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## **Mixed-Integer Programs (MIPs**

- We will concentrate on general MIPs of the form
  - min {  $c x : A x = b, x \ge 0, x_j$  integer for some j }
- Two main story characters
   The LP relaxation (beauty): easy to solve
  - The integer hull (the beast): convex hull of MIP sol.s, hard to describe



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#### Cutting planets the olution in the following way: choose some row is (1, 0) Once a dual feasible form is achieved, the dual simplex method planets once a dual feasible form is achieved, the dual simplex method planets on the olution in the following way: choose some row is (1, 4), of is negative, not select the columns of for which (1/4), of is negative, not select verified that this pivot step results in a Then pivot on a is the columns are still positive and since

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#### **Cuts:** linear inequalities valid for the integer hull (but not for the LP relaxation)

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#### How to compute?

Are they really useful?
If potentially useful, how to use them?



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## How to compute the cuts?

- **Problem-specific** classes of cuts (with nice theoretical properties)
  - Knapsack: cover inequalities, …
    TSP: subtour elimination, comb, clique tree, …
- **General** MIP cuts only derived from the input model
  - Cover inequalities
    - Flow-cover inequalities

## Gomory cuts (perhaps the most famous class of MIP cuts)



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## **Gomory cuts: basic version**

Basic version for pure-integer MIPs (no continuous var.s): **Gomory fractional cuts**, also known as **Chvàtal-Gomory cuts** 

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- Given any equation satisfied by the LP-relaxation points
  - 1. **relax** to its  $\leq$  form
  - 2. relax again by rounding down all left-hand-side coeff.s
  - 3. improve by rounding down the right-hand-side value

 $\sum \overline{a}_j x_j = \overline{b}$ 

 $\sum \overline{a}_j x_j \le \overline{b}$ 

 $\sum \lfloor \overline{a}_j \rfloor x_j \le \overline{b}$ 

 $\sum \lfloor \overline{a}_j \rfloor x_j \le \lfloor \overline{b} \rfloor$ 

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is a (lexicographically) due is pivot steps we choose pivphical dual simplex method al solution?  $\alpha_0^{k+1}$  which is predecessor  $\alpha_0^k$ .

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Note: all-integer coefficients (good for numerical stability)

## Gomory cuts: improved version

Gomory Mixed-Integer Cuts (GMICs):

$$\sum_{j} \lfloor \overline{a}_{j} \rfloor x_{j} \leq \lfloor \overline{b} \rfloor$$
$$\sum_{j} (\lfloor \overline{a}_{j} \rfloor + \epsilon_{j}) x_{j} + \sum_{x_{j} \text{ continuous}} \alpha_{j} x_{j} \leq \lfloor \overline{b} \rfloor$$

Some left-hand side coefficients can be increased by a fractional quantity  $\epsilon_j \ge 0 \rightarrow$  better cuts, though potentially less numerically stable

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- Can handle continuous variables, if any (a must for MIPs)

## **GMICs read from LP tableaux**

GMICs apply a simple formula to the coefficients of a starting equation

- Q. How to define this starting equation (crucial step)?
- A. The LP optimal tableau is plenty of equations, just use them!

|                                 |       |                 | $x_1$ | $x_2$          | $x_3$          | $x_4$          | $x_5$ | $x_6$ | $x_7$ | <i>x</i> <sub>8</sub> |
|---------------------------------|-------|-----------------|-------|----------------|----------------|----------------|-------|-------|-------|-----------------------|
|                                 | -z    | $-\frac{25}{3}$ | 0     | $\frac{4}{3}$  | $\frac{19}{6}$ | $\frac{9}{2}$  | 0     | 0     | 0     | $\frac{7}{6}$         |
| the equilibrium $k = a_{1,0}^k$ | $x_5$ | 1               | 0     | 1              | $-\frac{1}{2}$ | $-\frac{3}{2}$ | 1     | 0     | 0     | $\frac{3}{2}$         |
|                                 | $x_1$ | $\frac{11}{3}$  | 1     | $-\frac{2}{3}$ | $-\frac{1}{3}$ | 0              | 0     | 0     | 0     | $\frac{2}{3}$         |
| an i sú<br>ssociaic             | $x_6$ | $\frac{2}{3}$   | 0     | $\frac{1}{3}$  | $\frac{1}{6}$  | $-\frac{1}{2}$ | 0     | 1     | 0     | $\frac{7}{6}$         |
|                                 | $x_7$ | 1               | 0     | -3             | $\frac{1}{2}$  | $\frac{9}{2}$  | 0     | 0     | 1     | $-\frac{15}{2}$       |

