From Mixed-Integer Linear to Mixed-Integer Bilevel Linear Programming

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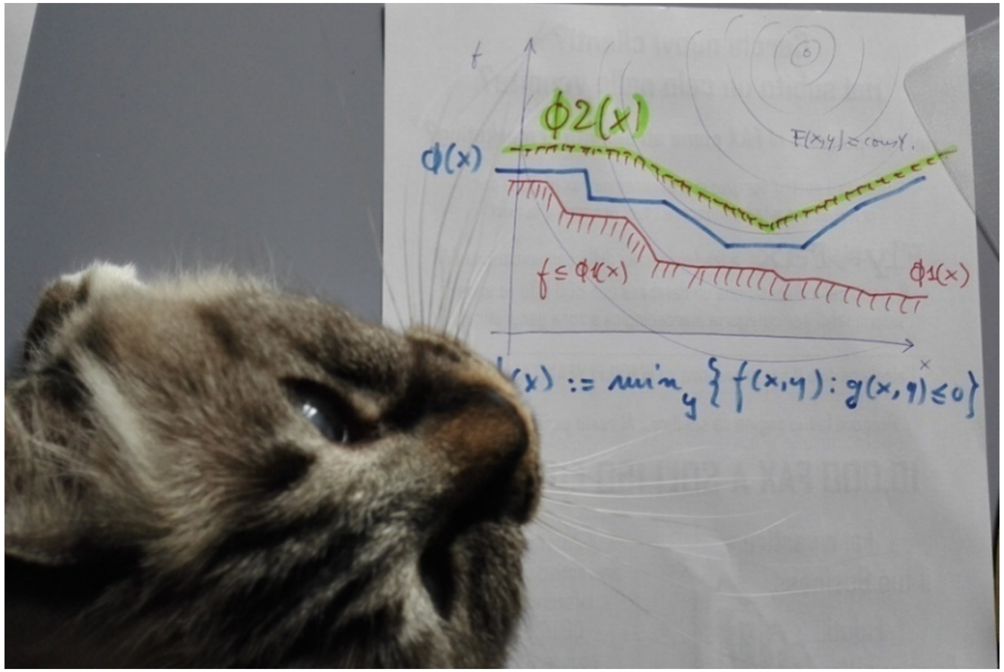


Bilevel Optimization

- The general **Bilevel Optimization Problem** (optimistic version) reads:

$$
\begin{aligned}
\min_{x \in \mathbb{R}^{n_1}, y \in \mathbb{R}^{n_2}} \; & F(x, y) \\
& G(x, y) \le 0 \\
& y \in \arg \min_{y' \in \mathbb{R}^{n_2}} \{ f(x, y') : g(x, y') \le 0 \}
\end{aligned}
$$

  where *x* var.s only are controlled by the **leader**, while *y* var.s are computed by another player (the **follower**) solving a different problem.

- A very very hard problem even in a **convex setting with continuous var.s** only

- **Convergent** solution algorithms are problematic and typically require additional assumptions (binary/integer var.s or alike)

Example: 0-1 ILP

- A generic 0-1 ILP can be reformulated as the following **linear & continuos bilevel problem**

$$\begin{aligned} \min c^T x \\ Ax = b \\ x \in \{0,1\}^n \end{aligned}$$

$$\begin{aligned} \min c^T x \\ Ax = b \\ x \in [0,1]^n \\ y = 0 \end{aligned}$$

$$y \in \arg\min_{y'} \{-\sum_{j=1}^{n} y'_j : \; y'_j \leq x_j, \; y'_j \leq 1 - x_j \quad \forall j = 1, \ldots, n\}$$

Note that y is fixed to 0 but it cannot be removed from the model!

Interdiction Problems

- A special case where *F(x,y) = - f(x,y)* and the action of the leader consists in the "**interdiction**" of some choices of the follower

- Typically stated as **min-max** optimization problems of the form:

$$
\begin{aligned}
\min_x \max_y \;& f(y) \\
& g(y) \leq 0 \\
& x_j \in \{0,1\}, && \forall j \in N \\
& 0 \leq y_j \leq UB_j(1 - x_j), && \forall j \in N \\
& F(x) \leq 0
\end{aligned}
$$

- E.g., the follower solves a max flow and the leader wants to keep the resulting flow as small as possible by **interdicting** (i.e., deleting) some arcs subject to a budget constraint $F(x) \leq 0$
- **Very very hard** both in theory (Sigma-2) and in practice

Reformulation

$$\min_{x\in\mathbb{R}^{n_1}, y\in\mathbb{R}^{n_2}} F(x,y)$$
$$G(x,y) \le 0$$
$$y \in \arg\min_{y'\in\mathbb{R}^{n_2}} \{f(x,y') : g(x,y') \le 0\}$$

