Mathematical Optimization for Social Distancing

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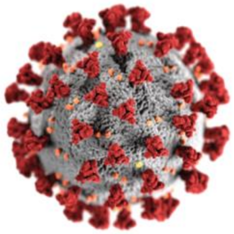
Picture from https://www.ted.com/playlists/735/the_fight_against_viruses

Based on joint work with Martina Fischetti and Jakob Soustroup

INFORMS annual meeting, November 7-13, 2020

Facility location under social distancing constraints



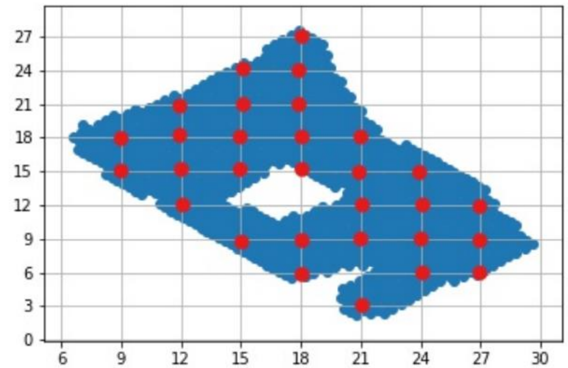
The spread of viruses such as SARS-CoV-2 brought new challenges to our society, including a stronger focus on safety across all businesses

In particular, many countries have imposed a **minimum social distance** between people (or, more generally, **facilities**) to ensure their safety

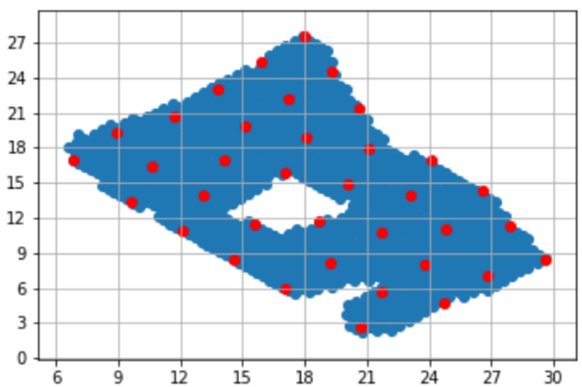
Can **Operations Research** help finding **more efficient** yet **safer** location patterns for facilities?

Example: outdoor table allocation

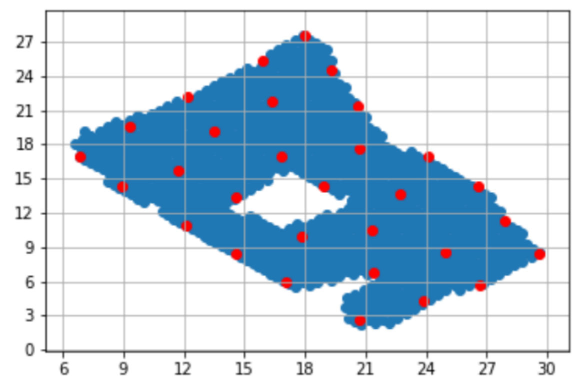
A brewpub in Copenhagen



Regular layout: 30 tables
on a regular 3m x 3m grid



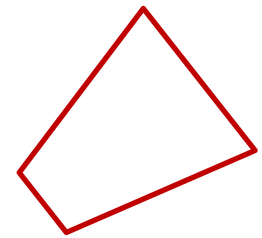
Optimized layout fitting 6 more tables



Still 30 tables, but in globally
safer configuration

The safest distance problem

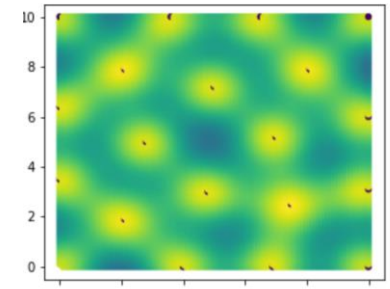
Given:

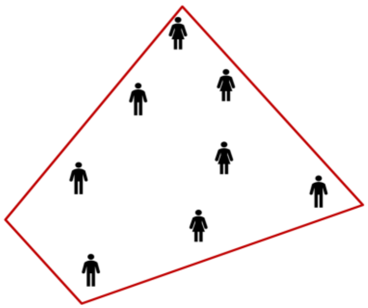


- An available (possibly disconnected/irregular) **area**
- A **minimum and maximum n. of facilities** to locate in that area

