A Benders Decomposition Approach to the Time Window Assignment Traveling Salesperson Problem with Stochastic Travel Times

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Joint work with
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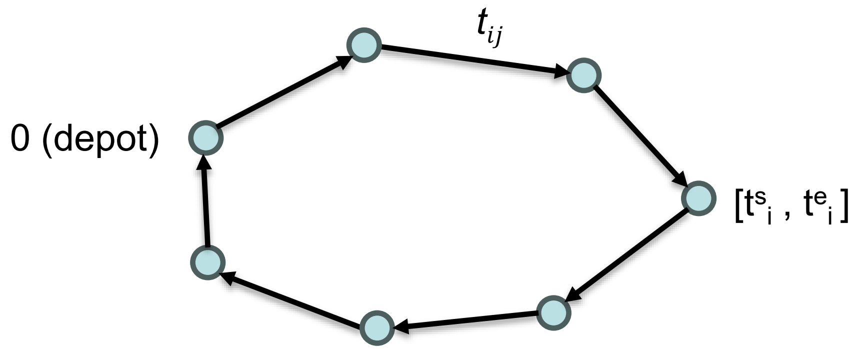
The TWATSP problem

Given:

- $G = (V, A)$ **directed** complete graph: $V = \{0,\dots,n\}$: 0 depot, $V^+ = \{1,\dots,n\}$ custom.s
- arc costs d_{ij} (deterministic) and travel times $t_{ij} \geq 0$ (**stochastic**)
- Maximum time *T* (deterministic "shift duration")

Find

- A Hamiltonian tour (ATSP sol. x_{ij}) of total cost ≤ T
- and assign each customer *i* a time window $[t^s_i , t^e_i]$ (s=start, e=end)



Note: time windows are decision var.s, not input data as in the classical TW-ATSP!

The TWATSP problem

Objective: minimize

1) the **tour cost** (i.e., the sum of the dij's of its arcs);
2) the sum of the **durations** t^e_i - t^e_i i of all time windows, weighed by an input penalty factor $\alpha \geq 0$;

For each scenario $\omega \in \Omega$, weighed by its probability p^ω:

3) the shift **overtime** o^ω (say) w.r.t. maximum shift duration T, weighed by $\beta \geq 0$;
4) the sum of the **time-window violations** e_i^ω (earliness) plus l_i^ω (lateness), weighed by $\delta \geq 0$

$$\min \sum_{i \in V} \sum_{j \in V} d_{ij} x_{ij} + \sum_{i \in V^+} \alpha (t_i^e - t_i^s) + \sum_{\omega \in \Omega} p^\omega \left[\beta\, o^\omega + \delta \sum_{j \in V^+} (e_j^\omega + l_j^\omega) \right]$$

MILP model from the literature

$$\min \sum_{i\in V}\sum_{j\in V} d_{ij}x_{ij} + \sum_{i\in V^+} \alpha(t_i^e - t_i^s) + \sum_{\omega\in\Omega} p^\omega \left[\beta\, o^\omega + \delta \sum_{j\in V^+}(e_j^\omega + l_j^\omega)\right]$$

$$\text{s.t.} \sum_{i\in V} x_{ij} = \sum_{i\in V} x_{ji} = 1 \qquad j \in V$$

$$\sum_{i\in S}\sum_{j\notin S} x_{ij} \geq 1 \qquad S \subset V : 0 \in S$$

$$t_i^e - t_i^s \geq 0 \qquad i \in V^+$$

$$w_j^\omega \geq w_i^\omega + t_{ij}^\omega - M(1 - x_{ij}) \qquad i \in V, j \in V^+, \omega \in \Omega$$

$$t_i^s - e_i^\omega \leq w_i^\omega \leq t_i^e + l_i^\omega \qquad i \in V^+, \omega \in \Omega$$

$$w_i^\omega + t_{i0}^\omega - M(1 - x_{i0}) \leq T + o^\omega \qquad i \in V^+, \omega \in \Omega$$

