On the knapsack closure of 0-1 Integer Linear Programs

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Motivation

- According to the recent computational analysis reported in

  M. Fischetti and M. Monaci, How tight is the corner relaxation?, Technical Report, 2005

  the Gomory's corner relaxation gives a **very good approximation** of the integer hull for MIPs with general-integer variables, but...

- ... the approximation is **less effective for problems with 0-1 variables only**, as observed already in

  E. Balas, A Note on the Group-Theoretic Approach to Integer Programming and the 0-1 Case, Operations Research 21, 1, 321-322 (1973).

- **Explanation:** for 0-1 ILPs, even the non-binding variable bound constraints $x_j \geq 0$ or $x_j \leq 1$ play an important role, hence their relaxation produces weaker bounds...

- **How can we take the variable bound constraints $0 \leq x_j \leq 1$ into account when generating Gomory-like cuts?**

- We introduce the concept of **knapsack closure** as a tightening of the classical Chavtal-Gomory (CG) concept:

  for **all** inequalities $w^T x \leq w_0$ valid for the LP relaxation ...

  ... add to the original system **all** the valid inequalities for the knapsack polytope

$$conv\{x \in \{0,1\}^n : w^T x \leq w_0\}$$

- **Question:** Is the knapsack closure **significantly tighter** than the classical CG closure?

- Answer (work in progress): actually **optimize** over the KP closure on a significant set of MIPLIB test instances.

The basic machinery

- We are interested in the 0-1 ILP

$$\min\{c^T x : x \in P \cap X\} \tag{1}$$

where

$$P := \{x \in \Re^n : Ax \leq b, x \geq 0\} \tag{2}$$

is a given polyhedron and

$$X \subseteq Z^n$$

is a "combinatorially simple" discrete set, e.g.,

$$X := \{x \in Z^n : 0 \leq x \leq 1\} \tag{3}$$

- Let $w^T x \leq w_0$ be any valid inequality for P, called **source KP inequality** in the sequel,

  and let
$$KP(w, w_0) := \{x \in X : w^T x \leq w_0\} \tag{4}$$
  define a corresponding **KP relaxation** of the original ILP problem.

- Given a (fractional) point $x^* \in \Re^n$, we are interested in the following

  Separation problem: **Find a linear inequality $\alpha^T x \leq \alpha_0$ that is valid for $KP(w, w_0)$ but violated by x^* (if any)**.

The "easy" case: the source KP inequality is given

- If the source KP inequality is given, the separation problem amounts to the solution of a series of knapsack problems, i.e., of optimizations of a linear function over the KP relaxation $KP(w, w_0)$.

- Indeed, one can in principle enumerate all the members of $KP(w, w_0)$, say $x^1, \ldots, x^K$, and write the following LP model for separation:

$$\max \quad \alpha^T x^* - \alpha_0 \tag{5}$$

$$\alpha^T x^i \leq \alpha_0, \quad \text{for all } i = 1, \ldots, K \tag{6}$$

$$-1 \leq \alpha_j \leq 1, \quad \text{for all } j = 0, \ldots, n \tag{7}$$

  where (7) are just normalization conditions.

- The above LP contains an exponential number of constraints $\Rightarrow$ standard run-time cut generation technique, where at each iteration the following steps are performed:

- consider explicitly **just a few** solutions in $KP(w, w_0)$, say solutions $x^1, \cdots, x^h$ for some $h \ll K$ (initially, $h := 0$)

- compute an optimal solution (α^*, α_0^*) of the corresponding restricted LP model

$$\max \quad \alpha^T x^* - \alpha_0 \tag{8}$$

$$\alpha^T x^i \leq \alpha_0, \quad \text{for all } i = 1, \ldots, h \tag{9}$$

$$-1 \leq \alpha_j \leq 1, \quad \text{for all } j = 0, \ldots, n \tag{10}$$

- if $\alpha^* x^* - \alpha_0^* \leq 0$, then the method can be stopped as no violated inequality $\alpha^T x \leq \alpha_0$ exists

- call an *oracle* to compute an optimal solution y^* of the KP problem

$$\max\{\alpha^* y : y \in KP(w, w_0)\}$$

- if $\alpha^* y^* \leq \alpha_0^*$, then the inequality $\alpha^* x \leq \alpha_0^*$ is valid for $KP(w, w_0)$ and maximally violated, so stop

- include y^* in the separation model by setting $h := h + 1$ and $x^h := y^*$, and repeat.

The "hard" case: the source KP inequality is not given

- We need to extend the method above to the case where the inequality $w^T x \leq w_0$ is **not given** a priori (nor read from the optimal LP tableau etc.), but is completely general and defined during the separation phase so as to maximize its effectiveness.

- This approach produces a much more powerful separation tool that goes far **beyond the separation over the first Chvátal closure**...

  ... but requires to use Farkas' Lemma to certify the validity of $w^T x \leq w_0$ for P, and a more involved MIP model to replace the "easy" LP separation model shown above.

