# Robustness by cutting planes and the Uncertain Set Covering Problem

**Matteo Fischetti**
**Michele Monaci**

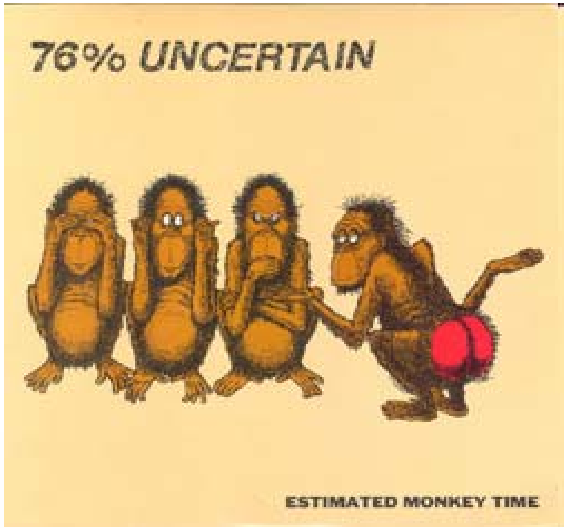
DEI, University of Padova
{fisch,monaci}@dei.unipd.it


AIRO 2008, Ischia, 10 September 2008 (work supported my MiUR and EU)

# Motivation

- A typical assumption in combinatorial optimization:

  **The exact values of all input data are known in advance**

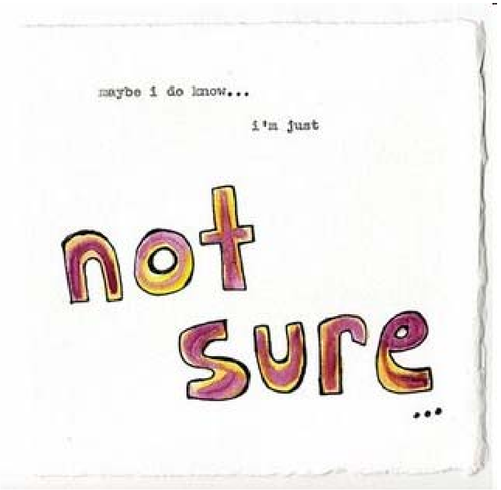



- This assumption can be violated in practical (real-world) applications
- The optimal solution, even if computed very accurately, can be very hard to implement accurately
- The solution found using nominal values of the parameters can be sub-optimal or even infeasible
- Small uncertainty in the data can make the nominal optimal solution completely meaningless from a practical viewpoint.

# Dealing with Uncertainty

- **Nominal** problem (LP)

$$\min \{ c^T x : Ax \le b, x \ge 0 \}$$




- We assume that each coefficient $a_{ij}$ of matrix A can take any value $\tilde{a}_{ij}$ in the interval $[a_{ij} - \hat{a}_{ij}, a_{ij} + \hat{a}_{ij}]$

- Bertsimas and Sim (BS'04): **a solution is considered robust if it remains feasible when at most $\Gamma_i$ coefficients in each row i take their worst value**

# Compact formulation

- Replace the i-th row with its robust counterpart

$$a_i^T x + \beta(\Gamma_i, x) \le b_i$$

where $\beta(\Gamma_i, x) = \max \{ \sum_{j \in S} a_{ij} x_j, \ |S| \le \Gamma_i \}$

represents the **level of protection** of the solution with respect to row

- Using LP duality, BS obtained a **compact formulation** for the robust LP involving a substantial n. of additional var.s and constraints

# Robustness by cutting planes


- Robustness of the solution with respect to row i can alternatively be imposed by means of **robustness cuts**

$$\sum_{j \in N} a_{ij}x_j + \sum_{j \in S} \hat{a}_{ij}x_j \le b_i, \qquad \text{for all } S: |S| \le \Gamma_i$$

- This allows one to work on the original variable space but requires to handle an exponential number of constraints.
  - Cutting planes approach
  - Separation of the current solution x* can be carried out in linear time: select the (at most) $\Gamma_i$ variables with largest positive values of