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strictly lexicographically similar We will diso assume that some <u>lower</u> bound is known for a colling is we assume it is known that if an integer solution e a z-value ≥ some known (possibly large negative) case if we are dealing with a bounded convex body

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we adopt the following pr

## The two available modules

#### The LP solver

- Input: a set of linear constraints & objective function
- Output: an optimal LP tableau (or basis)

#### The GMIC generator

- Input: an LP tableau (or a vertex x\* with its associated basis)
- Output: a *round* of GMICs (potentially, one for each tableau row with fractional right-hand side)



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## How to combine the two modules?

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A natural (??) interconnection scheme (Kelley, 1960):



In theory, this scheme **could** produce a finitely-convergent cutting plane scheme, i.e., an exact solution alg. only based on cuts (no branching) on of such ste only be a line stible sets of a ch. Conseq e are no negular a been oota has been oota has been oota



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## transforming matrices a no all zero columns can appear in any a statistic transforming matrices a statistic columns can appear in any a statistic transforming matrices a statistic columns can appear in any a statistic transforming matrices a statistic columns can appear in any a statistic transforming matrices a statistic columns can appear in any a statistic columns can appear a statistic columns can appear in any a statistic columns can appear a sta



that column which is the least negative.

assume then that we have obtained a (lexicogruphically) dua

 Stein15: toy set covering instance from MIPLIB
 I P bound After each pivot then we obtain a new "trial solution"

= 5

- LP bound

• multi cut generates **rounds** of cuts before each LP reopt.



### LP solution trajectories

• Plot of the LP-sol. trajectories for single-cut (red) and multi-cut (blue) versions (multidimensional scaling) (X,Y) = 2D representation of the x-space (multidimensional scaling)

Both versions collapse after a while  $\rightarrow$  why? CPAIOR 2010

THEMATICAL PROGRAMMING



#### **LP-basis determin**



s of maximal rank, no all zero columns can appear in any A\*. ant is achieved, the dual simplex method prozeeds in the following way: choose some row is (i. = 0) -orative. Then consider the columns  $\alpha_j$  for which  $(1/a(f_j)) \alpha_j$ 

After each pivot then we obtain a new "trial solution"  $\alpha_0^{k+1}$  which is lexicographically smaller than its predecessor  $\alpha_0^k$ 

#### Exponential growth $\rightarrow$ unstable behavior! z-value > some known (possibly large negative) (lexicographically) positive $(\beta > 0)$ if the first ponzero pricographical method is being used.

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from the top down, is positive. Negative is defined similar ) is greater than another column vector  $\beta^{\dagger}$  of  $\beta = \beta^{\dagger} > 0$ this sense we will set that **CPAIOR 2010** 

case if we are dealing with a bounded convex bod

### Intuition about saturation

- Cuts work reasonably well on the initial LP polyhedron
- ... however they create artificial vertices
- ... that tend to be very close one to each other
- ... hence they differ by small quantities and
  - have "weird entries"
- $\rightarrow$  very like using a smoothing plane on wood



- LP theory tells that small entries in LP basic sol.s x\*
   ... require a large basis determinant to be described
   ... and large determinants amplify the issue and create
   numerically unstable tableaux
- Kind of **driving a car on ice** with flat tires :
  - Initially you have some grip
    ... but soon wheels warm the ice and start sliding
    ... and the more gas you give the worse!

## **Gomory's convergent method**

- For pure integer problems (all-integer data) Gomory proved the existence of a finitely-convergent solution method only based on cuts, but one has to follow a **rigid recipe**:
  - use lexicographic optimization (a must!)
  - use the objective function as a source for GMICs
    be **really patient** (don't unplug your PC if nothing seems to happen...)
- Finite convergence guaranteed by an enumeration scheme hidden in lexicographic reoptimization (this adds anti-slip chains to Gomory's wheels...)



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xicographically) du steps we choose pir dual simplex methor tion"  $\alpha_0^{k+1}$  which i cessor  $\alpha_0^k$ .