- By defining the **value function**

$$\Phi(x) = \min_{y\in\mathbb{R}^{n_2}} \{f(x,y) : g(x,y) \le 0\},$$

the problem can be restated as

$$\begin{aligned} \min\, & F(x,y) \\ & G(x,y) \le 0 \\ & g(x,y) \le 0 \\ & f(x,y) \le \Phi(x) \\ & (x,y) \in \mathbb{R}^n. \end{aligned}$$

- Dropping the nonconvex condition $f(x,y) \le \Phi(x)$ one gets the so-called **High Point Relaxation** (HPR)

Mixed-Integer Bilevel Linear Problems

- We will focus the **Mixed-Integer Bilevel Linear** case (MIBLP) where *F, G, f* and *g* are **affine functions**, namely:

$$
\begin{aligned}
\min_{x,y}\ & c_x^T x + c_y^T y \\
& G_x x + G_y y \le q \\
& Ax + By \le b \\
& l \le y \le u \\
& x_j \text{ integer}, \quad \forall j \in J_x \\
& y_j \text{ integer}, \quad \forall j \in J_y \\
& d^T y \le \Phi(x)
\end{aligned}
$$

where for a given $x = x^*$ one computes the value function by solving the following **MILP**:

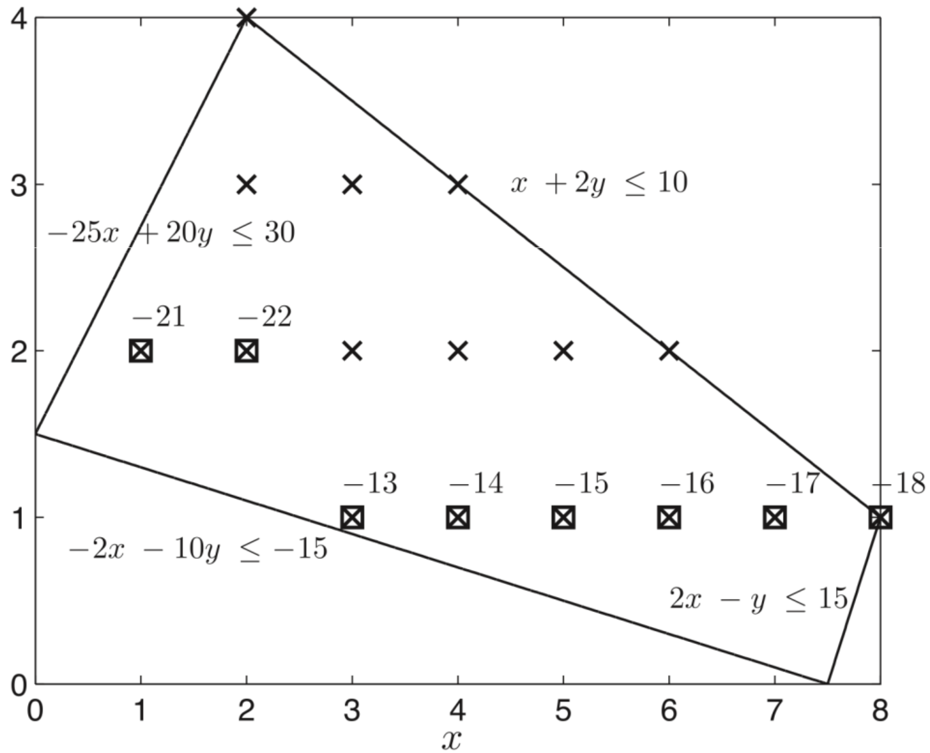
$$
\Phi(x^*) := \min_{y \in \mathbb{R}^{n_2}} \{ d^T y \ :\ By \le b - Ax^*, \quad l \le y \le u, \quad y_j \text{ integer } \forall j \in J_y \}.
$$

