and, in addition,

  2. $g(\cdot)$ is periodic in $[0, 1)$, i.e., $g(a+1) = g(a)$ for all $a \in \mathbb{R}$
  3. $g(0) = 0$

- Gomory and Johnson (1970) showed that, given the equation $\alpha^T x = \beta$, **all** the nontrivial facets of $G(\alpha, \beta)$ are defined by inequalities of the type

$$\sum_{j=1}^{n} g(\alpha_j) x_j \geq g(\beta) \tag{7}$$

  with $g(\cdot)$ subadditive $\Rightarrow$ g-SEP can be rephrased as follows

- **Separation problem (g-SEP): Given any point $x^* \geq 0$ and the equation $\alpha^T x = \beta$ with rational coefficients and such that $\phi(\beta) > 0$, find a subadditive function $g(\cdot)$ such that $\sum_{j=1}^{n} g(\alpha_j) x_j^* < g(\beta)$**

Examples

- Taking $g(\cdot) = \phi(\cdot)$ (fractional part) one obtains the well-know **Gomory fractional cut** (1958):

$$\sum_{j=1}^{n} \phi(\alpha_j) x_j \geq \phi(\beta) \ ,$$

- Taking the subadditive **GMI function** $\gamma^\beta(\cdot)$ defined as

$$\gamma^\beta(a) = \begin{cases} \phi(a) & \text{if } \phi(a) \leq \phi(\beta) \\ \phi(\beta)\frac{1-\phi(a)}{1-\phi(\beta)} & \text{otherwise} \end{cases} \quad \text{for all } a \in \mathbb{R} \tag{8}$$

one obtains the stronger **Gomory Mixed-Integer (GMI)** cut:

$$\sum_{j=1}^{n} \min\{\phi(\alpha_j),\ \phi(\beta)\frac{1-\phi(\alpha_j)}{1-\phi(\beta)}\}\, x_j \geq \phi(\beta) \ . \tag{9}$$