# LP instances: computational experiments

(instances from NETLIB)

| instance | T | Γ = 1 | | | Γ = 10 | | |
|---|---|---|---|---|---|---|---|
| | | %Δz | T(BS) | T(CP) | %Δz | T(BS) | T(CP) |
| | | | | | | | |
| BNL2 | 0.07 | 0.790 | 1.26 | 0.50 | 1.840 | 1.69 | 0.97 |
| D2Q06C | 2.71 | - | 18.76 | 3.84 | - | 41.77 | 5.26 |
| DEGEN3 | 0.43 | - | 58.34 | 1.45 | - | 88.69 | 1.84 |
| GANGES | 0.01 | 0.053 | 0.02 | 0.01 | 0.430 | 0.02 | 0.05 |
| PILOT | 3.05 | - | 661.72 | 12.95 | - | 58.95 | 4.85 |
| SCTAP2 | 0.01 | 1.533 | 0.12 | 0.24 | 2.814 | 0.18 | 0.62 |
| SCTAP3 | 0.02 | 1.602 | 0.17 | 0.39 | 2.995 | 0.24 | 1.46 |
| SHIP12L | 0.04 | 0.060 | 0.10 | 0.05 | 0.346 | 0.11 | 0.08 |
| SHIP12S | 0.02 | 0.062 | 0.06 | 0.03 | 0.386 | 0.09 | 0.05 |
| STOCFOR2 | 0.06 | 0.759 | 0.56 | 0.18 | 1.522 | 0.55 | 0.17 |
| STOCFOR3 | 1.58 | 0.733 | 11.68 | 12.95 | 1.482 | 10.41 | 11.61 |
| p17 | 0.09 | 0.447 | 0.12 | 0.12 | 1.280 | 0.15 | 0.11 |
| | | | | | | | |
| Average | 0.67 | | 62.73 | 2.73 | | 16.90 | 2.26 |

CPU seconds on a PC AMD Athlon 4200+ using Cplex 11.0

For each coefficient: $\hat{a}_{ij} = a_{ij}/100$ For each row, $\Gamma_i = \Gamma$ (constant)

# Cutting planes as the only option

There are situations where cutting planes are the **only** possible way to deal with uncertainty

Indeed, compact formulation by BS cannot be applied, e.g.,

- when the nominal formulation of the problem is itself noncompact (e.g., TSP)
- when the uncertainty domain cannot be fully described by a linear system
  - e.g., when the uncertainty domain involves yes-no decisions that cannot be modeled by continuous variables.

→**Uncertain set covering problems**

# The Uncertain Set Covering Problem

Classical **set covering problem**

- Given a n x m binary matrix A and a cost $c_j$ for each column j find a minimum-cost set S of columns such that each row is covered by (at least) one selected column

**Uncertain** counterpart (USCP)

- Each column j has a positive (independent) **probability** $p_j$ of disappearing, and each row must be covered with a probability larger than a given threshold P

**$\mathrm{Prob}(a_i^T x \geq 1) > P$, for all row i**


USCP arises, e.g., in **crew scheduling** applications where each pairing can be unavailable due to uncertainty (driver's nonshow)

# Modeling USCP

- **Basic** ILP model (**M1**): each row i must be covered by a set of columns having a small probability of disappearing all together

$$\sum_{j \in N \setminus S} a_{ij} x_j \geq 1 \qquad \text{for all } S: \prod_{j \in S} p_j > 1\text{-}P$$

  – Separation problem can be solved quickly (knapsack problem)

- **Alternative** compact ILP model (**M2**), exploiting $a_{ij}$ and $x_j$ binaries:

$\text{Prob}(a_i^T x \geq 1) > P$

⇕ Probabilities are independent one each other

$\prod_{j \in N} a_{ij} x_j p_j \leq 1 - P$ Probability that the selected col.s covering row i disappear all together

⇕

$\sum_{j \in N} a_{ij} w_j x_j \geq W \quad (*)$ Apply logarithm to both terms

with $w_j = -\log(p_j)$, $W = -\log(1\text{-}P)$

# Modeling USCP

- The models are equivalent in terms of integer solutions, but the associated LP relaxations are different (and no dominance exists among them)

- Constraints $\sum_{j \in N} a_{ij} w_j x_j \geq W$ (*)
  can be **strengthened**
  - Replacing $w_j$ with $w'_j = \min\{w_j, W\}$
  - With a rounding argument akin to Gomory's fractional cuts; given a positive integer k, the following inequality is valid for USCP

$$\sum_{j \in N} v_{ij} x_j \geq k \qquad (**)$$

where $v_{ij}$ are "small integer" defined as $(k-1)a_{ij}w_j/(W-\varepsilon)$ rounded up

No dominance exists between (*) and (**)

These new constraints make the model more stable from a numerical point of view.