Example

- A notorious example from

J. Moore and J. Bard. The mixed integer linear bilevel programming problem. *Operations Research*, 38(5):911–921, 1990.

$$
\begin{aligned}
\min_{x\in\mathbb{Z}} \quad & -x - 10y \\
& y \in \arg\min_{y'\in\mathbb{Z}} \{ \; y' : \\
& \qquad -25x + 20y' \le 30 \\
& \qquad x + 2y' \le 10 \\
& \qquad 2x - y' \le 15 \\
& \qquad 2x + 10y' \ge 15 \; \}
\end{aligned}
$$



where *f(x,y) = y*

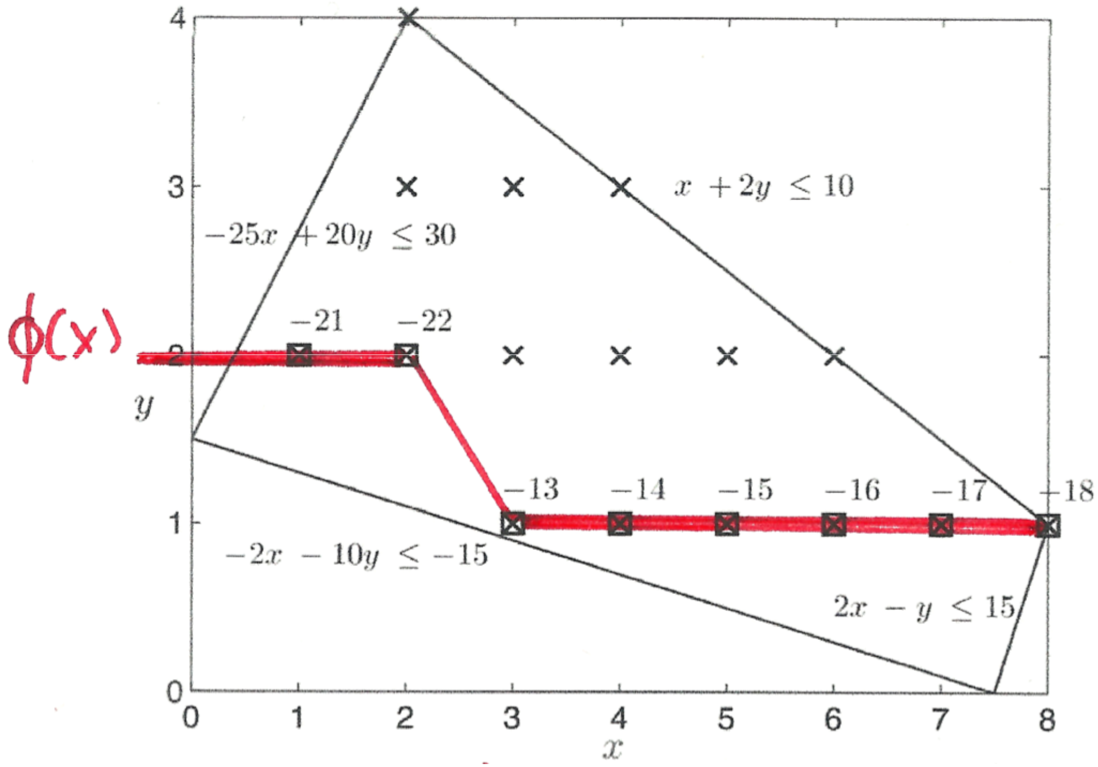
x points of HPR relax.

____ LP relax. of HPR

Example (cont.d)

Value-function reformulation

$$\begin{aligned} \min\ & -x - 10y \\ & -25x + 20y \leq 30 \\ & x + 2y \leq 10 \\ & 2x - y \leq 15 \\ & -2x - 10y \leq -15 \\ & x, y \in \mathbb{Z} \\ & y \leq \Phi(x) \end{aligned}$$



A convergent B&B scheme

Algorithm 2: A basic branch-and-bound scheme for MIBLP

Input : A MIBLP instance satisfying proper assumptions;
Output: An optimal MIBLP solution.

Apply a standard LP-based B&B to HPR, branching as customary on integer-constrained variables x_j and y_j that are fractional at the optimal LP solution; incumbent update is instead inhibited as it requires the bilevel-specific check described below;
for *each unfathomed B&B node where standard branching cannot be performed* **do**
  Let (x^*, y^*) be the integer HPR solution at the current node;
  Compute $\Phi(x^*)$ by solving the follower MILP for $x = x^*$;
  if $d^T y^* \leq \Phi(x^*)$ **then**
    The current solution (x^*, y^*) is bilevel feasible: update the incumbent and fathom the current node
  else
    if *not all variables x_j with $j \in J_F$ are fixed by branching* **then**
      Branch on any x_j $(j \in J_F)$ not fixed by branching yet, even if x_j^* is integer, so as to reduce its domain in both child nodes
    else
      let $(\hat{x}, \hat{y})$ be an optimal solution of the HPR at the current node amended by the additional restriction $d^T y \leq \Phi(x^*)$;
      Possibly update the incumbent with $(\hat{x}, \hat{y})$, and fathom the current node
    end
  end
end

Here J_F is the set of the leader x-variables appearing in the follower problem, all of which are assumed to be integer constrained (we also exclude HPR unboundedness)