$$w_0^\omega = 0 \qquad \omega \in \Omega$$

$$x_{ij} \in \{0, 1\} \qquad i, j \in V$$

$$t_i^s, t_i^e \in \mathbb{R}_+ \qquad i \in V^+$$

$$w_i^\omega \in \mathbb{R}_+ \qquad i \in V, \omega \in \Omega$$

$$e_i^\omega, l_i^\omega \in \mathbb{R}_+ \qquad i \in V^+, \omega \in \Omega$$

$$o^\omega \in \mathbb{R}_+ \qquad \omega \in \Omega,$$

Big-M constraint

w_i^ω = arrival time at node i

$$w_j^\omega \geq w_i^\omega + t_{ij}^\omega - M(1 - x_{ij})$$

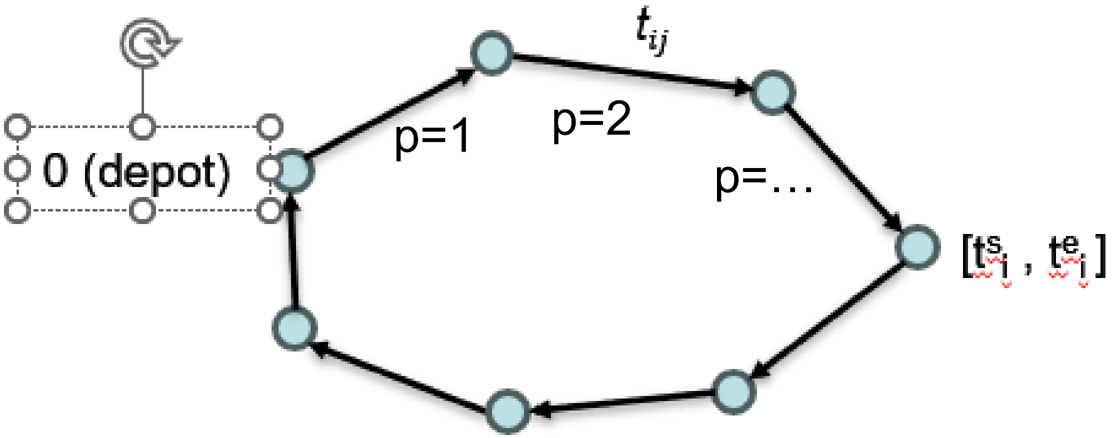
- Ş. Çelik, L. Martin, A. H. Schrotenboer, and T. Van Woensel. Exact two-step Benders decomposition for the time window assignment traveling salesperson problem. Transportation Science, 59(2):210–228, 2025.

A new 3-index formulation

- Idea: use **positional** (binary) variables

$x_{ij}^{p} = 1$ iff arc (i,j) selected in position p

as in: K.R. Fox, B. Gavish, and S.C. Graves. An n-constraint formulation of the (time-dependent) traveling salesman problem. Operations Research, 28 (4):1018–1021, 1980.



A new 3-index formulation

$$\min \sum_{i\in V}\sum_{j\in V}\sum_{p\in P} d_{ij}x_{ij}^p + \sum_{p\in P'} \alpha(t_p^e - t_p^s) + \sum_{\omega\in\Omega} p^\omega \left[\beta\, o^\omega + \delta \sum_{p\in P'} (e_p^\omega + l_p^\omega)\right]$$

$$\text{s.t.} \sum_{i\in V}\sum_{j\in V} x_{ij}^p = 1 \qquad p \in P \setminus \{1, n+1\}$$

$$\sum_{j\in V} x_{0j}^1 = 1$$

$$\sum_{i\in V} x_{i0}^{n+1} = 1$$

$$\sum_{i\in V} x_{ih}^{p-1} = \sum_{j\in V} x_{hj}^p \qquad p \in P \setminus \{1\},\ h \in V^+$$

$$\sum_{i\in V}\sum_{p\in P} x_{ij}^p = \sum_{i\in V}\sum_{p\in P} x_{ji}^p = 1 \qquad j \in V$$

$$t_p^e - t_p^s \geq 0 \qquad p \in P'$$

$$w_1^\omega \geq \sum_{j\in V} d_{0j}^\omega x_{0j}^1 \qquad \omega \in \Omega$$

$$w_{p+1}^\omega \geq w_p^\omega + \sum_{i\in V}\sum_{j\in V} d_{ij}^\omega x_{ij}^{p+1} \qquad p \in P', \omega \in \Omega$$