Illustration

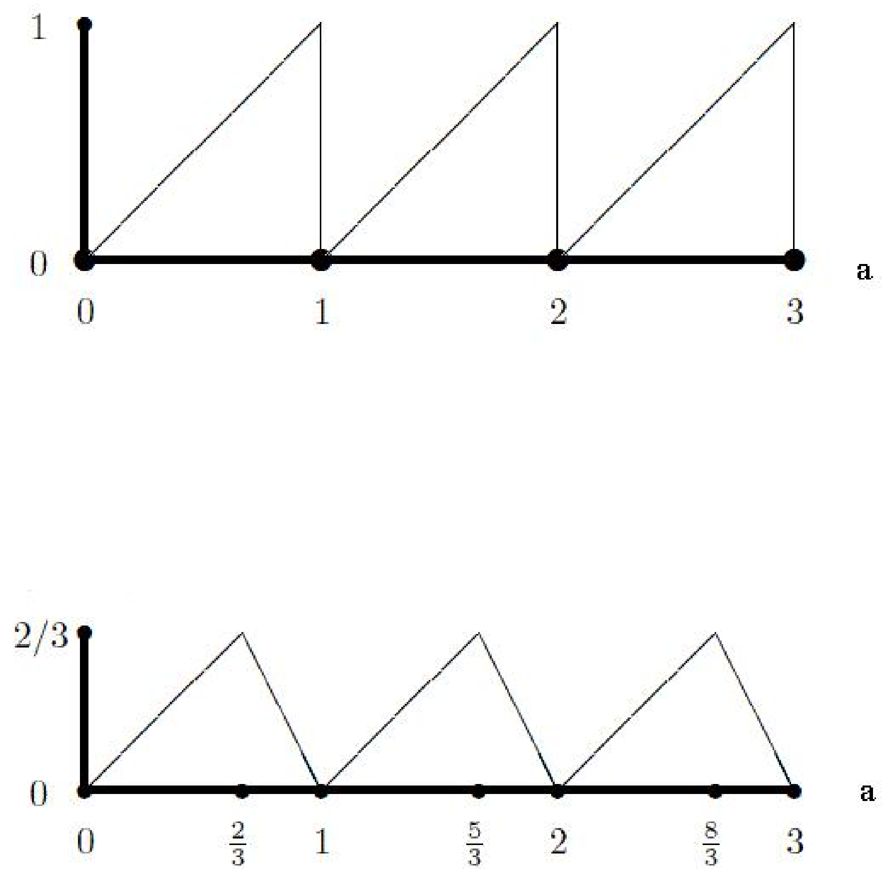

Figure 1: Two subadditive functions: the fractional part $\phi(\cdot)$ (top) and the GMI function $\gamma^{2/3}(\cdot)$ (bottom).

A separation algorithm for subadditive cuts

- Given the equation $\alpha^T x = \beta$, let $k \geq 2$ be the smallest integer such that $k(\alpha, \beta)$ is integer (**ideal** k)

- The subadditivity of $g(\cdot)$ implies that the same property holds over the discrete set $\{0, 1/k, 2/k, \cdots, (k-1)/k\}$ $\Rightarrow$ a necessary condition for subadditivity is that the "sampled" values $g_i := g(i/k)$ satisfy the following g-**system**:

$$\begin{cases} g_h \leq g_i + g_j, & 1 \leq i, j, h \leq k-1 \text{ and } i + j \equiv h \pmod{k} \\ g_0 = 0, & \\ 0 \leq g_i \leq 1, & i = 1, \cdots, k-1 \end{cases} \tag{10}$$

where bounds $0 \leq g_i \leq 1$ play a normalization role.

- **However ... we also need to compute the value of $g(\cdot)$ outside the sample points $1/k, 2/k, \cdots, (k-1)/k$ so as to get the required subadditive function $g : \mathbb{R} \rightarrow \mathbb{R}_+$**

Interpolation

- Any solution $(g_0, \cdots, g_{k-1})$ of the g-system above can be completed so as to define a subadditive function $g : \mathbb{R} \longrightarrow \mathbb{R}_+$ through a simple **interpolation procedure** due to Gomory and Johnson (1972):
  1. take a linear interpolation of the values $g_0, \cdots, g_{k-1}$ over $[0, 1)$,
  2. extend the resulting piecewise-linear function to $\mathbb{R}$, in the obvious periodic way

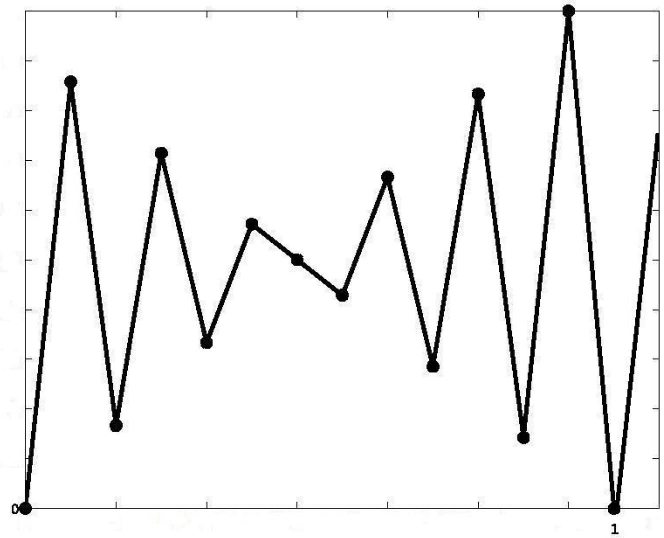

Figure 2: The Gomory-Johnson interpolation procedure

T-space separation

- The given x^* violates a cut of the form

$$\sum_{j=1}^{n} g(\alpha_j)x_j \geq g(\beta) \rightarrow \sum_{j=1}^{n} g(\alpha_j)x_j + g(\beta)(-1) \geq 0$$

  iff

$$\sum_{j=1}^{n} g(\alpha_j)x_j^* + g(\alpha_0)x_0^* = \sum_{j=0}^{n} g(\alpha_j)x_j^* < 0$$

  where $\alpha_0 := \beta$ and $x_0^* := -1$ to simplify notation

- Observation: k **ideal** $\Rightarrow$ the value of $g(\cdot)$ outside the sample points i/k is immaterial

$$\sum_{j=0}^{n} g(\alpha_j)x_j^* = \sum_{i=1}^{k-1} g(i/k)\,[\sum_{j:\phi(\alpha_j)=i/k} x_j^*] =: \sum_{i=1}^{k-1} g(i/k)t_i^*$$

