Looking inside Gomory

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(joint work with Egon Balas and Arrigo Zanette)







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Gomory cuts

- Modern branch-and-cut MIP methods heavily based on Gomory cuts, used to reduce the number of branching nodes needed to reach optimality
- However, pure cutting plane methods based on Gomory cuts alone are typically not used in practice, due to their poor convergence properties
- Branching as a **symptomatic cure** to the well-known drawbacks of Gomory cuts---saturation, bad numerical behavior, etc.
 - From the cutting plane point of view, however, the cure is even worse than the disease—it hides the **trouble source**!

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The pure cutting plane dimension

- The purpose of our project is to try to come up with a viable pure cutting plane method (i.e., one that is not knocked out by numerical difficulties)...
- ... even if on most problems it will not be competitive with the branch-and-bound based methods
- First step: Gomory's fractional cuts (FGCs), for two reasons:
 - simplest to generate, and
 - when expressed in the structural variables, all their coefficients are integer → easier to work with them and to assess how nice or weird they are (numerically more stable than GMI cuts)
- This talk: looking inside the chest of FGC convergent method



Rules of the game: cuts from LP tableau

- Main requirement: reading (essentially for free) the FGCs directly from the optimal LP tableau
- Cut separation heavily entangled with LP reoptimization!
- Intrinsically different from the recent works on the first closure by Fischetti and Lodi (Chvatal-Gomory) and Balas and Saxena (GMI/split closure) where separation is **decoupled** from optimization
- Subtle side effects! The FGC

2 X1

n work much better than its GMI (dominati

 $X_2 +$

can work much better than its GMI (dominating but numerically less stable) counterpart

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 $3 X_3 <= 5$

2.2727272726 $x_1 - x_2 + 3.1818181815 x_3 \le 5$



Cuts and Pivots: an entangled pair

- Long sequence of cuts that eventually lead to an optimal integer solution \rightarrow cut **side effects** that are typically underestimated when just a few cuts are used within an enumeration scheme
- A must! Pivot strategies aimed to keep the optimal tableau clean so as generate clean cuts in the next iterations
- In particular: avoid cutting LP optimal vertices with a **weird fractionality** (possibly due to numerical inaccuracy)
 - \rightarrow the corresponding LP basis has a large determinant (needed to describe the weird fractionality)

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→ the tableau contains weird entries that lead to weaker and weaker Gomory cuts





Role of degeneracy

- Keep a clean tableau \rightarrow deal with numerical issues ...
- Avoid cutting weird optimal vertices \rightarrow ??

Exploit dual degeneracy!

- Dual degeneracy is notoriously massive in cutting plane methods
- It can play an important role and actually can **favor** the practical convergence of the method...
 - ... provided that it is exploited to choose the cleanest LP solution (and tableau) among the equivalent optimal ones...
 - Unfortunately, the highly-correlated sequence of reoptimization pivots performed by a generic LP solver leads invariably to an **uncontrolled growth** of the basis determinant \rightarrow method out of control after just a few iterations!

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Dura lex, sed lex ...

- In his proof of convergence, Gomory used the **lexicographic simplex** to cope with degeneracy
- The dual lexicographic simplex is a modified version of the dual simplex algorithm:
 - instead of considering the minimization of the objective function $x_0 = c^T x$
 - one is interested in the lexicographic minimization of the entire solution vector (x₀,x₁,..., x_n)

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- Rigid pivoting rules with ratio tests involving vectors \rightarrow cumbersome and **slow/unstable implementation**
- Useful in theory for convergence proofs, but ...
 ... also in practice?