A MILP-based solver

- We want to apply a standard Branch-and-Cut MILP solver to **HPR,** by generating **bilevel-specific cuts** on the fly to approximate the missing nonlinear condition $d^T y \leq \Phi(x)$ by a sequence of (local) **linear cuts**

- Forget for a moment about internal heuristics (i.e., deactivate all of them), and assume the LP relaxation at each node is solved by the simplex algorithm → each relevant sol. *(x*,y*)* comes with an **LP basis**

- At each B&C node, let *(x*,y*)* be the current **LP optimal vertex**:

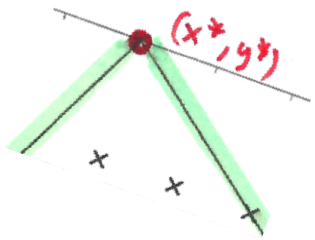
  if (x,y*)* is fractional → cut it by a MILP cut, or branch **as usual**

  if (x,y*)* is integer and $f(x^*, y^*) \leq \Phi(x^*)$ → *(x*,y*)* is bilevel-feasible and integer → update the incumbent **as usual**

i.e., no bilevel-specific actions are needed (the MILP solver already knows what to do)

The difficult case

- But, what can we do in third possible case, namely ***(x*,y*)* is integer but** not bilevel-feasible, i.e., when $f(x^*, y^*) > \Phi(x^*)$?



- **How can we cut this infeasible but integer *(x*,y*)* ?**

  Possible answers from the literature

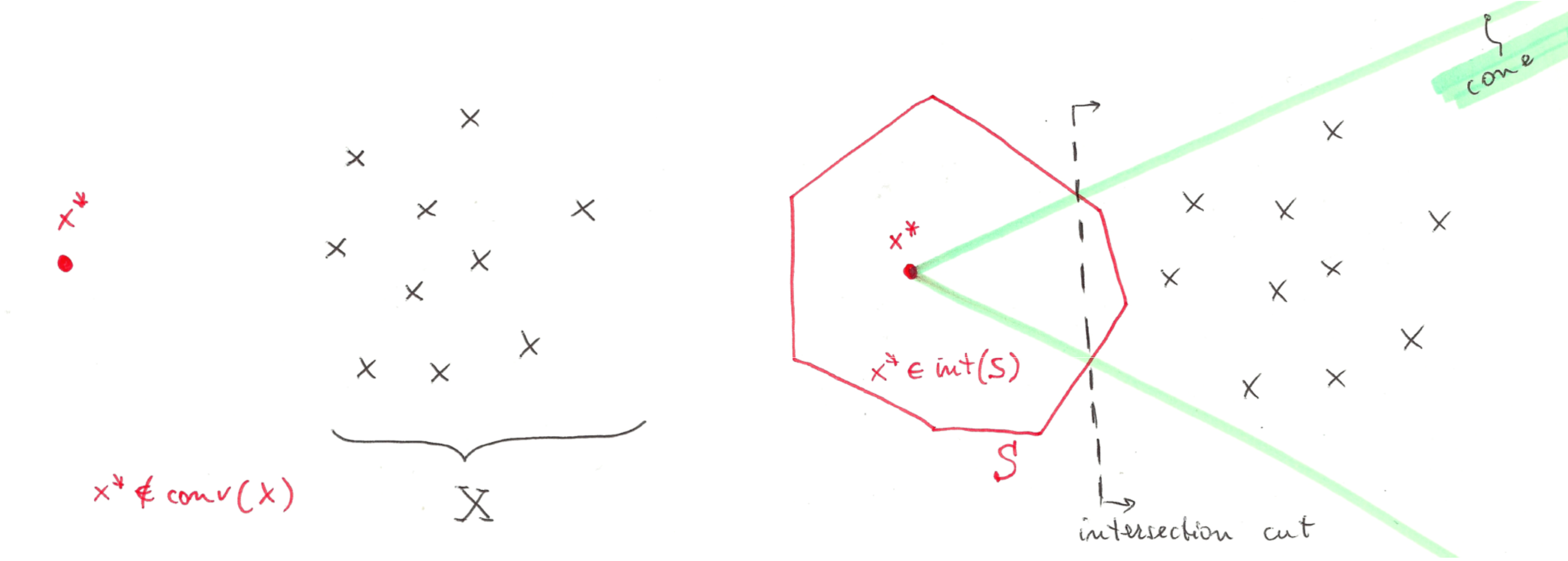
  – If *(x,y)* is restricted to be **binary**, add **a no-good** linear cut requiring to flip at least one variable w.r.t. *(x*,y*)* or w.r.t. *x**

  – If *(x,y)* is restricted to be **integer** and all MILP coeff.s are integer, add a cut requiring a slack of 1 for the sum of all the inequalities that are tight at *(x*,y*)*

- Is there a better way to enforce $f(x^*, y^*) \leq \Phi(x^*)$?

