Branch-and-cut implementation of Benders' decomposition

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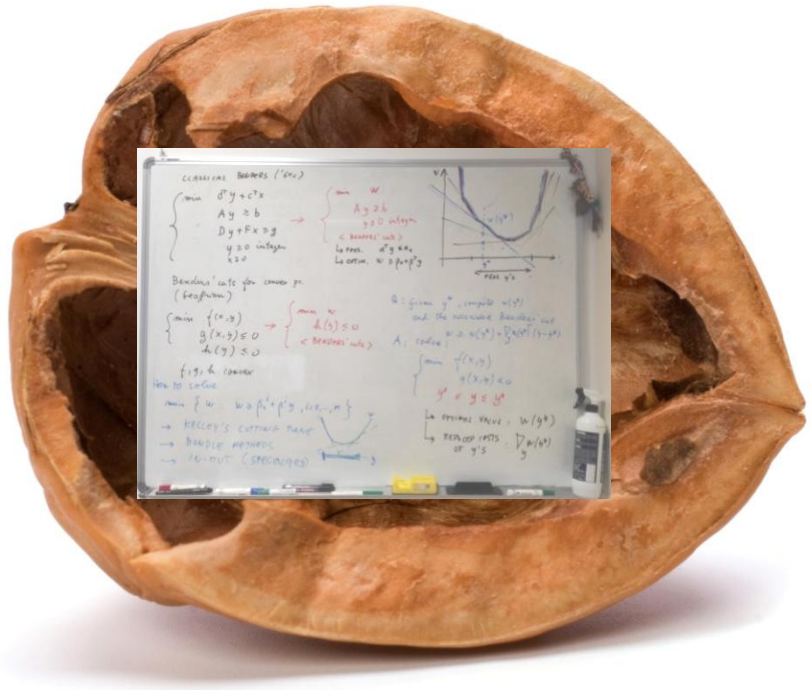


What do you actually mean by "Benders decomposition"?

- The original Benders decomposition from the '60s uses **two** distinct ingredients for solving a Mixed-Integer Linear Program (MILP):
  1) A **cut loop strategy** where a relaxed **(NP-hard)** MILP on a variable **subspace** is solved exactly (i.e., to **integrality**) by a black-box solver, and then is iteratively tightened by means of additional **"Benders" linear cuts**
  2) The **technicality** of how to actually compute those cuts (Farkas' projection)
  - Papers proposing "a new Benders-like scheme" typically refer to 1)
  - Students scared by "Benders implementations" typically refer to 2)

  Later developments in the '70s:
  - Folklore (Miliotios for TSP?): generate Benders cuts within a **single B&B tree** to cut any infeasible integer solution that is going to update the incumbent
  - McDaniel & Devine (1977): use Benders cuts to cut **fractional sol.s** as well (root node only)
- Everything fits very naturally within a modern **Branch-and-Cut** (B&C) framework.

Modern Benders

- Consider the convex MINLP in the *(x,y)* space

$$\min f(x,y)$$
$$g(x,y) \leq 0$$
$$Ay \leq b$$
$$y \text{ integer}$$

Warning: the important var.s are the y's !

and assume for the sake of simplicity that $S := \{y : Ay \leq b\}$ is nonempty and bounded, and that

$$X(y) := \{x : g(x,y) \leq 0\}$$

is **nonempty**, closed and bounded for all y ∈ S

→ the **convex function** $\Phi(y) := \min_{x \in X(y)} f(x,y)$ is well defined for all y ∈ S

→ no "feasibility cuts" needed (this kind of cuts will be discussed later on)

Working on the y-space (projection)

(1)

$$\min_y \min_x f(x,y)$$
$$g(x,y) \le 0$$
$$Ay \le b$$
$$y \text{ integer}$$

(2)

"isolate the inner minimization over *x*"

$$\Phi(y) := \min_x f(x,y)$$
$$g(x,y) \le 0$$

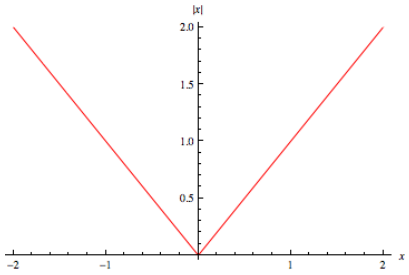
(3)

$$\min \Phi(y)$$
$$Ay \le b$$
$$y \text{ integer}$$

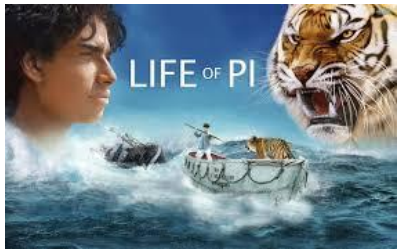
Original MINLP in the (x,y) space → Benders' **master** problem in the *y* space