- Hence we can model *g-SEP* **exactly** as the following LP (in the T-space):

$$g - SEP_k : \quad \min\{\sum_{i=1}^{k-1} t_i^* g_i : \text{ ``}g\text{-system''} \} , \tag{11}$$

Dealing with a nonideal k

- Unfortunately, the ideal k is very often too large to be used in practice $\Rightarrow$ choose a smaller value in order to produce a manageable g-system

- In this case, the interpolation procedure **does restrict** (often considerably) the range of subadditive functions that can be captured by $g - SEP_k$

- **Modified definition of the weights t_i^* needed to take interpolation into account**

- For any given integer $k \geq 2$ (not necessarily ideal), the separation weights t_i^* are defined through the following "splitting" algorithm:

1. define the fictitious values $\alpha_0 := \beta$ and $x_0^* := -1$;
2. initialize $t_0^* := t_1^* := \cdots := t_{k-1}^* := 0$;
2. for $j = 0, 1, \cdots, n$ such that $x_j^* > 0$ and $\phi(\alpha_j) > 0$ do
3. let $i := \lfloor k\,\phi(\alpha_j) \rfloor$ and $h = i + 1 \bmod k$;
4. let $\theta := k\phi(\alpha_j) - i$;
5. update $t_i^* := t_i^* + (1 - \theta)x_j^*$ and $t_h^* := t_h^* + \theta x_j^*$
6. enddo

Weakness of interpolation

- Observe that, for the interpolated function $g(\cdot)$, we sometimes have $g(a) > g(\beta) \ \Rightarrow$ an interpolated subadditive cut $\sum_{j=1}^{n} g(\alpha_j)x_j \geq g(\beta)$ can easily be improved to its **clipped** form:

$$\sum_{j=1}^{n} \min\{g(\alpha_j), g(\beta)\}x_j \geq g(\beta) \tag{12}$$

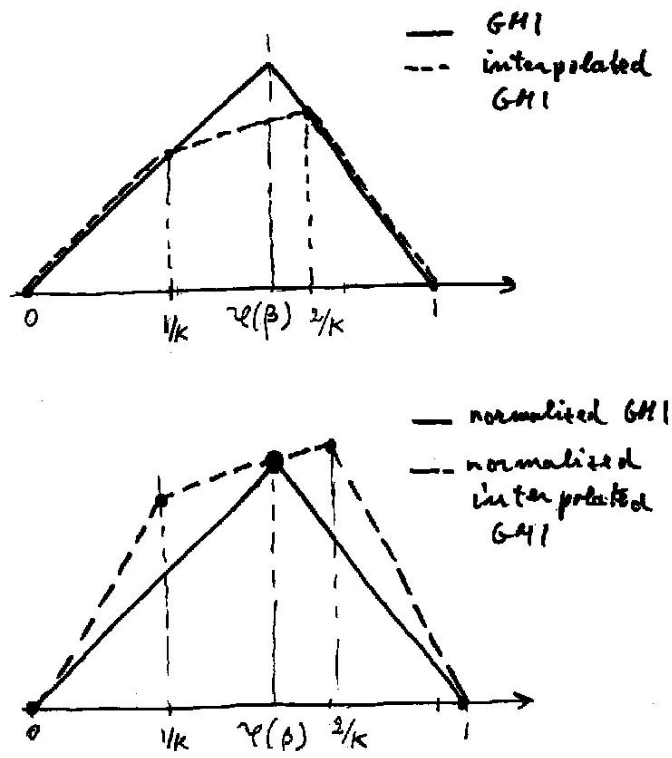

Figure 3: GMI and interpolated GMI functions (normalization of the rhs value)