Lex-FGC: fractional vertices (stein15)

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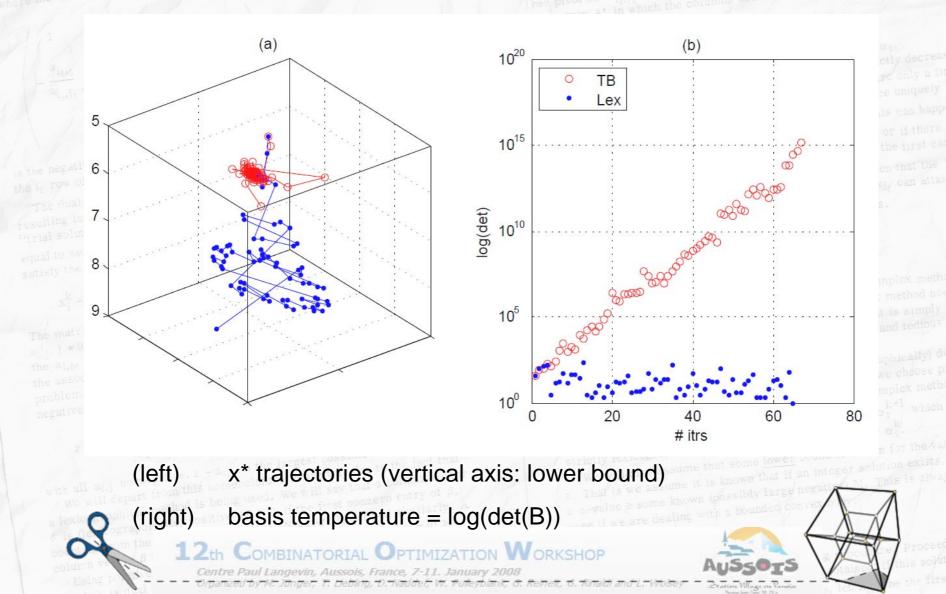
A consequence of the nice "sign pattern" of lex-optimal tableau,

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Textbook vs Lex (stein15)



A reliable and fast lex. implementation

- Based on the use of a black-box LP solver
 - Step 0. Optimize $x_0 \rightarrow x_0$ --> optimal value x_0^*
 - Step 1. Fix $x_0 = x_0^*$, and optimize $x_1 \rightarrow optimal value x_1^*$
 - Step 2. Fix also $x_1 = x_1^*$, and optimize x_2^* --> optimal value x_2^*
 - Simple, but ... does it work?

Forget! ... just a nightmare!



- <u>Clever version</u>: at each step, instead of adding the equation $x_j = x_j^*$
 - ... fix out of the basis all the nonbasic variables with nonzero reduced cost
- → sequence of fast (and clean) reoptimizations on smaller and smaller degeneracy subspaces, leading to the required lexoptimal tableau

 \rightarrow to be compared with Balas-Perregaard L&P pivoiting...

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Role of lex reoptimization (stein15)

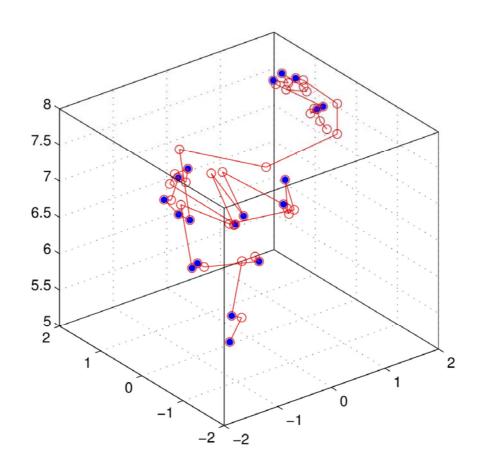
The negative of the inverse of the l_0 row of A^6 and all the unit row. The dual simplex method donsist esulting in a sequence of matrice 'trial solution' obtained by setting equal to zero, and then choosing t satisfy the equations, i.e.,

The matrix and trial solution and 1 + j = 0 are all nonnegative. The algo 1 = 0 are nonnegative. The associated trial solution is problem since the primal feas negative and, since