Warning: projection changes the objective function (e.g., linear → convex nonlinear)

$$\min x$$
$$x \ge y$$
$$x \ge -y$$
$$y \in [-1,1]$$

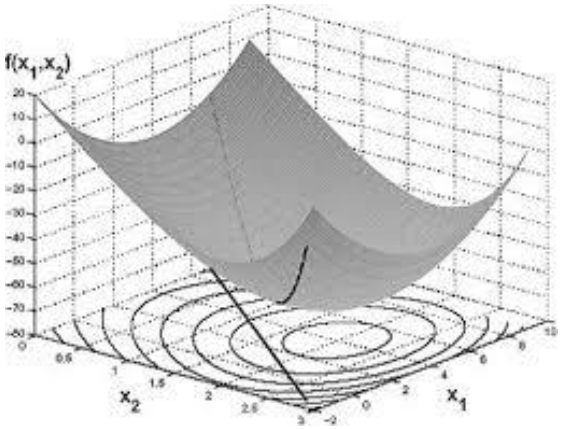


$$\min \Phi(y) = |y|$$
$$y \in [-1,1]$$



Life of P(H)I

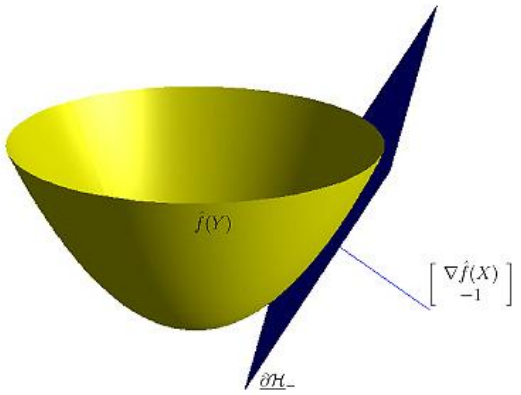
- Solving Benders' master problem calls for the minimization of a **nonlinear** convex function **(even if you start from a linear problem!)**



- Branch-and-cut MINLP solvers generate a sequence of **linear cuts** to approximate this function from below (**outer-approximation**)

$$\begin{aligned} \min\ & w \\ \text{s.t. } & w \geq \Phi(y) \\ & Ay \leq b \\ & y \text{ integer} \end{aligned}$$

subgradient
(aka Benders) cut →



$$w \geq \Phi(y) \geq \Phi(y^*) + \xi(y^*)^T(y - y^*)$$

Benders cut computation

- **Benders** (for linear) and **Geoffrion** (general convex) told us how to compute a **subgradient** to be used in the cut derivation, by using the optimal primal-dual solution *(x*,u*)* available after computing $\Phi(y^*)$

$$\xi(y^*) = \nabla_y f(x^*, y^*) + u^* \nabla_y g(x^*, y^*)$$

- The above formula is **problem-specific** and perhaps **#scaring**
- Introduce an **artificial variable vector q** (acting as a copy of *y*) to get

$$\Phi(y^*) = \min\{f(x, \mathbf{q}) \mid g(x, \mathbf{q}) \leq 0,\ y^* \leq \mathbf{q} \leq y^*\}$$

and to obtain the following **simpler** and **completely general** cut-recipe:

1) solve the original convex problem with new var. bounds $y^* \leq y \leq y^*$
2) take opt_val and reduced costs r_j's
3) write $w \geq opt\_val + \sum_j r_j(y_j - y_j^*)$

Benders feasibility cuts

- For some important applications, the set

$$X(y) := \{x : g(x,y) \leq 0\}$$

can be empty for some "**infeasible**" $y \in S$

→ $\Phi(y) := \min_{x \in X(y)} f(x,y)$ undefined

- This situation can be handled by considering the "phase-1" feasibility condition

$$0 \geq \Psi(y) := \min\{1^T s \,|\, g(x,y) \leq s,\ s \geq 0\}$$

where the function $\Psi(y)$ is **convex**
→ it can be approximated by the usual subgradient **"Benders feasibility cut"**