Dealing with continuous variables

- Mixed-integer case: some variables x_j with $j \in \mathcal{C}$ (say) are not restricted to be integer valued

- Gomory and Johnson (1972) showed that, for any subadditive function $g(\cdot)$, it is enough to modify cut

$$\sum_{j=1}^{n} g(\alpha_j) x_j \geq g(\beta)$$

into

$$\sum_{j \in \mathcal{I}}^{n} g(\alpha_j) x_j + \sum_{j \in \mathcal{C}: \alpha_j > 0} slope_+ \, \alpha_j x_j + \sum_{j \in \mathcal{C}: \alpha_j < 0} slope_- \, \alpha_j x_j \geq g(\beta) \ , \qquad (13)$$

where

$\mathcal{I} := \{1, \cdots, n\} \setminus \mathcal{C}$ is the index set of the integer variables,
$slope_+ := \lim_{\delta \to 0^+} g(\delta)/\delta$ is the slope of $g(\cdot)$ in 0^+, and
$slope_- := \lim_{\delta \to 0^-} g(\delta)/\delta$ is the slope of $g(\cdot)$ in 0^- (or, equivalently, in 1^-)

- Intuitive explanation based on a simple **scaling argument** $\Rightarrow$ one can deal with continuous variables without any modification of the separation procedure (used as a black box)

Computational experiments

- Extensive computational analysis aimed at comparing the quality of Gomory mixed-integer cuts with that of the interpolated sudadditive cuts, when embedded in a pure cutting plane method

- Test-bed includes MIPLIB 3.0/2003 instances

- After the solution of first LP relaxation of our model, we store in our equation pool all the tableau rows $\alpha^T x = \beta$ with fractional right-hand side β.

- This pool is never updated during the run, i.e., we deliberately avoid generating subadditive cuts of rank greater than 1

- At each round of separation, at most 200 cuts are generated

- Each run is aborted at the root node, i.e., no branching is allowed.

Computational results: MIPLIB Pure Integer Problems

Problem	Type of Cuts	Final LB	Closed Gap	Separation Time	Total Time	Number of Cuts
air04	GMI	55583.78	8.04%	2.12	352.16	202
56137.00	K=10	55580.69	7.52%	3.16	539.35	283
55535.44	K=20⋆	55583.86	8.05%	6.38	537.86	300
37.98	K=30⋆	55585.19	8.27%	10.61	614.55	370
	K=60⋆	55586.21	8.44%	208.45	757.56	389
l152lav	GMI	4664.41	12.25%	0.07	0.41	51
4722.00	K=10	4664.03	11.67%	0.21	0.85	88
4656.36	K=20⋆	4664.60	12.54%	0.54	1.20	88
0.08	K=30⋆	4665.26	13.55%	2.29	3.65	237
	K=60⋆	4665.87	14.48%	90.01	93.08	349
lseu	GMI	991.87	55.09%	0.00	0.00	13
1120.00	K=10⋆	996.29	56.64%	0.01	0.02	22
834.68	K=20⋆	997.34	57.01%	0.08	0.08	25
0.00	K=30⋆	998.64	57.47%	0.41	0.43	42
	K=60⋆	1000.29	58.04%	8.78	8.78	31
mod010	GMI	6535.50	21.47%	0.06	0.44	34
6548.00	K=10	6535.46	21.24%	0.13	0.70	38
6532.08	K=20	6535.46	21.24%	0.16	0.66	36
0.08	K=30⋆	6535.75	23.04%	0.25	0.74	36
	K=60⋆	6536.00	24.61%	3.07	3.61	40
harp2	GMI	−74251958.32	22.35%	0.03	0.20	30
−73899798.00	K=10⋆	−74247224.08	23.40%	0.20	0.73	58
−74353341.50	K=20⋆	−74236993.08	25.65%	0.36	0.99	62
0.03	K=30⋆	−74236058.30	25.86%	1.02	1.76	71
	K=60⋆	−74225928.01	28.09%	28.39	29.32	75