- A **minimum distance** between facilities

- A **virus-spread model** to measure infection probability among facilities

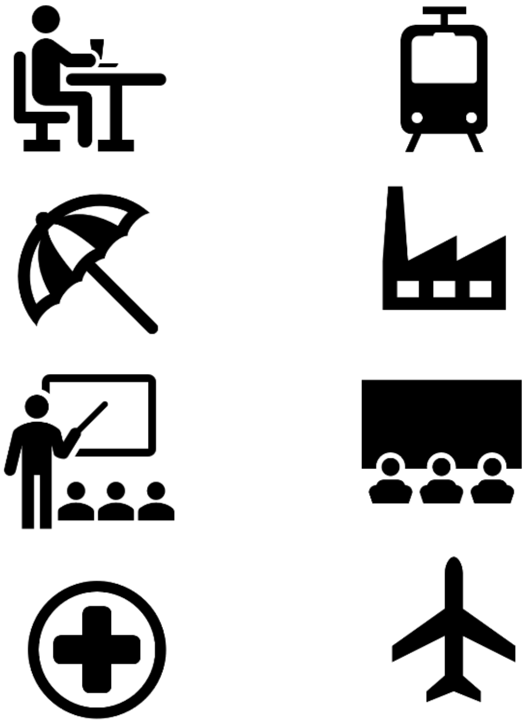




Find:

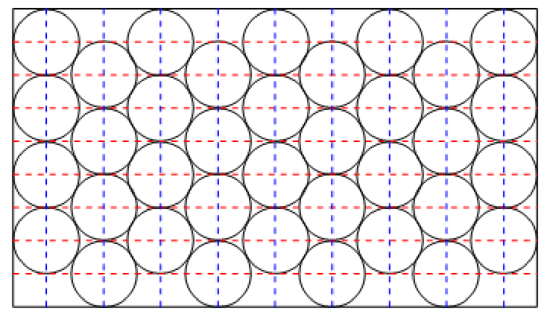
A facility location pattern that minimizes the overall infection probability (sum of all pairwise probabilities)

Applications



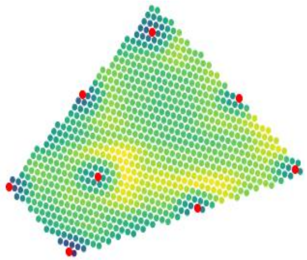
2 variants of the problem:

- Fit as many customers as possible while respecting social distancing → a familiar **packing problem**



- Fit a fixed number of customers **maximizing their safety →** a new (?) quadratic optimization problem

A mathematical optimization model



- The available **area** is sampled to get a set $V = \{1,\ldots,n\}$ of possible allocation points
- Let N_{MIN} and N_{MAX} be min and max n. of facilities to allocate, resp.
- Let P_i denote the **profit** of allocating a facility at point *i* (*e.g.,* $P_i \equiv 1$)
- Define an **incompatibility** graph $G_I = (V, E_I)$ whose edges *[i,j]* correspond to infeasible pairs with *distance(i,j)* < *minimum_distance*
- Let I_{ij} denote the **infection probability** that *j* is infected by *i* (assuming *i* is positive)

i → *j*

The safest distance problem

$$\max \sum_{i \in V} P_i x_i - \sum_{i \in V} \sum_{j \in V} I_{ij} x_i x_j \quad (1)$$

$$\text{s.t.} \quad N_{MIN} \leq \sum_{i \in V} x_i \leq N_{MAX} \quad (2)$$

$$x_i + x_j \leq 1 \qquad \forall [i, j] \in E_I \quad (3)$$

$$x_i \in \{0, 1\} \qquad \forall i \in V \quad (4)$$

An analogy: wind turbine allocation

An analogy: wind turbine allocation

Wind farm layout optimization

Given

- a site (**offshore** or onshore)
- characteristics of the turbines to build
- measurements of the wind in the site

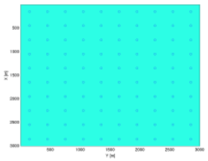
Determine a turbine allocation that **maximizes power production**

Taking into account:

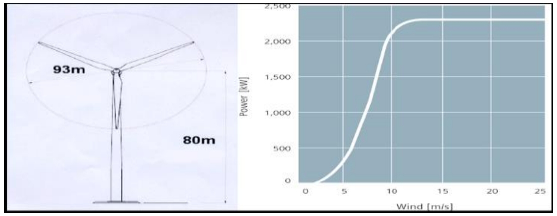
- proximity constraints (no collisions)
- minimum/maximum number of turbines
- paiwise interference due to wake effects

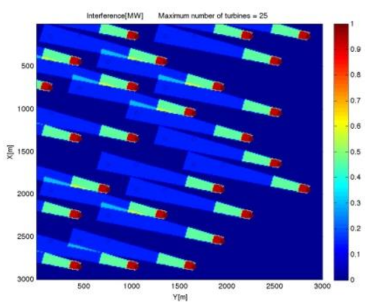
The wind farm layout problem

- Define a grid of candidate **positions** for turbine allocation

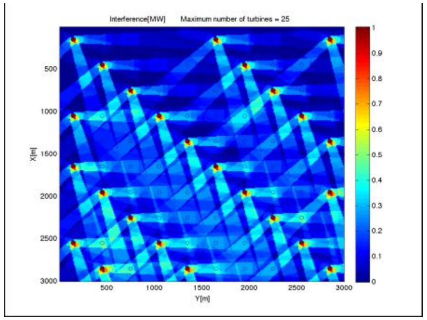


- For each pair of positions *i,j,* let I_{ij} denote the average **interference** (power loss) experienced at position *j* if a turbine is built in position *i* → it depends on average wind speed/direction, turbine power curve, etc.

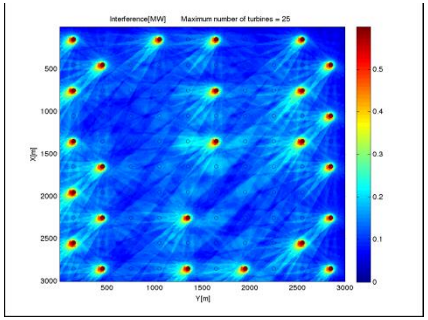




only one wind scenario



three wind scenarios



several wind scenarios

- Assume overall interference is **cumulative** (sum of pairwise interf.s)

Basic (noncovex) quadratic MIP

$$\max \sum_{i \in V} P_i x_i - \sum_{i \in V} \sum_{j \in V} I_{ij} x_i x_j \tag{1}$$

$$\text{s.t.} \quad N_{MIN} \leq \sum_{i \in V} x_i \leq N_{MAX} \tag{2}$$

$$x_i + x_j \leq 1 \qquad \forall [i,j] \in E_I \tag{3}$$

$$x_i \in \{0,1\} \qquad \forall i \in V \tag{4}$$

A standard MILP reformulation

- Introduce a quadratic n. of var.s $z_{ij} = x_i \, x_j$

$$
\begin{aligned}
\max \sum_{i \in V} P_i x_i - \sum_{i \in V} \sum_{j \in V, i<j} (I_{ij} + I_{ji}) z_{ij} & & (1)\\
\text{s.t.} \quad N_{MIN} \le \sum_{i \in V} x_i \le N_{MAX} & & (2)\\
x_i + x_j \le 1 \quad & \forall [i,j] \in E_I & (3)\\
x_i + x_j - 1 \le z_{ij} \quad & \forall i,j \in V, i<j & (4)\\
x_i \in \{0,1\} \quad & \forall i \in V & (5)\\
z_{ij} \in \{0,1\} \quad & \forall i,j \in V, i<j & (6)
\end{aligned}
$$

An alternative MILP reformulation

Glover's trick: the objective function

$$\sum_{i \in V} P_i x_i - \sum_{i \in V} (\sum_{j \in V} I_{ij} x_j)\, x_i \qquad (11)$$

is restated as

$$\sum_{i \in V} (P_i x_i - w_i) \qquad (12)$$

where

$$w_i := (\sum_{j \in V} I_{ij} x_j)\, x_i = \begin{cases} \sum_{j \in V} I_{ij} x_j & \text{if } x_i = 1; \\ 0 & \text{if } x_i = 0. \end{cases}$$

→ the new continuous variable w_i is the product between a continuous term (∑ …) and a binary variable (x_i)

An alternative MILP reformuation

A linearized model with linear n. of additional var.s ***w_i*** and BIGM constr.s

$$\max z = \sum_{i \in V} (P_i x_i - w_i) \tag{13}$$

s.t.