$$t_p^s - e_p^\omega \leq w_p^\omega \leq t_p^e + l_p^\omega \qquad p \in P',\ \omega \in \Omega$$

$$w_{n+1}^\omega \leq T + o^\omega \qquad \omega \in \Omega$$

$$x_{ij}^p \in \{0,1\} \qquad i, j \in V,\ p \in P$$

$$t_p^s, t_p^e \in \mathbb{R}_+ \qquad p \in P'$$

$$e_p^\omega, l_p^\omega \in \mathbb{R}_+ \qquad p \in P',\ \omega \in \Omega$$

$$w_p^\omega \in \mathbb{R}_+ \qquad p \in P,\ \omega \in \Omega$$

$$o^\omega \in \mathbb{R}_+ \qquad \omega \in \Omega.$$

no big-M required:

$$w_{p+1}^\omega \geq w_p^\omega + \sum_{i\in V}\sum_{j\in V} t_{ij}^\omega x_{ij}^{p+1}$$

A computational comparison

Aggregated results (CPLEX black box):

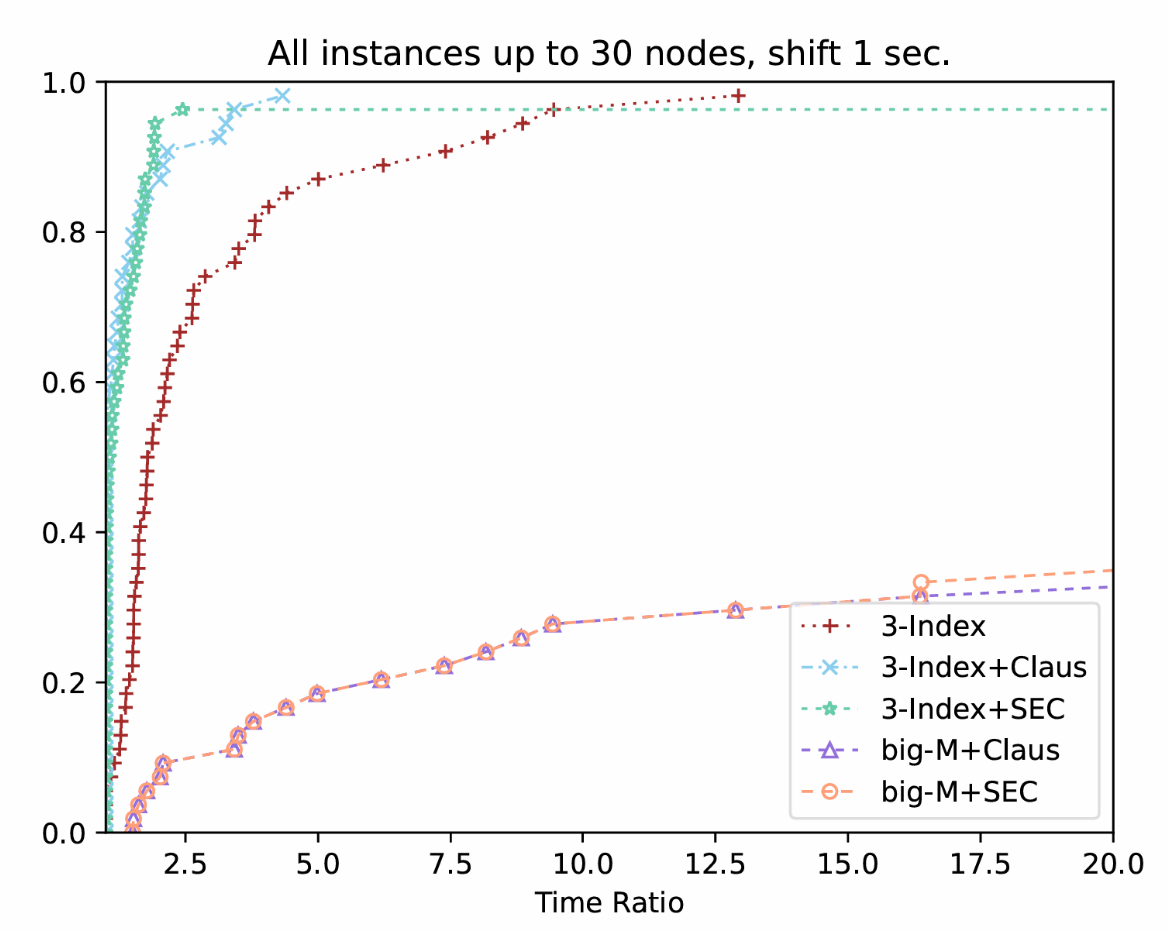
- ***big-M:*** model form the literature
- ***3-index**:* our new model
- ***+SEC:*** SECs have been separated on the fly,
- ***+Claus:*** model augmented using Claus' 3-index ATSP formulation