 $= a_{0,0} + \sum_{j=1}^{j=n} a_{0,j} (-t_j^k)$

with all apply nonnegative We will depart from this horizon applical method is

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 α_{0} it follows that $\alpha_{0} < \alpha_{0}$, is in a succession of strictly decreas ber of these since there are only a finvariables, and any choice uniquely be process must stop. This can happen nents in the current α_{0}^{k} , or if there columns $(1/a_{0,0}) \alpha_{1}^{k}$. In the first curthe second it is easily seen that the value that the variable x_{0} can attain unregative numbers exists.

e lexicographical dual simplex method implied that this simplex method need necessary to the proof. It is simply to the original rather long and tedious ;

have/obtained a (lexicographically) us l_succeeding plvot steps we choose pi lexicographical dual simplex metho i a new "trial solution" α_0^{k+1} which i lier than its predecessor α_0^k . ome <u>lower</u> bound is known for the value we that if an integer solution exists, we that if an integer solution exists, but large negative M. This is alway



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Rinaldi and L. Wrolsey - Brailen Villa

Heuristic pivoting rules

- Alternative pivoting rules to **mimic** the lexicographic dual simplex
- Useful to try to highlight the crucial properties that allow the lexicographic method to produce stable Gomory cuts

HEUR1

Just a truncated lex. method on the arguably most-important variable

After the addition of a FGC, lex. minimize (x_0, x_i) where x_i^* is the basic fractional variable generating the FGC

HEUR2

Try to select optimal vertices where the previously generated FGCs are slack (having a cut slack into the basis avoids it appears in the cutgeneration row and hence reduces cut correlation)

After the addition of the i-th FGC with slack variable s_i (say), try to keep all slacks s_1, \ldots, s_i inside the opt. basis by lex. minimizing

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 $(x_0, -s_{i_1}, ..., -s_2, -s_1)$



Experiments

- Pure ILP instances from MIPLIB 3 and MIPLIB 2003 (none solved by previous pure cutting plane methods based on FGCs)
- Input data is assumed to be integer. Cuts also derived from the objective function tableau row, as prescribed by Gomory's proof of convergence.
- Once a FGC is generated, we put it in its **all-integer form** in the space of the structural variables.
- Numerical stabilization: we use a threshold of 0.1 to test whether a coefficient is integer or not:
 - a coefficient with fractional part smaller than 0.1 is rounded to its nearest integer
 - cuts with larger fractionality are viewed as unreliable and hence discarded.

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Experiments (one cut at a time)

 First set of experiments addressed the single-cut version of Gomory's algorithm

Runs on a PC Intel Core 2 Q6600, 2.40GHz, with a time limit of 1 hour of CPU time and a memory limit of 2GB for each instance

ual to zero, and then choose atisfy the equations, i.e.,

 $x_1 - x_{1,0}$ is matrix and trial solution are usually said to be "unaddifference" is j = 0 are all nonnegative. They are feasible (or primal and dual feasible $x_1 = 0$ are nonnegative. If A^k is both primal and dual feasible is associated trial solution is the solution to the linear programming roblem, since the primal feasible property makes the x_1 values nonroblem, since the primal feasible property makes the x_1

negative and, sin

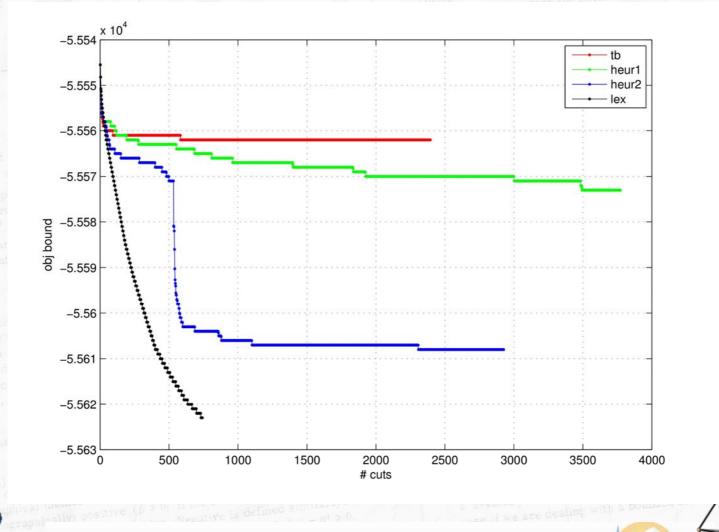
egative, $z \neq a_{1,1}$ is the largest possible value of z. from this nomenclature in one way due to the fact that method is being used. We will say that a column vector method is being used. We will say that a column vector for the first ponzero entry of β , for the first ponzero entry of β .