Intersection Cuts (ICs)

- Try and use of **intersection cuts** (Balas, 1971) instead
- ICs are a powerful tool to separate a point **x*** from a set **X** by a linear cut



- All you need is
  - a **cone** pointed at **x***, containing all **x ε X**
  - a **convex set S** with x* (but no **x ε X**) in its **interior**
- If x* **vertex** of an LP relaxation, a suitable cone comes for the **LP basis**

ICs for bilevel problems

- Our idea is first illustrated on the Moore&Bard example

$$\min_{x \in \mathbb{Z}} \; -x - 10y$$

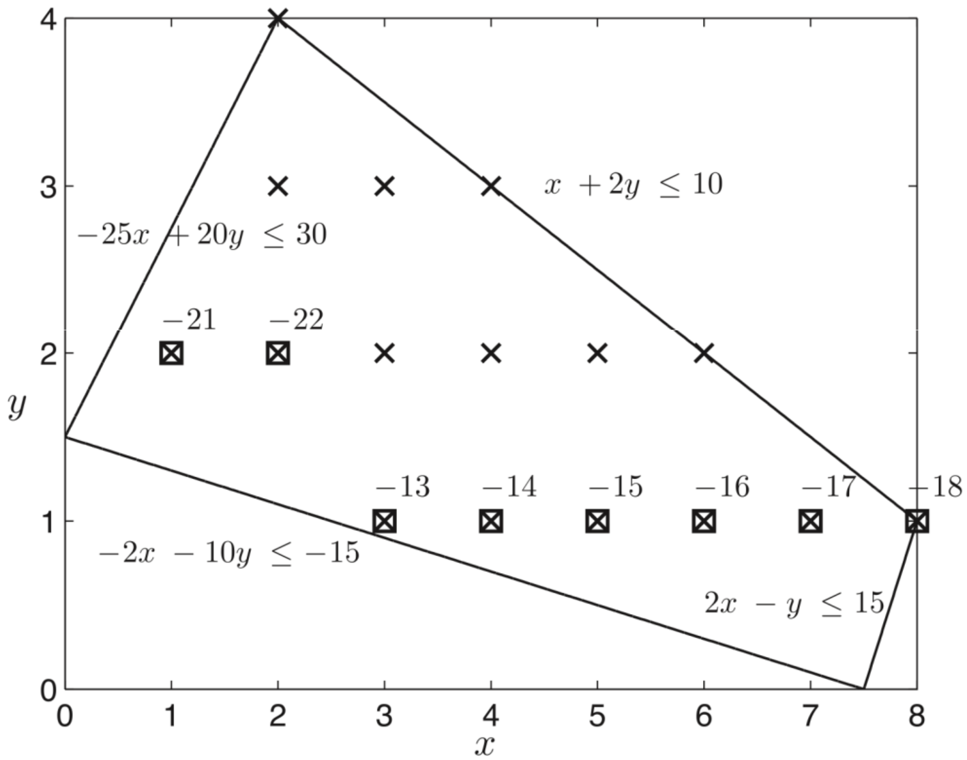
$$y \in \arg\min_{y' \in \mathbb{Z}} \{ \, y' :$$

$$\begin{aligned} -25x + 20y' &\leq 30 \\ x + 2y' &\leq 10 \\ 2x - y' &\leq 15 \\ 2x + 10y' &\geq 15 \; \} \end{aligned}$$