$$0 \geq \Psi(y) \geq \Psi(y^*) + \xi(y^*)^T (y - y^*)$$

to be computed using reduced costs as before

Successful Benders applications

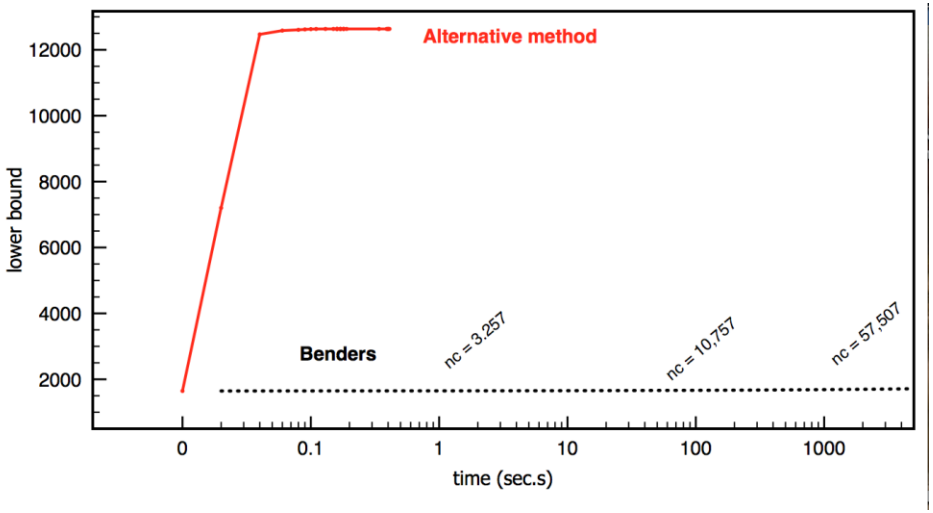
- Benders decomposition works well when fixing $y = y^*$ for computing $\Phi(y^*)$ makes the problem **much simpler to solve**.

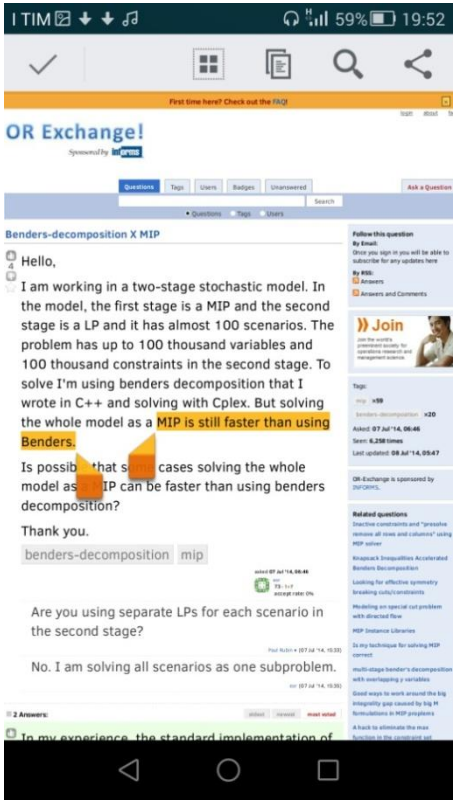
- This usually happens when
  - The problem for $y = y^*$ decomposes into a number of **independent subproblems**
    - Stochastic Programming
    - Uncapacitated Facility Location
    - etc.
  - Fixing $y = y^*$ **changes the nature** of some constraints:
    - in Capacitated Facility Location, tons of constr.s of the form $x_{ij} \leq y_j$ become just variable bounds
    - Second Order Constraints $x_{ij}^2 \leq z_{ij}\, y_i$ become quadratic constr.s
    - etc.

$$\begin{aligned} \min & \sum_{i\in I} f_i y_i + \sum_{i\in I}\sum_{j\in J} c_{ij} x_{ij} & \\ \text{s.t.} & \sum_{i\in I} x_{ij} = 1 & \forall j \in J \\ & x_{ij} \leq y_i & \forall i \in I, j \in J \\ & x_{ij} \geq 0 & \forall i \in I, j \in J \\ & y_i \in \{0,1\} & \forall i \in I \end{aligned}$$

That's it … or not?

- In practice, Benders decomposition can work quite well, but sometimes it is **desperately slow**
  … as the root node bound does not improve even after the addition of tons of Benders cuts





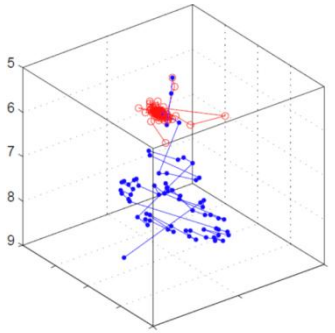
- Slow convergence is generally attributed to the **poor quality** of Benders cuts, to be cured by a more clever **selection policy** (Pareto optimality of Magnanti and Wong, 1981, etc.) but **there is more**…

Role of the cut loop

- B&C codes generate cuts, on the fly, in a **sequential** fashion
- Consider e.g. the **root B&C node** (arguably, the most critical one)
- A classical **cut-loop scheme** (described here for MILPs)

  J. E. Kelley. The cutting plane method for solving convex programs, Journal of the SIAM, 8:703-712, 1960.

  – Find an optimal **vertex** x^* of the current LP relaxation
  – Invoke a separation function on x^*, add the returned violated cut (if any) to the current LP, and repeat



- Can be very **ineffective** in the **first iterations** when few constraints are specified, and x^* moves along an **unstable zig-zag trajectory**

  ... which is precisely what often happens with Benders cuts

Stabilizing Benders can be easy!