$$N_{MIN} \le \sum_{i \in V} x_i \le N_{MAX} \tag{14}$$

$$x_i + x_j \le 1 \qquad \forall [i,j] \in E_I \tag{15}$$

$$\sum_{j \in V} I_{ij} x_j \le w_i + M_i(1 - x_i) \qquad \forall i \in V \tag{16}$$

$$x_i \in \{0,1\} \qquad \forall i \in V \tag{17}$$

$$w_i \ge 0 \qquad \forall i \in V \tag{18}$$

where $M_i >> 0$ (BIGM)

Solution methods

- **No interference →** packing problem → exactly solvable by modern MILP solvers up to hundreds of possible positions

- **With interference →** very hard quadratic problem

- Proximity search matheuristics from the wind-farm opt. literature


J Heuristics (2016) 22:459–474
DOI 10.1007/s10732-015-9283-4

CrossMark

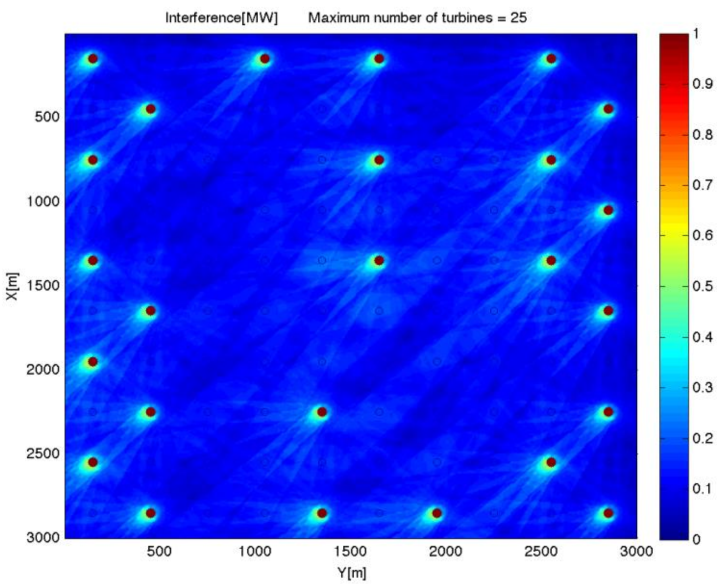
Proximity search heuristics for wind farm optimal layout

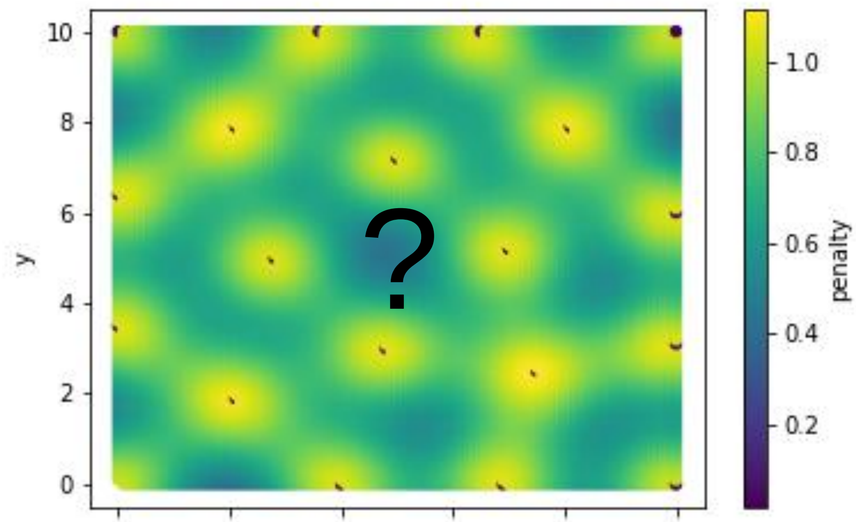
Martina Fischetti · Michele Monaci

Martina Fischetti and Michele Monaci (2016), Proximity search heuristics for wind farm optimal layout, *Journal of Heuristics* 22 (4), pp. 459-474.

- Results in the present talk have been obtained by an ad-hoc wind-farm layout optimization software developed by *Double-Click SRL*, Padua, Italy

Modelling virus spread





Reliable wind turbine interference models in the literature (e.g., Jensen's model)

Virus spread models?