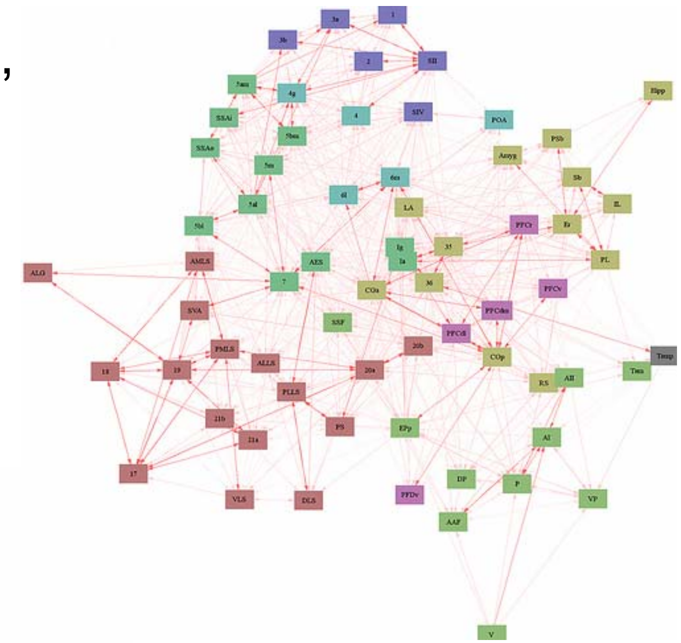
# USCP: computational experiments

- Test instances randomly derived from some SCP instances in the ORlibrary.
  - All probabilities are randomly generated in [0, 0.2].
  - Required probability: P = {0.90, 0.95}
  - Both ILP models solved using Cplex 11.0

| instance | z* | T | P = 0.90 | | | P = 0.95 | | |
|---|---|---|---|---|---|---|---|---|
| | | | z | T(M1) | T(M2) | z | T(M1) | T(M2) |
| | | | | | | | | |
| scp41 | 429 | 0,01 | 701 | 0,06 | 0,09 | 921 | 1,32 | 0,17 |
| scp51 | 253 | 0,10 | 391 | 0,95 | 0,24 | 467 | 20,76 | 0,60 |
| scp61 | 138 | 0,45 | 199 | 0,84 | 0,71 | 236 | 11,86 | 1,05 |
| scpa1 | 253 | 1,47 | 383 | 8,68 | 3,88 | 472 | 1152,59 | 11,60 |
| scpb1 | 69 | 1,57 | 109 | 77,24 | 33,21 | 125 | 1455,35 | 12,85 |
| scpc1 | 227 | 1,12 | 360 | 54,56 | 16,06 | 442 | >1800.00 | 63,91 |
| scpd1 | 60 | 2,64 | 89 | 129,47 | 21,57 | 106 | >1800.00 | 70,01 |
| scpe1 | 5 | 0,39 | 5 | 74,92 | 5,40 | 6 | 110,30 | 9,72 |
| scpnrf1 | 14 | 113,12 | 16 | >1800.00 | 291,87 | 19 | >1800.00 | 184,62 |
| | | | | | | | | |

# Uncertain graph connectivity

- Given an undirected graph with costs and probabilities associated with edges, find a min-cost partial graph that is **connected with a given probability P** (NP-hard)



- Arises e.g. in the design of reliable **telecommunication network**

- ILP formulation

$$\min \sum_{e \in E} c_e x_e$$

$$\text{Prob}\left(\sum_{e \in \delta(S)} x_e \geq 1\right) > P \quad \text{for each proper nodeset } S$$

$$x \text{ binary}$$

- Assuming probabilities to be independent, the constraint associated with a given subset S reads

$$\sum_{e \in \delta(S)} w_e x_e \geq W$$

# Computational experiments

- Test instances randomly derived from some TSP instances in the TSPLIB.
  - All probabilities are randomly generated in [0, 0.2].
  - Required probability: P = {0.85, 0.90, 0.95, 0.99}
  - Branch-and-cut embedded within Cplex 11.0