→ safe but slow (like driving on a highway with chains...)

## The underlying enumeration tree

- Any LP solution  $x^*$  can be visualized on a lex-tree ( $x_o = c x = objective$ )
- The structure of the tree is fixed (for a given lex-order of the var.s)
- Leaves correspond to integer sol.s of increasing lex-value (left to right)





# The "bad" Gomory at (1/4)(1) of the column of the least begative.



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#### LP bound = 5; ILP optimum = 8 its in accordance with the lexicographical dual simplex method pivot then we obtain a new "trial solution" $\alpha_0^{k+1}$ which is

### **TB** = no-lex **multi-cut** vers. (as before) nat is we assume it is known that if an integer solution exists, it

#### value > some known (possibly large negative) LEX = single-cut with lex-optimization

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### tion in the following way: choose some row $i_0$ $(i_0 \neq \emptyset)$ Good Gomory: Stein15 (LP sol.s) at column which the least positive.

from the top down, is positive. Negative is defined similarly.

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#### ler than its predecessor Plot of the LP-sol. trajectories for TB (red) and LEX (black) versions wicographical method is being used. We will say that a o

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**TB** = multi-cut vers. (as before)

#### **LEX** = single-cut with lex-opt.

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## So, what is wrong with Gomory?

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- GMICs are not bad by themselves
- What is problematic is their use in a naïve Kelley's scheme
  - A main issue with Kelley is the closed-loop nature of the interconnection scheme
- Closed-loop systems are intrinsically prone to instability...

... unless a filter (like lex-reopt) is used for input-output decoupling



Open and closed loop systems

## **Brainstorming about GMICs**

- Ok, let's think "laterally" about this cutting plane stuff
- We have a cut-generation module that needs an LP tableau on input

... but we cannot short-cut it directly onto the LP-solver module (soon the LP determinant burns!)



olean cally) hoose

- Shall we forget about GMICs and look for more fancy cuts,
  - ... or we better design a different scheme to exploit them?

## **Brainstorming about GMICs**

This sounds like *déjà vu*...

## ... we have a **simple module** that works well in the beginning

but soon it gets stuck in a corner

#### ... Where did I hear this?



In these proofs we will use the lexicographical dear bear method use leserabed in Section 7. It is not implied that this simplex method use be used in practice or that it is necessary to the proof. It is simply be used in practice or that it is necessary to the proof. It is simply to be used in practice or that it is necessary to the proof.

#### Oh yeah! It was about heuristics and metaheuristics.

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## We need a META-SCHEME for cut generation

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### **Toward a meta-scheme for MIP cuts**

- We stick with **simple** cut-generation modules; if we get into trouble...
  - ... we don't give-up but apply a **diversification step** (isn't this the name, Fred?) to perturb the problem and explore a different "**cut neighborhood**"







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## A diving meta-scheme for GMICs

• A main source of feedback is the presence of previous GMICs in the LP  $\rightarrow$  avoid modifying the input constr.s, use the obj. function instead

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Contr

CU

POOL

SOLVER

GMIC generator

 A kick-off (very simple) scheme:

Dive & Gomory

Idea: Simulate enumeration by adding/subtracting a bigM to the **cost** of some var.s and apply a classical GMIC generator to each LP

... but **don't add the cuts to the LP** (just store them in a cut pool for future use...)

**D&G** results





columns can appear in any A".

|              | MIPLIB 2003 |            |      |  |
|--------------|-------------|------------|------|--|
| method       | cl.gap      | time $(s)$ | en t |  |
| 1gmi         | 18.3%       | 0.54       | 449  |  |
| Lift&Project | 30.7%       | 95.23      |      |  |
| Dive&Gomory  | 31.5%       | 7.45       |      |  |



## **A Lagrangian filter for GMICs**

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• As in Dive&Gomory, diversification can be obtained by changing the objective function passed to the LP-solver module so as to produce LP tableaux that are only **weakly correlated** with the LP optimal solution x\* that we want to cut

• A promising framework is *relax-and-cut* where GMICs are not added to the LP but immediately relaxed in a **Lagrangian** fashion

→ very interesting results to be reported by Domenico (Salvagnin) in his Friday's talk about "LaGromory cuts"...



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