where *f(x,y) = y*

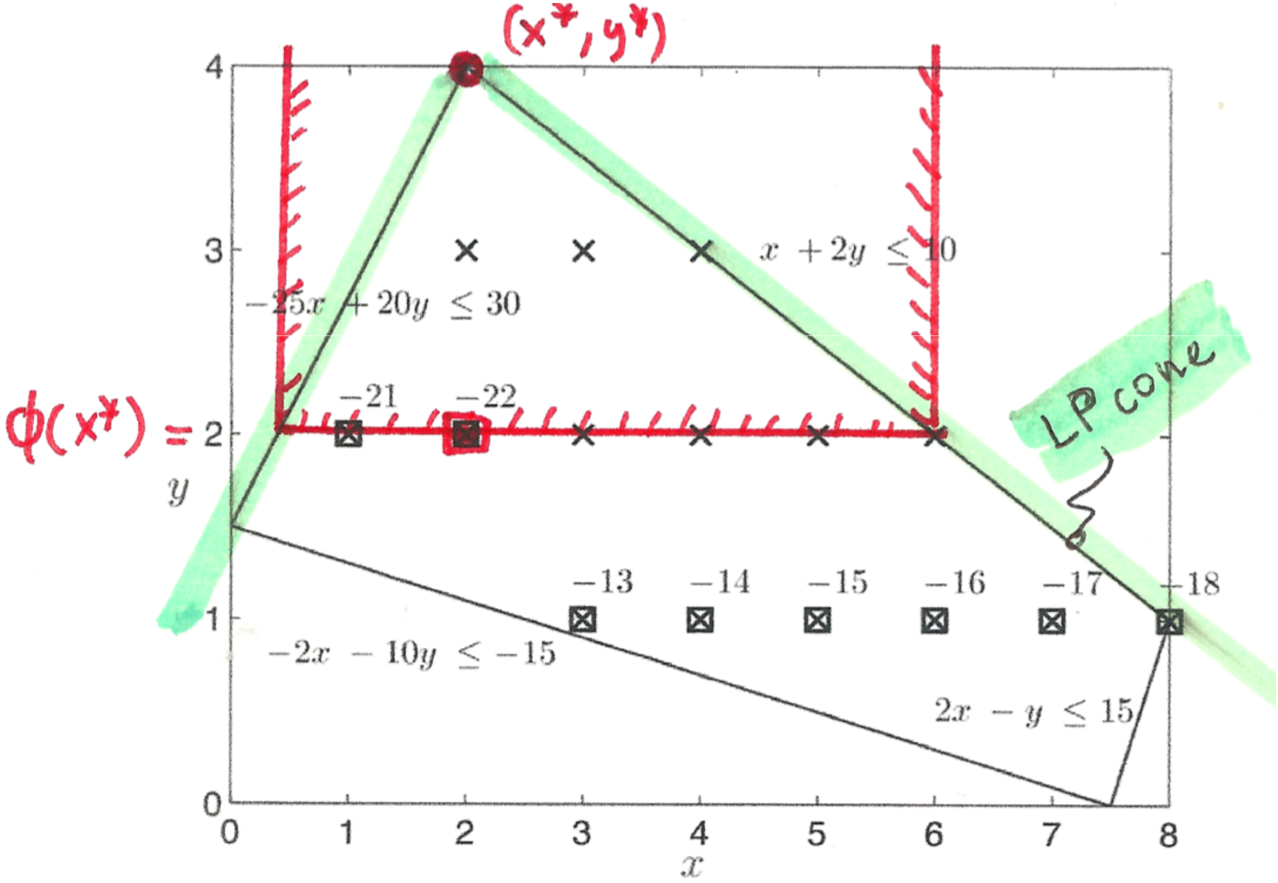
x points of HPR relax.