# Nodes	3-Index		3-Index+SEC		3-Index+Claus		big-M+SEC		big-M+Claus	
	Time	Opt	Time	Opt	Time	Opt	Time	Opt	Time	Opt
10	2.88	18	1.64	18	2.49	18	2.49	18	128.02	18
20	84.15	18	50.68	18	51.15	18	51.15	0	3600.00	0
30	3331.93	1	1424.93	18	997.35	13	997.35	0	3600.00	0

Time: computing time in sec.s (shifted geo. mean)

Opt: number of instances solved to proven optimum (1h time limit).

Performance profile plot



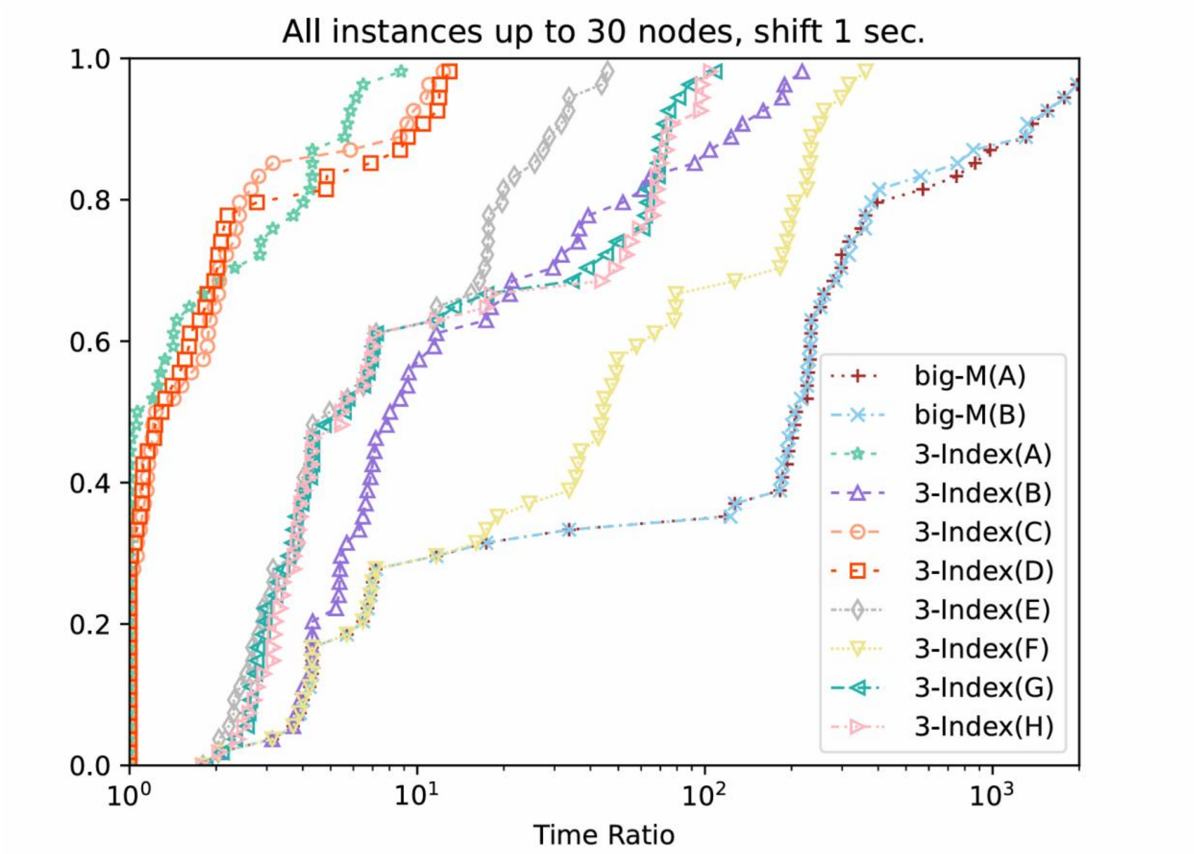
Ready for Benders?