Modelling virus spread

Let d_{ij} be the distance (in m) between points *i* and *j,* and let d_{max} be the max. distance in the given area

Alternative definitions for the virus-interference I_{ij}

$$I_{ij} = d_{max} - d_{ij}$$

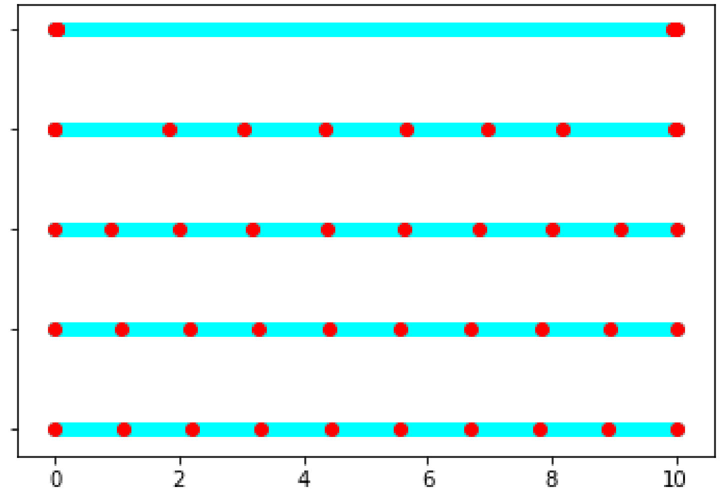
Proportional to distance

$$I_{ij} = e^{-d_{ij}^2/2}$$

Gaussian (σ = 1m)

$$I_{ij} = 1/d_{ij}$$
$$I_{ij} = 1/d_{ij}^2$$
$$I_{ij} = 1/d_{ij}^3$$

Uniform concentration inside a 1- or 2- or 3-dimensional sphere



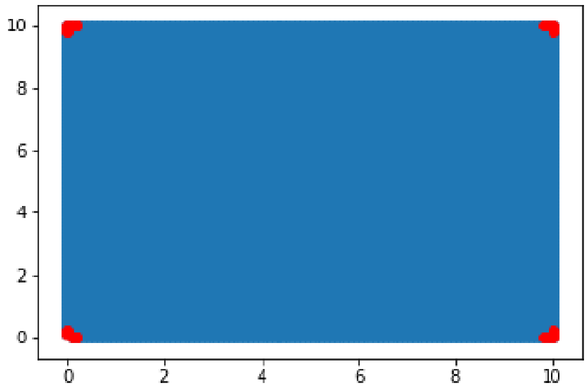
- **Proportional** seems a natural choice
- **Gaussian** fits droplet (negligible after 3 σ)
- **Uniform concentration** fits aerosol spread

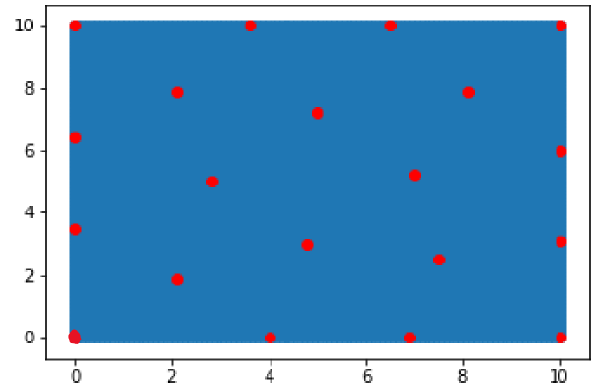
Example: optimal allocation of **10 facilities** on a line segment (no minimum distance imposed)