| instance | P = 0.85 | | P = 0.90 | | P = 0.95 | | P = 0.99 | |
|---|---|---|---|---|---|---|---|---|
| | z | T | z | T | z | T | z | T |
| | | | | | | | | |
| burma14 | 2671 | 1,06 | 2811 | 0,49 | 3182 | 0,14 | 3881 | 1,94 |
| fri26 | 780 | 8714,49 | 841 | 1548,62 | 874 | 2,80 | 1080 | 281,15 |
| gr17 | 1496 | 0,25 | 1599 | 0,21 | 1807 | 0,12 | 2466 | 925,66 |
| gr21 | 2330 | 5,42 | 2554 | 58,67 | 2584 | 0,29 | 3113 | 23,15 |
| ulysses16 | 4985 | 1,97 | 5540 | 10,53 | 6414 | 0,21 | 7770 | 0,63 |
| ulysses22 | 4952 | 18,93 | 5410 | 18,22 | 5999 | 0,30 | 7061 | 133,77 |
| | | | | | | | | |

# Uncertain (?!) Plots

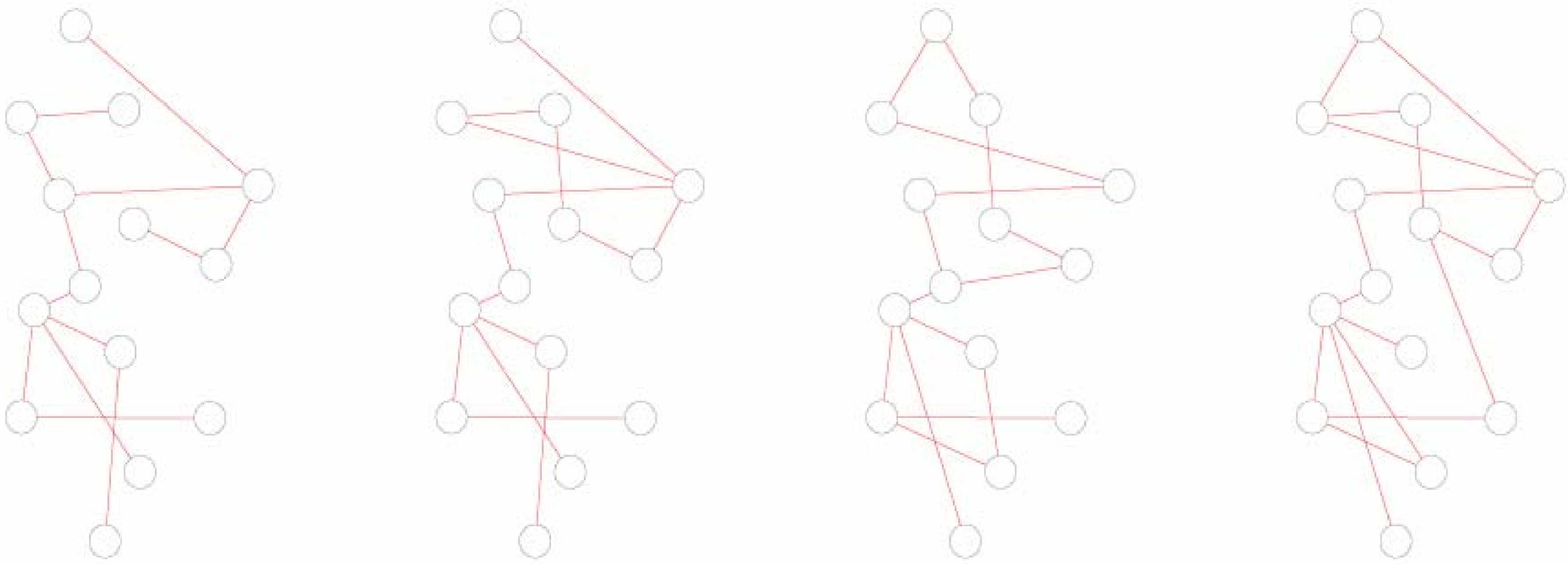


Figure 2: Optimal solutions of instance burma14 for $P^{\mathtt{min}} \in \{0.85, 0.90, 0.95, 0.99\}$.