Computational results: MIPLIB Mixed-Integer Problems

Problem	Type of Cuts	Final LB	Closed Gap	Separation Time	Total Time	Number of Cuts
bell5 8966406.49 8608417.95 0.01	GMI	8660422.46	14.53%	0.00	0.00	40
	K=10	8654669.96	12.92%	0.03	0.07	59
	K=20	8657274.83	13.65%	0.12	0.14	60
	K=30	8658662.54	14.04%	0.44	0.45	59
	K=60⋆	8661152.43	14.73%	13.15	13.16	77
mas74 11801.20 10482.80 0.00	GMI	10570.72	6.67%	0.00	0.01	12
	K=10⋆	10570.94	6.69%	0.01	0.04	33
	K=20⋆	10576.54	7.11%	0.14	0.17	44
	K=30⋆	10581.80	7.51%	0.86	0.94	71
	K=60⋆	10585.87	7.82%	27.94	28.00	79
mas76 40005.10 38893.90 0.00	GMI	38965.29	6.42%	0.00	0.01	11
	K=10⋆	38968.36	6.70%	0.00	0.04	25
	K=20⋆	38972.76	7.10%	0.09	0.14	43
	K=30⋆	38975.64	7.36%	0.48	0.52	34
	K=60⋆	38977.76	7.55%	20.09	20.16	52
mkc −563.85 −611.85 0.11	GMI	−609.41	5.09%	0.73	1.22	142
	K=10⋆	−609.32	5.27%	5.18	7.09	367
	K=20⋆	−609.32	5.27%	8.70	11.40	463
	K=30⋆	−609.08	5.76%	16.68	20.28	600
	K=60⋆	−608.92	6.11%	408.78	416.19	958
qnet1 16029.69 14274.10 0.04	GMI	14445.72	9.78%	0.10	0.27	55
	K=10⋆	14446.24	9.80%	0.20	0.48	59
	K=20⋆	14446.24	9.80%	0.37	0.63	63
	K=30⋆	14446.24	9.80%	1.04	1.34	69
	K=60⋆	14447.10	9.85%	24.51	24.86	75

Lessons learned

- As reported by other authors (k-cuts etc.), **GMI cuts are hard to beat** by just using clever subadditive functions

- For a given equation $\alpha^T x = \beta$, a GMI cut often captures (alone) the power of the whole family of subadditive cuts based on that equation $\Rightarrow$ **a single GMI cut is often sufficient to bring x^* inside the corresponding group polyhedron** $G(\alpha, \beta)$

- Negative role of interpolation: interpolated subadditive cuts typically become competitive with (or slightly better than) GMI cuts for $k \geq 20$, though their separation requires a substantial computing-time overhead

- Future work should address the possibility of exploiting 2 (or more) tableau rows at the same time, so as to better approximate the optimization over **Gomory's corner polyhedron**:

$$\min\{c^T x : x_B + B^{-1} N x_N \equiv B^{-1} b \text{ (mod 1)}, x \geq 0 \text{ integer}\} \tag{14}$$

- **But ... is this worth doing? In other words: who knows how tight is this relaxation?**

Just ask!

			LP relaxation	Corner	GMI	1^{st} closure
ID	B	I	% gap	% gap closed	% gap closed	% gap closed
air03	10757	0	0.38	100.00	100.00	100.00
cap6000	6000	0	0.01	21.42	41.65	26.90
l152lav	1989	0	1.39	14.68	12.25	69.20
mitre	10724	0	0.36	$\geq$ 46.02	82.20	100.00
mod008	319	0	5.23	[23.66 - 62.66]	20.88	100.00
mod010	2655	0	0.24	100.00	21.47	100.00
p0033	33	0	18.40	[31.86 - 52.85]	54.60	85.40
p0282	282	0	31.56	9.28	3.70	99.90
stein27	27	0	27.78	100.00	0.00	0.00
stein45	45	0	26.67	100.00	0.00	0.00
gt2	24	164	36.41	46.79	71.88	100.00

... more in the forthcoming paper

M. F. and M. Monaci, "On the optimal value of Gomory's corner relaxation", Technical Report DEI, University of Padova, 2005.