Modeling virus spread

Optimal allocation of 20 facilities on a square

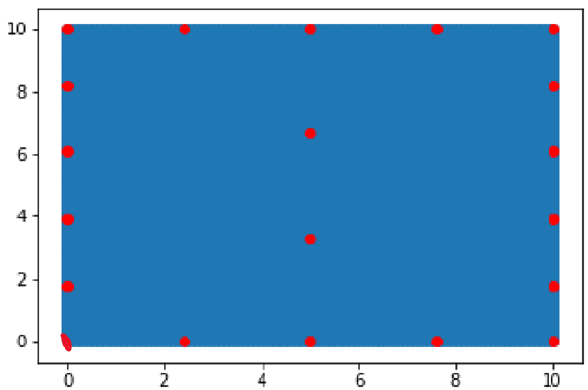
Proportional to distance

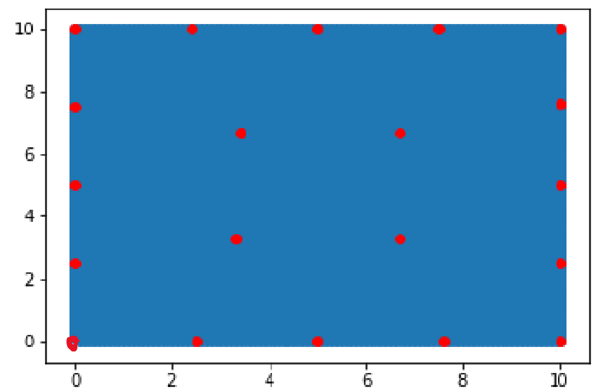




Gaussian
(σ = 1m)

