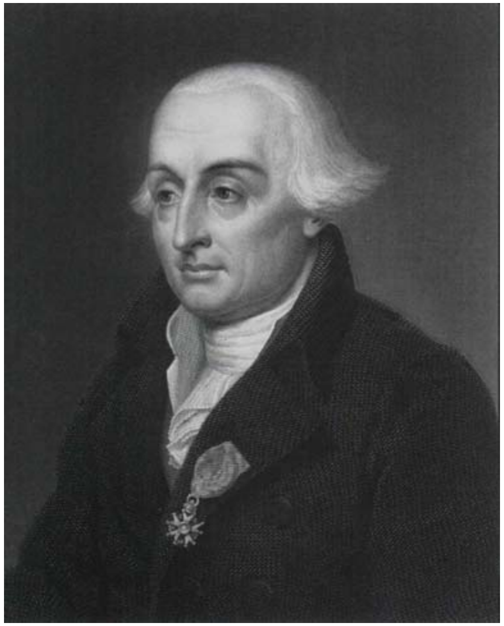

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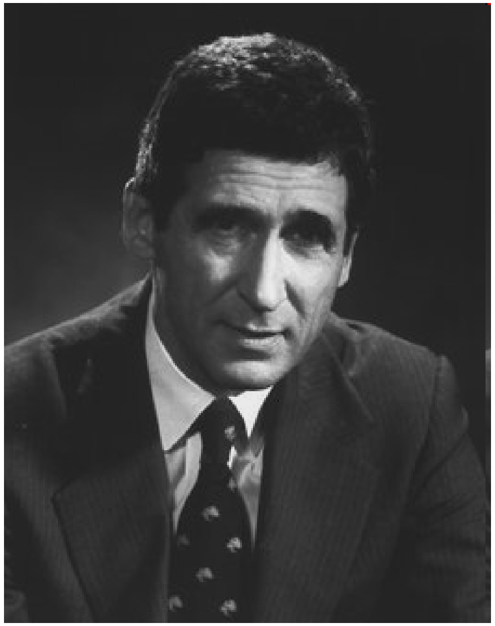

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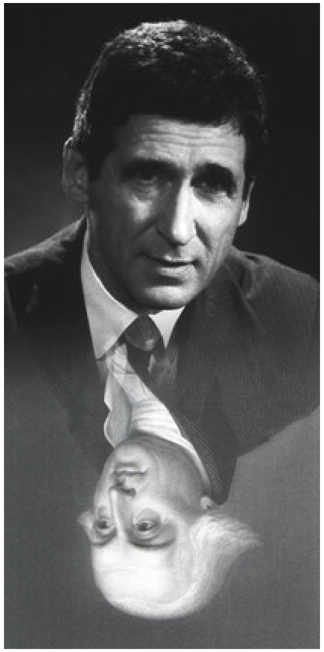

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Aussois, 4-8/1/2010

Cutting plane methods

- **Cutting plane** methods widely used in convex optimization and to provide bounds for Mixed-Integer Programs (MIPs)
- Made by two equally important components:
  - (i) the **separation procedure** (oracle) that produces the cut(s) used to tighten the current relaxation, and
  - (ii) the overall **search framework** that actually uses the generated cuts and determines the next point to cut
- In the last 50 years, considerable research effort devoted to the study of (i) → families of cuts, cut selection criteria, etc.
- Search component (ii) much less studied by the MIP community → the standard approach is to always cut an optimal LP vertex

The problem

- Consider a generic MIP: $z(MIP) := min \{ c^T x : x \varepsilon X \}$

  and its LP relaxation

$$z(LP) := min \{c^T x: x \varepsilon P \}$$

$$P := \{ x : A x \le b \}, \qquad X := \{ x \varepsilon P: x_j \text{ integer for } j \varepsilon J \}$$

- We are also given a convex set **P_1** with **conv(X) ≤ P1 ≤ P** (e.g., the first GMI closure) described only implicitly through a separation function:

  ***oracle(y)* returns a valid linear ineq. for *P_1* violated by *y* (if any)**

- We want to (approximately) compute $z_1 := min \{c^T x: x \varepsilon P_1 \}$

Kelley's cutting plane method

- A classical search scheme

  J. E. Kelley. The cutting plane method for solving convex programs, Journal of the SIAM, 8:703-712, 1960.

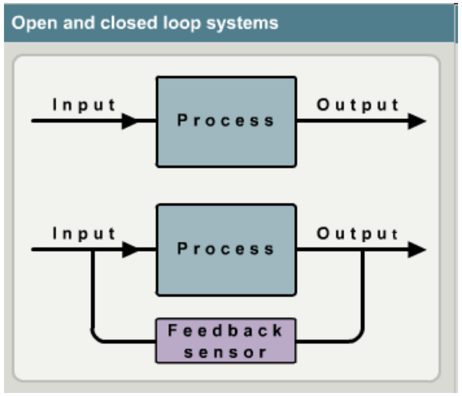
1. Let *P' := { x ε P : x satisfies all cuts generated so far }*
2. Find an optimal **vertex** *x** of the current LP: *min {c^T x: x ε P' }* ,
3. Invoke *oracle(x*)* and repeat (if a violated cut is found)

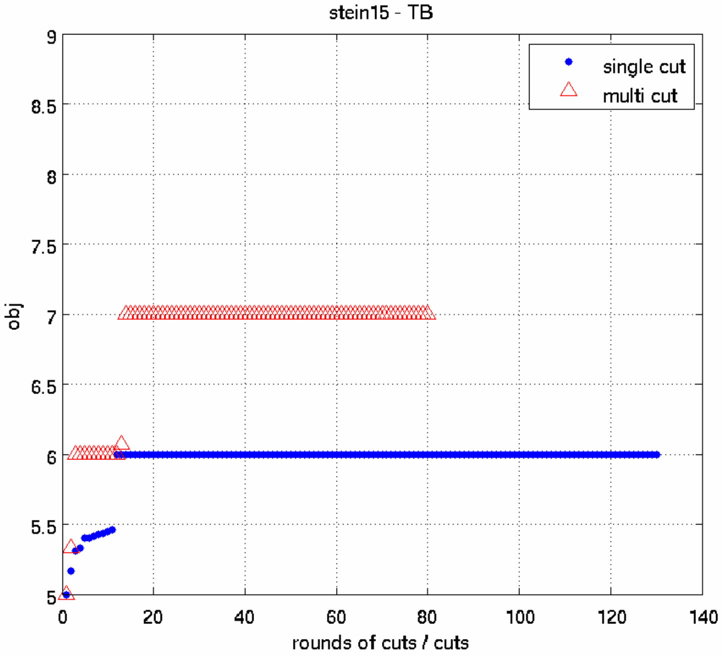
- Practically satisfactory only if the oracle is able to find "**deep**" cuts

- Very ineffective in case shallow cuts are generated

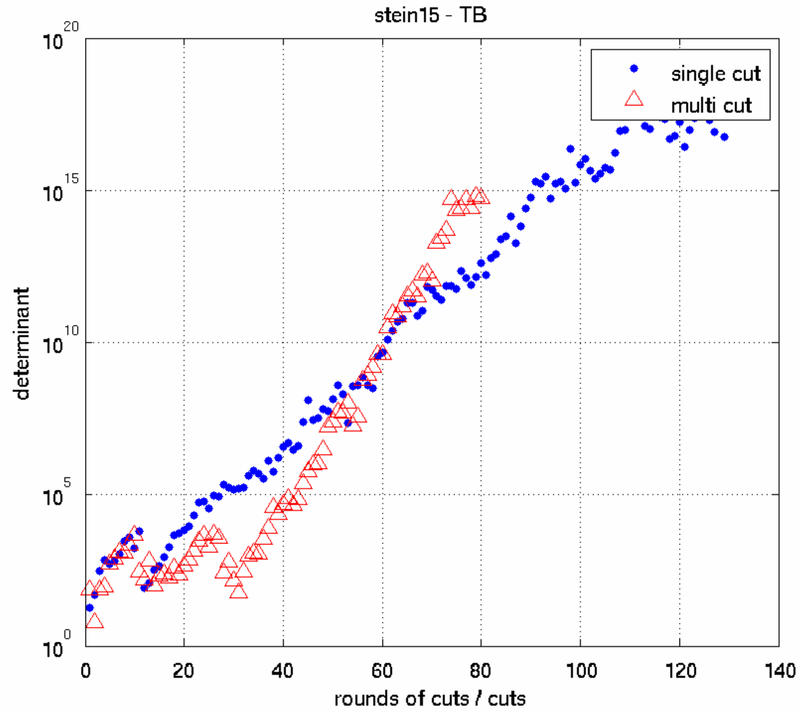
- **May induce a dangerous correlation between *x** and the returned cut (e.g. when the cuts are read from the LP tableau)**

Kelley and Gomory: a problematic marriage





LP bound = **5**; ILP optimum = **8**



exponential determinant growth → unstable system!

But… is it all Gomory's fault?