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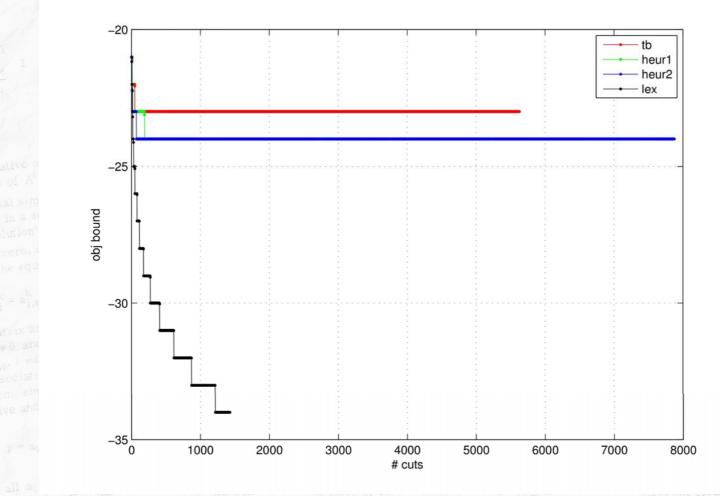


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Once a dual feasible form is achieved, the boose some row is (by p) tobtain the solution in the following way: choose some row is (by p). Then consider the columns of for which (1/4). Then consider the columns of for which (1/4). Then consider the column which is the least population in the fastily verified that this pivot step results to be and since



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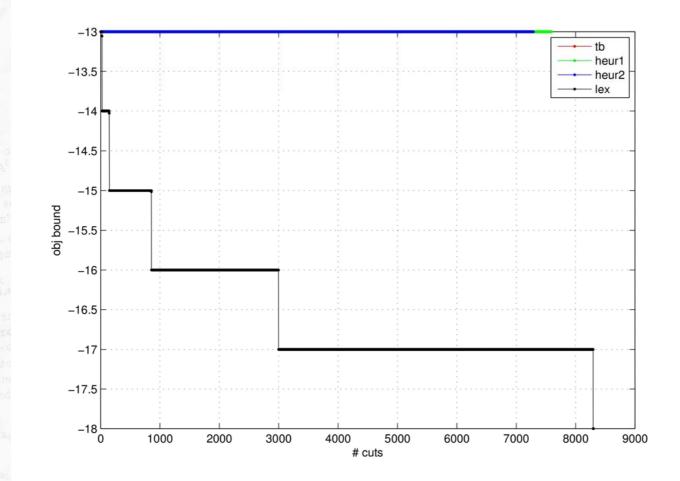
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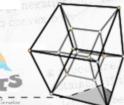
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Multi-cut version