____ LP relax. of HPR



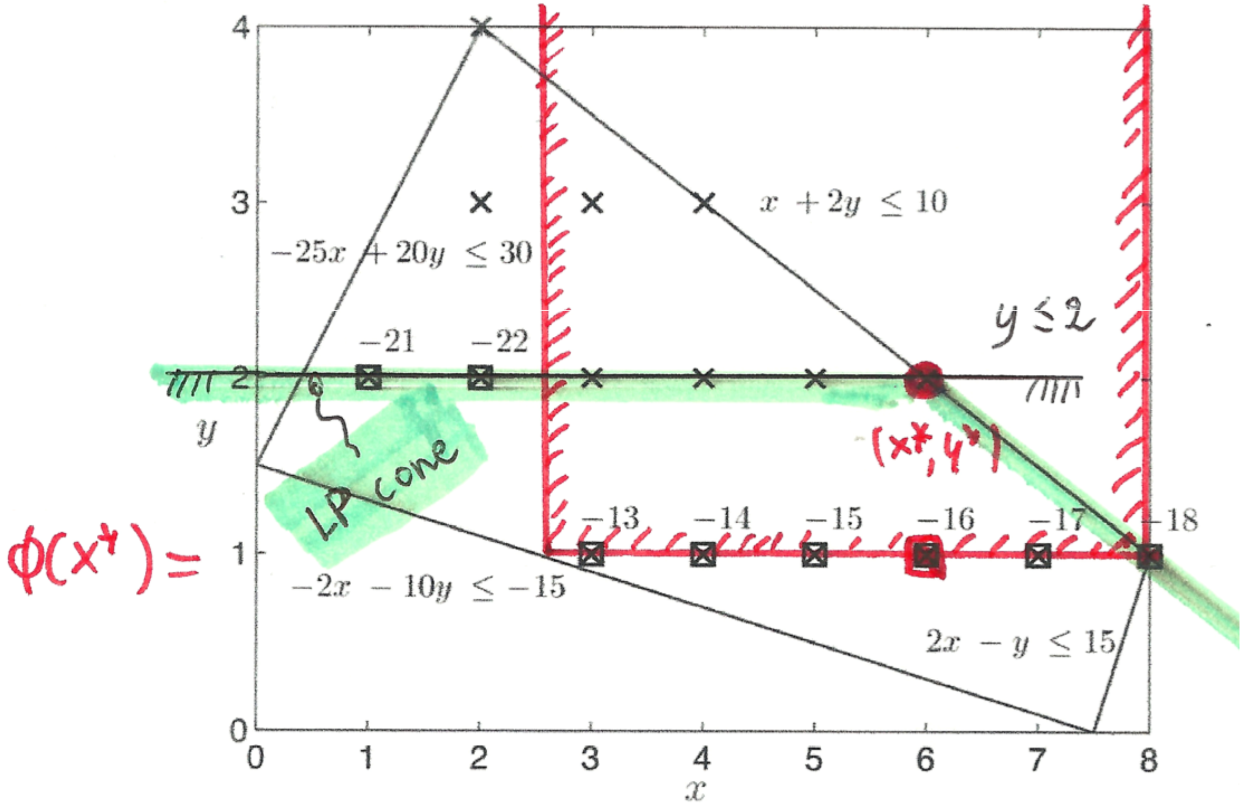
Define a suitable bilevel-free set

- Take the LP vertex $(x^*,y^*) = (2,4)$ → $f(x^*,y^*) = y^* = 4 > Phi(x^*) = 2$



Intersection cut

- We can therefore generate the intersection cut *y <= 2* and repeat



Constructing a bilevel-free set

Lemma 1. *For any feasible solution $\hat{y}$ of the follower, the set*

$$S(\hat{y}) = \{(x,y) \in \mathbb{R}^n : f(x,y) \geq f(x,\hat{y}),\ g(x,\hat{y}) \leq 0\} \tag{10}$$

does not contain any bilevel-feasible point in its interior.

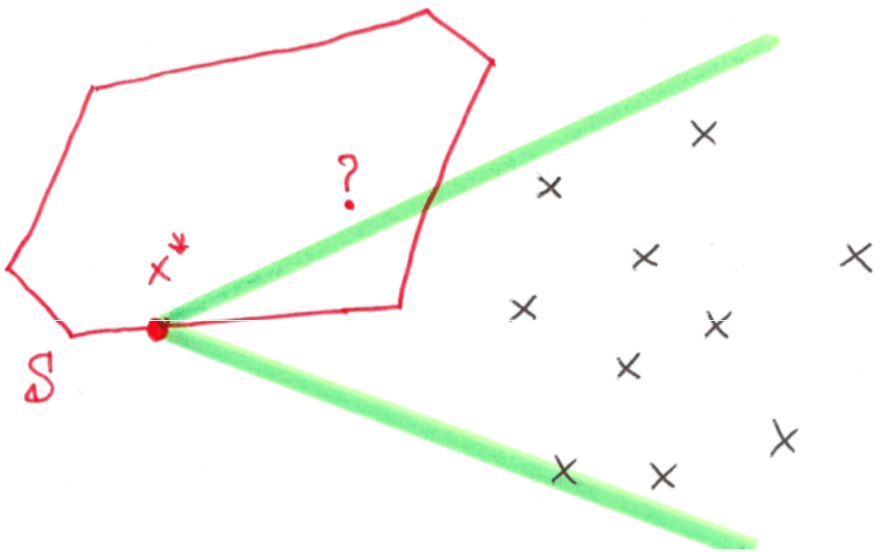
- **Note**: $S(\hat{y})$ is a convex set (actually, a **polyhedron**) in the MIBLP case

- **Separation algorithm**: given an optimal vertex *(x*,y*)* of the LP relaxation of HPR

  - Solve the follower for *x=x** and get an optimal sol., say $\hat{y}$
  - **if** *(x*,y*)* strictly inside $S(\hat{y})$ **then**
    generate a violated IC using the LP-cone pointed at *(x*,y*)*
    together with the bilevel-free set $S(\hat{y})$

However…

- The above Lemma does exclude that ***(x*,y*)*** can be on the **frontier** of the bilevel-free set $S(\hat{y})$, so we cannot guarantee to cut it …



- We need to define an **enlarged** bilevel-free set if we want be sure to cut *(x*,y*),* though this requires additional assumptions