- Models are well suited for Benders decomposition…
- … but several options are available
- … which perform very differently!

In particular: how to split master and subproblem var.s?

Layout	Master var.s	Subproblem (continuous) var.s
big-M(A)	2-Index only	Time windows, Scenarios
big-M(B)	2-Index, Time windows	Scenarios*
3-index(A)	3-Index only	Time windows, Scenarios
3-index(B)	2-Index only	3-Index, Time windows, Scenarios
3-index(C)	3-Index only	2-Index, Time windows, Scenarios
3-index(D)	2-Index, 3-Index	Time windows, Scenarios
3-index(E)	3-Index, Time windows	Scenarios*
3-index(F)	2-Index, Time windows	3-Index, Scenarios
3-index(G)	3-Index, Time windows	2-Index, Scenarios*
3-index(H)	2+3-Index, Time windows	Scenarios*

Comparison using CPLEX's Benders



Worst: big-M formulations are (by far) the worst ones

Best: 3-index (D): 2- and 3-index var.s in the master, all time-related var.s (including time windows) in the subproblem

And now: coding!

- We designed our own **specialized** Benders' decomposition based on the outcomes of the previous experiments

- **Branch-and-cut** based on CPLEX callbacks

- Cut separation:
  - **SECs** on the fly (all nodes, both fract. and integer sols.)
  - **Benders cuts** for integer sol.s only (CANDIDATE callback)
  - **Scenario aggregation** to speedup Benders separation (not in the master)

- **Initial primal heuristic**: solve a relaxation without time windows (only overtime $\sum_{i \in V} \sum_{j \in V} t^{\omega}{}_{ij}\, x_{ij} \leq T + o^{\omega}$ for all ω)
- …

Computational results

Extensive computational analysis on a mixture of:

- **Localized Disruption Scenarios:** small disruption probability (on average, less than one arc per route gets disrupted) -- customary in the literature
- **Spread Disruption Scenarios:** disruption applied to each arc of the graph, using a gamma distribution: small travel-time differences are distributed in the whole network

# Nodes	3-Index		big-M	
	Time	Opt	Time	Opt
10	0.88	90	2408.60	35
20	5.86	90	3600.00	0
30	92.65	90	3600.00	0
40	334.30	73	3600.00	0
50	3069.73	29	3600.00	0
60	2669.00	18	3600.00	0

Thanks for your attention!

Paper:

F. Cavaliere, M. Fischetti, R. Roberti, D. Salvagnin, "Models and Algorithms for the Time Window Assignment Traveling Salesperson Problem with Stochastic Travel Times", **European Journal of Operational Research**, to appear, 2025

Slides available at http://www.dei.unipd.it/~fisch/papers/slides/

Models and Algorithms for the Time Window Assignment Traveling Salesperson Problem with Stochastic Travel Times

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ABSTRACT

We study the Time Window Assignment Traveling Salesperson Problem with Stochastic Travel Times, a two-stage stochastic problem where the first-stage decisions involve the routing aspects and the customer time window definition. Second-stage decisions follow, which integrate travel time uncertainties into the optimization process. The objective is to minimize the combined routing and time window cost, including penalties for earliness and lateness, marking a shift from a cost-focused routing strategy to a more balanced approach that considers both cost and service quality aspects in delivery operations.

We introduce a novel formulation inspired by a 3-index formulation for the Time-Dependent Traveling Salesperson Problem, and we report an extensive computational comparison of alternative models and solution methods from the literature. Additionally, we provide a set of benchmark instances characterized by two opposite scenario types, intended to facilitate future research. Our results show that the (by far) most effective solution method is an ad-hoc Benders Decomposition algorithm that leverages our new model, demonstrating substantial improvements over prior state-of-the-art exact solution methods.