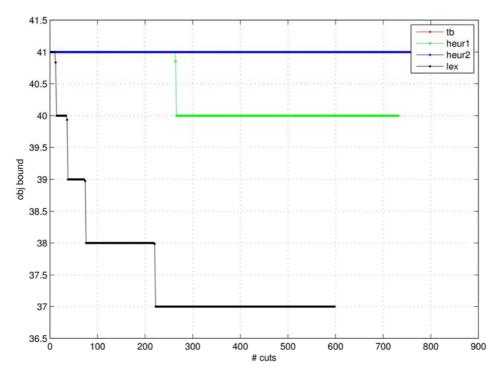
	Textbook								Lex							
Problem		Itrs	Cuts	Time	ClGap	Coeff.	Det.	κ		Itrs	Cuts	Time	ClGap	Coeff.	Det.	κ
air04	M	35	11912	1209.41	11.23	2.4e+03	6.9e + 11	9.3e + 09	tL	150	47995	3606.61	19.21	7.3e+03	2.2e+09	$3.9e{+}11$
air05	M	30	9648	955.09	5.11	7.1e+02	4.2e + 11	4.9e + 09	tL	261	76263	3598.20	14.00	1.5e + 03	4.9e + 10	9.6e + 10
bm23	E	144	2168	9.33	<mark>5.28</mark>	2.3e + 15	5.5e + 30	8.7e+24	0	659	<mark>8298</mark>	1.67	100.00	3.7e + 02	4.6e + 10	3e+14
cap6000	tL	979	11693	3614.59	10.34	1.4e+09	7e + 15	4.3e + 20	E	966	6769	2110.12	24.66	4.5e + 06	5.1e + 09	1.6e + 15
hard_ks100	tL	1950	10770	3603.03	100.00	5e+04	2.2e + 05	1.5e + 12	0	98	439	0.63	100.00	5e + 05	1.5e+04	2.8e+09
hard_ks9	0	269	1678	0.30	100.00	7.8e+04	1.7e + 05	6.9e + 09	0	139	614	0.12	100.00	3e + 04	3.6e + 04	2.9e+10
krob200	0	44	2017	153.92	100.00	1.6e + 02	1e + 06	9.4e + 07	0	17	282	22.18	100.00	7.2	1e+06	3.7e+04
l152lav	M	525	31177	2715.78	40.59	5.3e + 03	8e + 08	1.6e + 10	0	681	22364	206.82	100.00	1.8e + 04	6.7e + 05	2.4e+10
lin318	0	15	250	12.82	100.00	7	3.4e + 07	9.7e + 05	0	19	416	124.27	100.00	17	3.2e + 07	1.8e+08
lseu	E	149	3710	34.35	44.58	1 <mark>.7e+14</mark>	9.8e + 33	2e+19	0	1330	18925	6.62	100.00	9.6e + 03	1.2e + 14	2.6e+14
manna81	0	1	270	8.88	100.00	1	5.2e + 92	3.7e + 06	0	2	280	29.58	100.00	1	1.7e + 97	3.7e+06
mitre	M	94	30016	1262.24	85.52	8.9e + 08	Inf	4.2e + 20	tL	232	20972	3619.30	97.83	6.2e + 06	Inf	9.1e+13
mzzv11	M	33	17936	1735.42	38.56	4.9e+02	Inf	2.6e + 11	tL	61	25542	3739.41	40.60	1.8e + 02	2.8e + 51	8.5e+11
mzzv42z	M	62	14664	1515.50	33.80	3e + 06	Inf	6.5e + 14	tL	61	14830	3616.85	25.50	8.5e + 02	1e+38	8.7e+12
p0033	M	1529	23328	2493.35	81.53	1.8e + 14	1.1e + 34	2.2e + 24	0	87	1085	0.14	100.00	4.5e + 02	1.6e + 13	6e + 14
p0201	tL	1332	55408	3602.21	19.32	2e+12	2e + 34	2.5e + 26	E	27574	629004	1119.71	74.32	2.3e+05	3.6e + 22	7.4e+11
p0548	M	825	35944	2223.35	48.98	1.3e+09	2.4e + 137	2e + 24	E	461	27067	131.69	47.50	4.8e + 06	7.1e + 135	5.5e + 14
p2756	M	740	19925	2423.48	78.86	4.5e + 12	Inf	1.3e + 27	E	642	14645	509.72	79.09	2.4e + 05	Inf	1.1e+13
pipex	M	2929	49391	1921.78	51.25	$1.3e{+}14$	3.3e + 27	4e + 28	0	50000	488285	188.11	74.18	2.8e + 05	8.5e + 11	1e+14
protfold	M	7	5526	1058.84	<mark>8.76</mark>	4.6e + 06	Inf	5.1e + 13	tL	159	38714	3607.08	45.26	30	1.5e + 32	1e+07
sentoy	M	1953	34503	2586.38	18.25	$5.1e{+}14$	5.3e + 43	7e + 30	0	5331	68827	25.85	100.00	7.6e + 04	3.2e + 34	3.9e+14
seymour	tL	18	11748	7517.40	21.67	3.3e + 08	1.3e + 159	1.4e + 16	tL	87	48339	3610.62	26.89	78	1e+23	3.6e+07
stein15	E	116	3265	9.50	20.92	2.5e+15	2.4e + 26	3.2e + 28	0	64	676	0.14	100.00	15	2e+02	2.3e+06
stein27	E	57	2399	20.01	0.00	1.7e + 15	2.8e + 36	8.3e + 18	0	3175	37180	19.72	100.00	7.5e+02	1.9e+04	1.9e+05
timtab1-int	M	231	73188	585.72	23.59	1.6e + 11	Inf	2.1e+21	cL	3165	1000191	841.56	50.76	3.5e+06	Inf	1.7e+16