uniform
on a 1-
dim

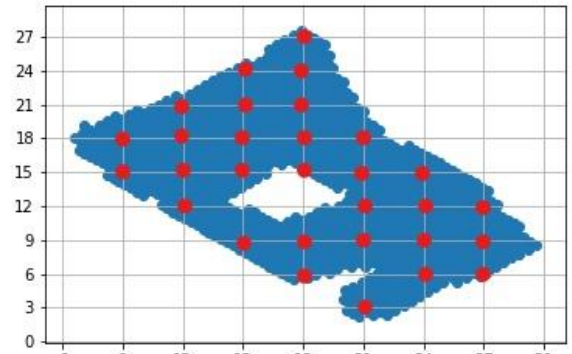




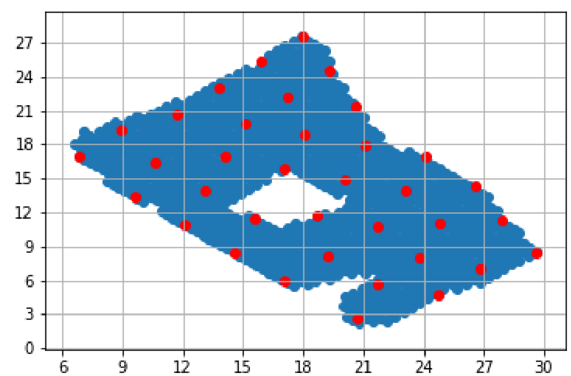
uniform
2- or 3-dim

Applications: a pub in Copenhagen

Minimum distance of 3m



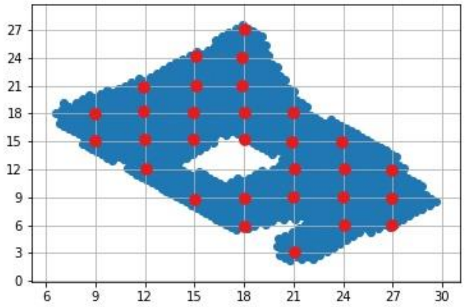
Regular solution



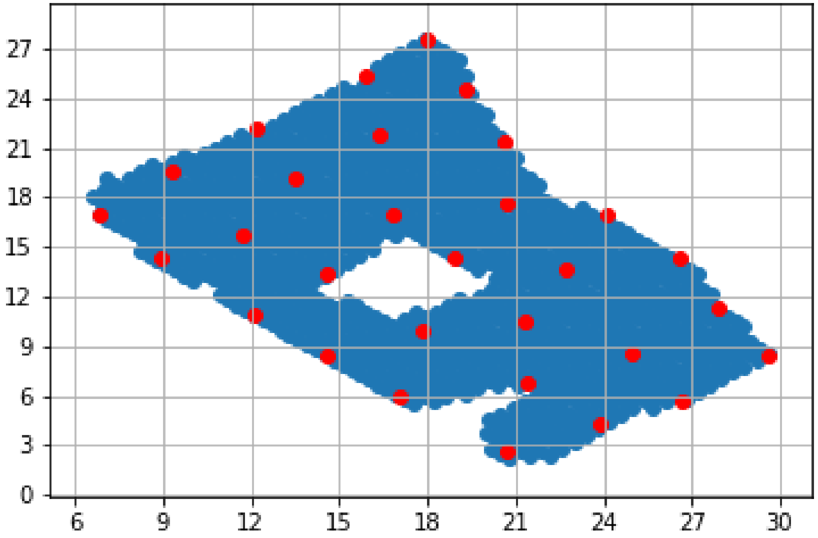
Optimized solution with 6 more tables

Applications: a pub in Copenhagen

Min dist. 3m but fixed n. of tables

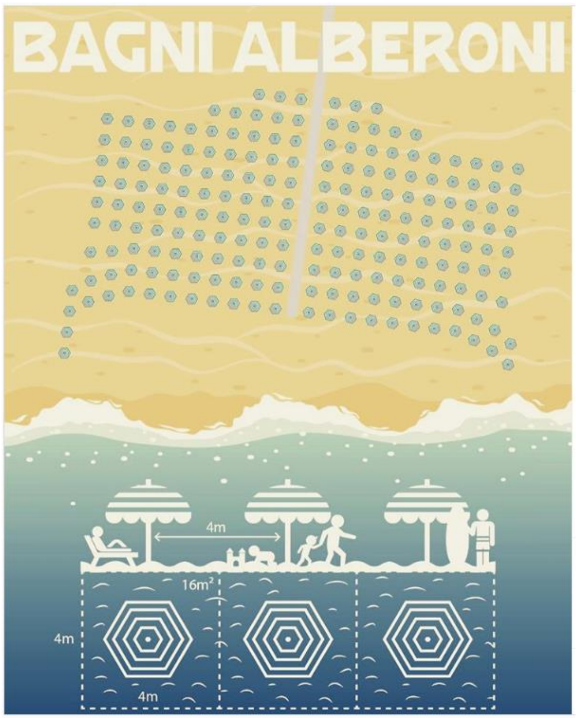


Regular solution



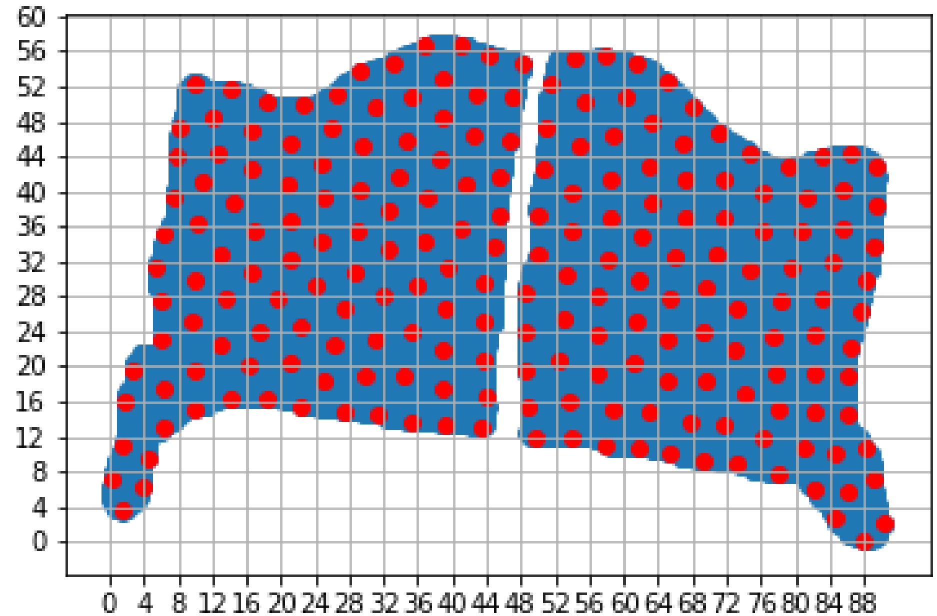
Optimized solution (safer)

Applications: beach umbrellas in Venice



Min distance of 4m

Optimized solution



8 more umbrellas

Lessons learned

- **Facility location under social distance** constraints and wind farm layout optimization are "similar" problems → we can borrow models and algorithms from wind farm literature

- **Operations Research** can make a big impact for businesses and customers, both in terms of profit and safety

- Fair layouts are often **not regular** **and not easy to find "by hand"**

- Mathematical optimization is instrumental to finding **more profitable** yet **safer** configurations

Full paper available at **www.dei.unipd.it/~fisch/papers**

Martina Fischetti, Matteo Fischetti, Jakob Stoustrup, Mathematical optimization for social distancing, Tech. Rep. UniPD, 2020.