An enlarged bilevel-free set

- Assuming ***g(x,y)* is integer** for all integer HPR solutions, one can "move apart" by 1 the frontier of $S(\hat{y})$ so as be sure that the point *(x*,y*)* belongs to its interior

Theorem 1. *Assume that $g(x,y)$ is integer for all HPR solutions (x,y). Then, for any feasible solution $\hat{y}$ of the follower, the extended set*

$$S^+(\hat{y}) = \{(x,y) \in \mathbb{R}^n : f(x,y) \geq f(x,\hat{y}),\ g(x,\hat{y}) \leq 1\} \tag{11}$$

does not contain any bilevel-feasible point in its interior, where 1 *denotes a vector of all one's.*

- The above result leads to a "minimalist" **B&C solver for MIBLP**
- **Notes** (see the full papers for details)
  - **branching** on integer variables can be required to break tailing-off and to ensure finite convergence
  - **alternative** bilevel-free sets can be defined to produce hopefully deeper ICs
  - **additional features** (preprocessing, heuristics etc.) available

IC-separation numerical issues

- IC separation can be problematic, as we need to read the cone rays from the LP tableau → **numerical accuracy** can be a big issue here!

- For **MILP**s, ICs like Gomory cuts are **not mandatory** (so we can skip their generation in case of numerical problems), but for **MIBLP**s they are instrumental **#SeparateOrPerish**

- **Notation change**: let $\xi = (x, y) \in \mathbb{R}^n$

$\min\{\hat{c}^T \xi : \hat{A}\xi = \hat{b}, \xi \geq 0\}$ be the LP relaxation at a given node

$S = \{\xi : g_i^T \xi \leq g_{0i},\ i = 1, \ldots, k\}$ be the bilevel-free set

$\bigvee_{i=1}^{k} (g_i^T \xi \geq g_{i0})$ be the corresp. disjunction (valid for all feas. sol.s)

Numerically safe ICs

A **single** valid inequality can be obtained by taking, for each variable, the worst LHS Coefficient (and RHS) in each disjunction

$$\bigvee_{i=1}^{k} (g_i^T \xi \geq g_{i0})$$

$$\bigvee_{i=1}^{k} (\overline{g}_i^T \xi \geq \overline{g}_{i0})$$

To be applied to a **reduced form** of each disjunction where the coefficient of all basic variables is zero (kind of LP reduced costs)

$$\bigvee_{i=1}^{k} (\frac{\overline{g}_i^T}{\overline{g}_{i0}} \xi \geq 1)$$

Algorithm 1: Intersection cut separation

Input : An LP vertex ξ^* along with its a associated LP basis $\hat{B}$;
the feasible-free polyhedron $S = \{\xi : g_i^T \xi \leq g_{0i},\ i = 1, \ldots, k\}$ and the associated valid disjunction $\bigvee_{i=1}^{k} (g_i^T \xi \geq g_{i0})$ whose members are violated by ξ^*;

Output: A valid intersection cut violated by ξ^*;

1 **for** $i := 1$ **to** k **do**
2 $\quad(\overline{g}_i^T, \overline{g}_{i0}) := (g_i^T, g_{i0}) - u_i^T(\hat{A}, \hat{b})$, where $u_i^T = (g_i)_{\hat{B}}^T \hat{B}^{-1}$
3 **end**
4 **for** $j := 1$ **to** n **do** $\gamma_j := \max\{\overline{g}_{ij} / \overline{g}_{i0} : i \in \{1, \ldots, k\}\}$;
5 **return** *the violated cut* $\gamma^T \xi \geq 1$

Conclusions

- Mixed-Integer Bilevel Linear Programming is a **MILP** plus additional constr.s
- **Intersection cuts** can produce valuable information at the B&B nodes
- Sound MIBLP **heuristics**, **preprocessing** etc. (not discussed here) available
- Many instances from the literature can be **solved in a satisfactory way**
- Our **binary code** is available on request (research purposes)

Slides http://www.dei.unipd.it/~fisch/papers/slides/

Reference papers:

M. Fischetti, I. Ljubic, M. Monaci, M. Sinnl, "Intersection cuts for bilevel optimization", in Integer Programming and Combinatorial Optimization: 18th International Conference, IPCO 2016 Proceedings, 77-88, 2016 (to appear in *Mathematical Programming*)

M. Fischetti, I. Ljubic, M. Monaci, M. Sinnl, "A new general-purpose algorithm for mixed-integer bilevel linear program", to appear in *Operations Research*.

M. Fischetti, I. Ljubic, M. Monaci, M. Sinnl, "Interdiction Games and Monotonicity", Tech. Report 2016 (submitted)