Table 1: Comparison between textbook and lexicographic implementation of Gomory's algorithm (multi-cut version)



the negative of the inverse of in 1₂ row of A⁴ and all the unit The dual simplex method conresulting in a sequence of matr "trial solution" obtained by set equal to zero, and then choosinsatisfy the equations, i.e.,

Instance protfold



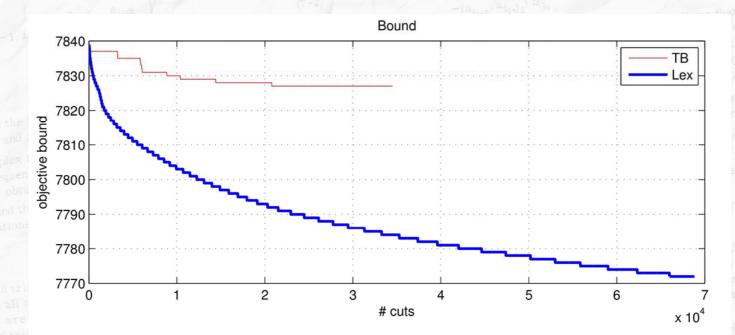
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Very hard protein-folding instance (max; opt=31; 9x2 days using tuned Xpress2006b)

• Cplex: after 14 days, memory overflow (3GB) after 4,000,000 nodes; best bound 36

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• lexFGC: after 14 days, about 3,000,000 cuts in 9,500 rounds; best bound 34, still alive and kicking (max 100 MB memory)



of strictly decreas here are only a fin y choice uniquely op. This can happe at α_0^k or if there k. In the first can ly seen that the table x_{ij} can attain exists.

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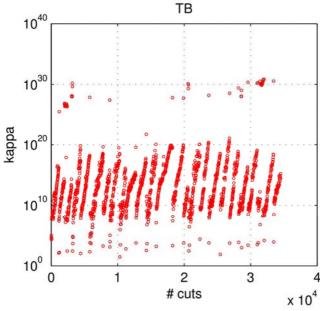
Upper bound (max problem; multi-cut version)

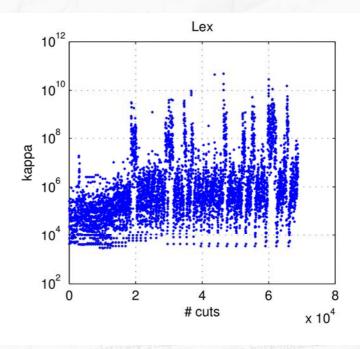
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Condition number of the optimal basis