- Here is how the MIP separation model looks like:

$$\max \quad \alpha^T x^* - \alpha_0 \tag{11}$$

$$w^T \leq u^T A, \; w_0 \geq u^T b, \; u \geq 0 \tag{12}$$

$$\alpha^T x^i \leq \alpha_0 + M\delta_i, \quad \text{for all } i = 1, \ldots, Q \tag{13}$$

$$w^T x^i \geq w_0 + \epsilon - M(1 - \delta_i), \quad \text{for all } i = 1, \ldots, Q \tag{14}$$

$$\delta_i \in \{0, 1\}, \quad \text{for all } i = 1, \ldots, Q \tag{15}$$

$$-1 \leq \alpha_j \leq 1, \quad \text{for all } j = 0, \ldots, n \tag{16}$$

where $X =: \{x^1, \ldots, x^Q\}$, and M and ϵ are a large and a small positive value, respectively.

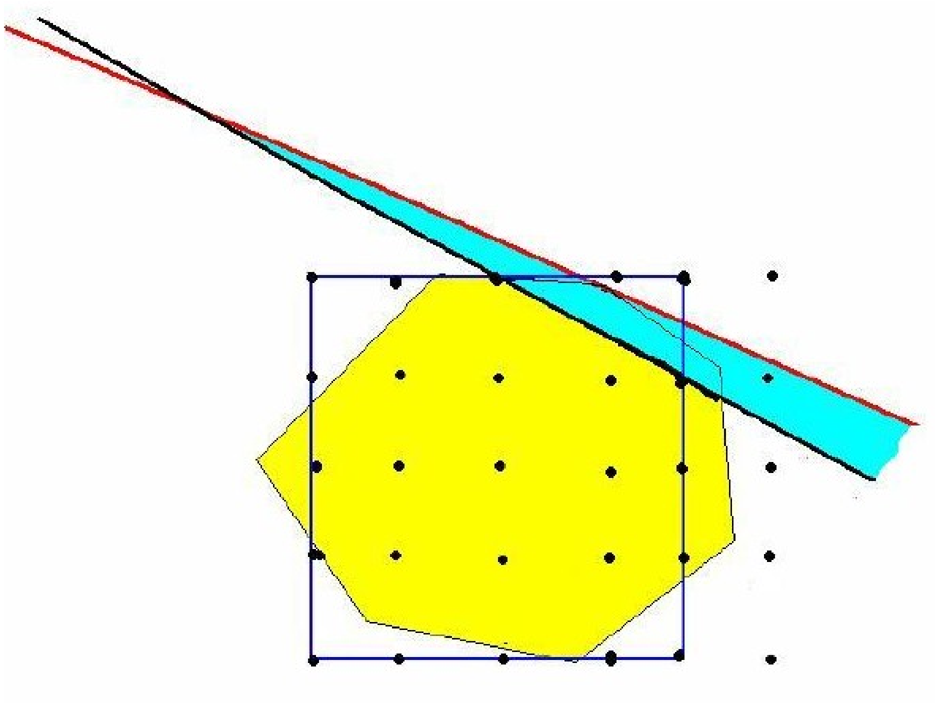
Notice that $u, w, w_0, \alpha, \alpha_0, \delta$ are all variables.

- The idea of the model above is to certify the validity of $w^T x \leq w_0$ for P (where w and w_0 are now variables) by using Farkas' characterization (12).

  Because of (13), a point $x^i \in X$ can violate the inequality $\alpha^T x \leq \alpha_0$ only by setting $\delta_i = 1$

  in which case (14) imposes that the valid inequality $w^T x \leq w_0$ cuts it off (hence this point cannot be feasible for the original ILP model).

A geometrical interpretation



$$\max \quad \alpha^T x^* - \alpha_0 \tag{17}$$

$$w^T \leq u^T A, \; w_0 \geq u^T b, \; u \geq 0 \tag{18}$$

$$\alpha^T x^i \leq \alpha_0 + M\delta_i, \quad \text{for all } i = 1, \ldots, Q \tag{19}$$

$$w^T x^i \geq w_0 + \epsilon - M(1 - \delta_i), \quad \text{for all } i = 1, \ldots, Q \tag{20}$$

$$\delta_i \in \{0, 1\}, \quad \text{for all } i = 1, \ldots, Q \tag{21}$$

$$-1 \leq \alpha_j \leq 1, \quad \text{for all } j = 0, \ldots, n \tag{22}$$

- The solution of the MIP separation model can be obtained along the same lines as for its LP counterpart:

  Find an optimal solution $(u^*, w^*, w_0^*, \alpha^*, \alpha_0^*, \delta^*)$ of a **restricted** MIP separation problem taking into account only a subset of points $x^1 \cdots x^h$.

  Invoke the KP oracle to solve

$$\max\{\alpha^* y : y \in KP(w^*, w_0^*)\}$$

  so as to certify the validity of $\alpha^* x \le \alpha_0^*$ for the current KP relaxation $KP(w^*, w_0^*)$...

  ... or else to produce a new point x^{h+1} to be inserted in the MIP separation model (along with the corresponding variable δ_{h+1}), and repeat.

Very preliminary experiments (small cases)

- **Single 0-1 knapsack problems**: **NO GAP**, all solved to optimality (as expected)
- **Multiple 0-1 knapsack problems**: about 20% more gap closed than the CG closure
- More results at MIP 2006, Miami, June 5–8, 2006.