- To summarize:
  - Benders cut machinery is easy to implement …
  - … but the root node cut loop can be **very critical**
    → many implementations sank here!

  - Kelley's cut loop can be **desperately slow**
  - Stabilization using "interior points" is a **must**
    → this is well-known in subgradient optimization and Dantzig-Wolfe decomposition (column generation), but holds for Benders as well

- E.g., for facility location problems, we implemented a very simple **"chase the carrot"** heuristic to determine a stabilized path towards the optimal *y*

- Mimics **Frank-Wolfe** alg. for minimizing a convex function over a convex set

Our #ChaseTheCarrot heuristic

- We (the donkey) start with ***y = (1,1,…,1)*** and optimize the master LP as in Kelley, to get optimal ***y**** (the carrot on the stick).

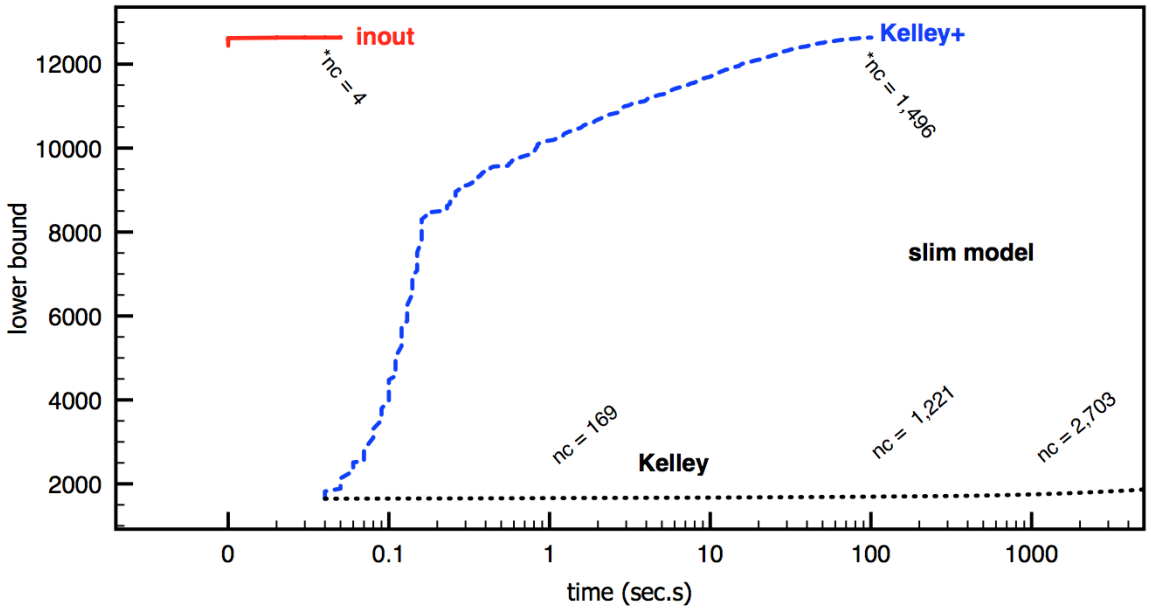
- We move ***y*** just **half-way** towards ***y**.*** We then separate a point *y'* in the segment **[*y, y**]** close to the new ***y***.

- The generated Benders cut is added to the master LP, which is reoptimized to get the new optimal ***y**** (carrot moves).

- Repeat until bound improves, then switch to Kelley for final bound refinement (kind of cross-over)

- **Warning: adaptations needed if feasibility Benders cuts need to be generated…**

Effect of the improved cut loop



- Comparing **Kelley** cut loop at the root node with **Kelley+** (add epsilon to y*) and with our chase-the-carrot method (**inout**)
- Koerkel-Ghosh **qUFL** instance gs250a-1 (250x250, quadratic costs)
- ***nc** = n. of Benders cuts generated at the end of the root node
- times in **logarithmic scale**

Conclusions

To summarize:

- Benders cuts are **easy** to implement within modern B&C (just use a callback where you solve the problem for $y = y^*$ and compute reduced costs)
- Kelley's cut loop can be **desperately slow** hence stabilization is a **must**
- Implemented in CPLEX **general** MIP solver since version 12.7

Slides available at http://www.dei.unipd.it/~fisch/papers/slides/

Reference papers:

M. Fischetti, I. Ljubic, M. Sinnl, "Benders decomposition without separability: a computational study for capacitated facility location problems", European Journal of Operational Research, 253, 557-569, 2016.

M. Fischetti, I. Ljubic, M. Sinnl, "Redesigning Benders Decomposition for Large Scale Facility Location", Management Science 63 (7), 2146-2162, 2017.