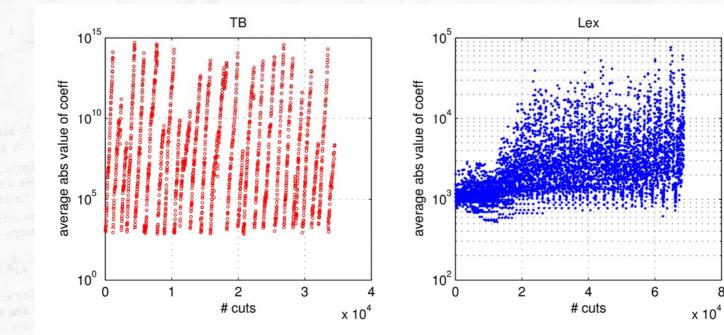
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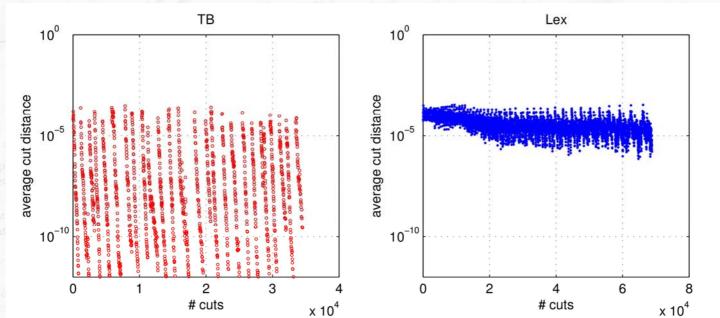
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Average absolute value of cut coefficients





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dual simplex method simplex method need proof. It is simply th er long and tedious p

Avg. geometric distance of x^* from the FGC





(A4 P-1 where the $(n + 1) \times (n + 1)$ matrix P^{-1}

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he $(n + 1) \times (n + 1)$ is the negative of the inver rows execut the on the i_0 row of A^0 and all th The dual simplex method resulting in a sequence of m "trial solution" obtained by se equal to zero, and then choosing the c satisfy the equations, i.e.,

 $\mathbf{x}_{i}^{k} = \mathbf{a}_{i,0}^{k}$ The matrix and trial solution are usually said to be at i = 0 are all nonnegative. They are least the algo, $i \neq 0$ are nonnegative. If A^k is b Hadar progra the associated trial solution is the soluti akes the ti value problem, since the primal feasible prop negative and, since

 $z = a_{0,0} + \sum_{i=1}^{n} a_{0,i} (-t_{j}^{k_{ij}})$ nonnegative, $z \neq a_{0,0}$ is the largest possible value of z. ienclature in one way due to the fact that a column vector

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with aio, negative. Then consider the columns of 101 wanted to then is negative, and select from these that column which is the least negative. Then pivot on Big. It is easily verified that this pivot step results in a new matrix A' in which the column

-(a_{i0,0}/a_{i0,j0}) α_j it follow that $\alpha_0 < \alpha_0$. Lex, dura lex ion of strictly decreasing since there are only a finit (a negative any choice uniquely st stop. This can happen ant age, or if there lonseque In the first case is easily seen that the sed lex he largest value that the variable xir can attain. and consequently no solution in nonnegative numbers exists.

> 8, FINITENESS PROOFS In these proofs we will use the lexicographical dual simplex method be used in practice of the simplified that this simplex method need its use in the proof has reduced the original rather long and tedious pr

Let us assume then that we have obtained a (lexicographically) dua to relatively simple ones. feasible solution, and that in all succeeding pivot steps we choose pivo elements in accordance with the lexicographical dual simplex method After each pivot then we obtain a new "trial solution" α_0^{k+1} which is strictly lexicographically smaller than its predecessor α_0^k . We will also assume that some lower bound is known for the value it is known that if an integer solution exists, it