- Experiment: take a given LP problem (e.g. root node relaxation of a MIP)

$$\min \{ c^T x : A' x \le b', A'' x = b'', l \le x \le u \}$$

  and define:
  - $P := \{ x : A'' x = b'', l \le x \le u \}$
  - constr. list $A' x \le b'$ can only be accessed through the ***separation oracle***

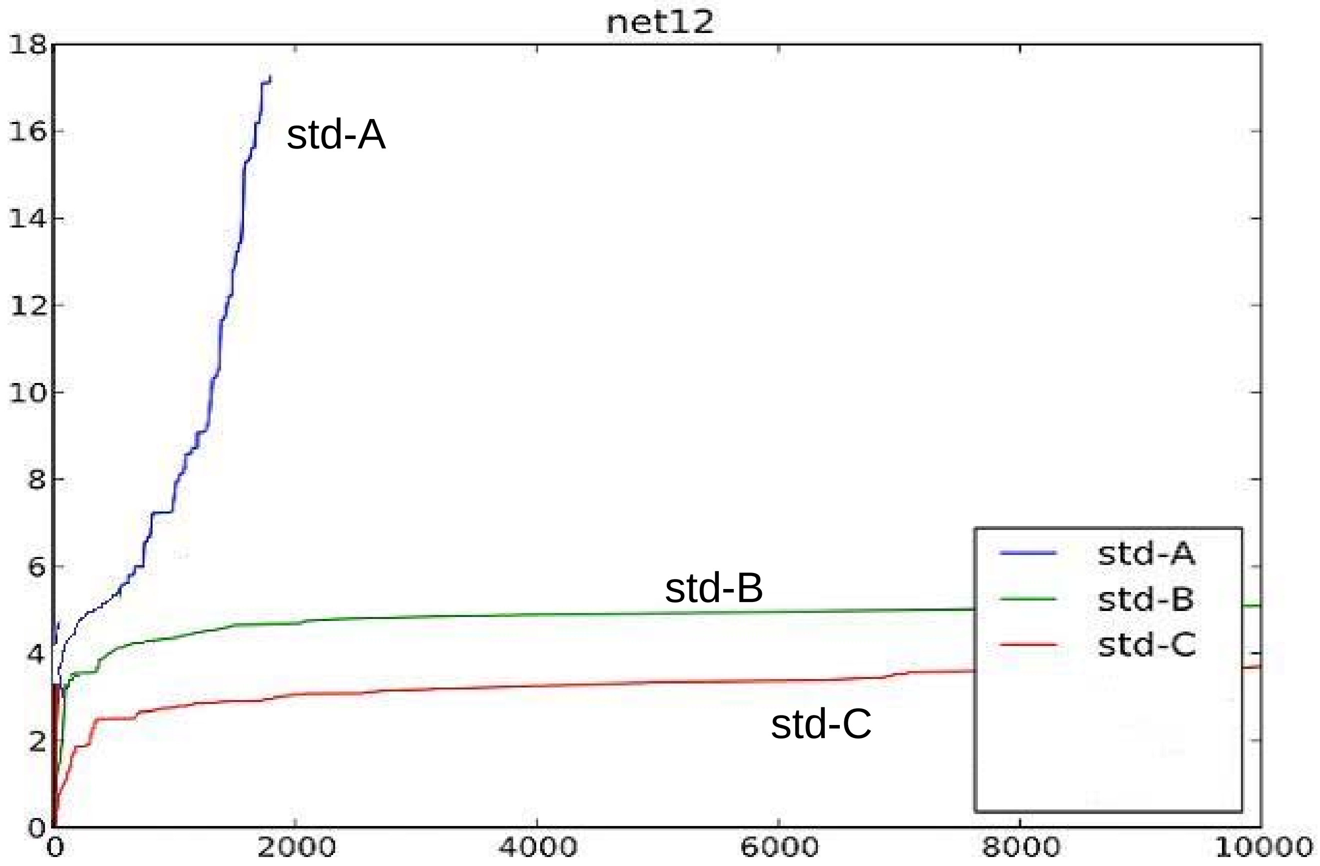
- 3 cut **selection criteria** implemented for separation: for a given x^* the oracle returns:

  A) the **deepest** violated cut in the list (Euclidean distance) → "best" facet

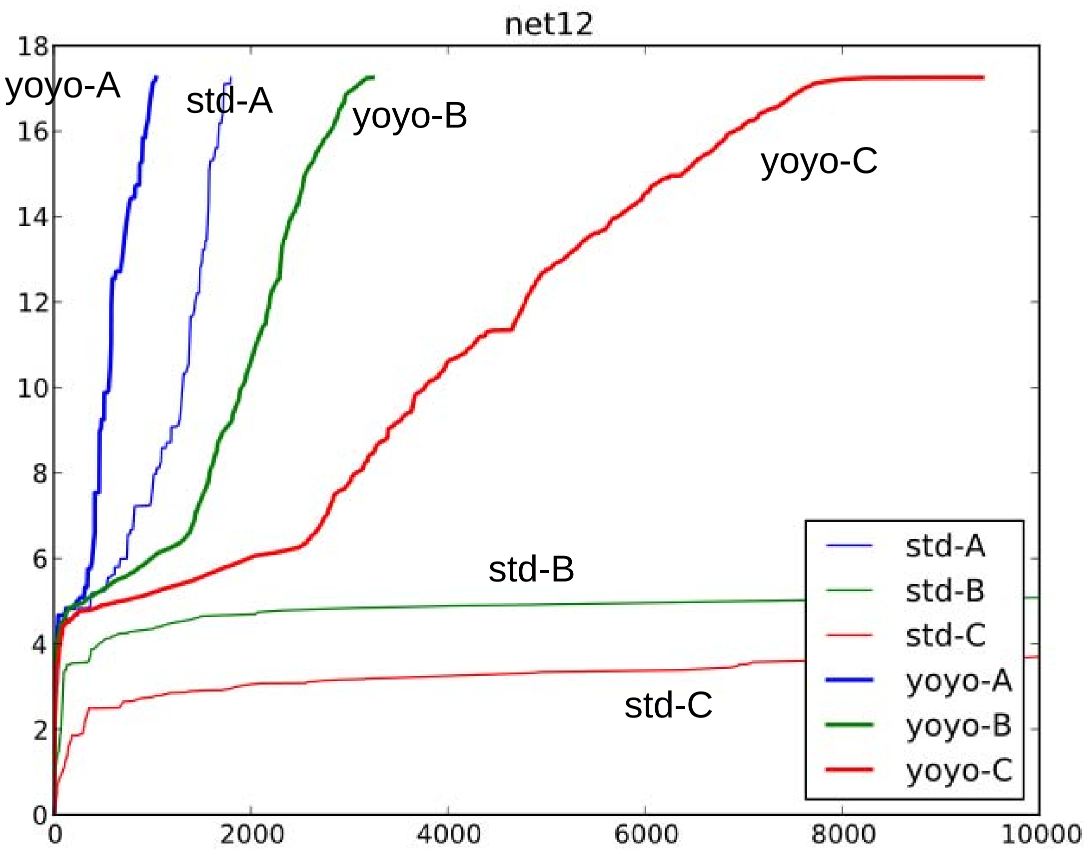
  B) a convex combination of the deepest one and of the (at most) first 10 **violated** or **tight** cuts encountered when scanning the list

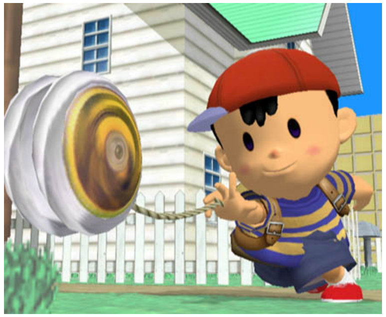
  C) the cut is first defined as in case B, but then its rhs is **weakened** so as to **half** the degree of violation

Output with Kelley's search (std)



But other search schemes work better…





e.g. yo-yo search…

Lessons learned

- Kelley's method is intrinsically **nonrobust**
- Other search methods can survive even with **weak cuts** (analytic center, yo-yo search, etc.)
- As to GMI cuts, reading **both** the vertex x^* to cut and the cuts themselves from the **same** tableau soon creates a dangerous **feedback** (unstable dynamic system)
- Kelley + Gomory is problematic in the long run, but … it is not all Gomory's fault!
- In fact, F. and Lodi (2007) report very good bounds by separating (fractional) Gomory cuts of rank 1 by means of an **external** MIP solver → a key ingredient was the new search scheme that decoupled optimization and separation
- Results confirmed for GMI cuts by Balas & Saxena and Dash, Gunluk & Lodi

Reading GMIs from LP bases

- If you insist on reading GMI cuts from an LP basis … at least don't use an optimal one!

- Steps in this direction: given an optimal LP **vertex** x^* of the "large LP" (original+cuts) and the associated **optimal** basis B*:

  - Balas and Perregaard (2003): perform a sequence of pivots leading to a (possibly non-optimal or even infeasible) basis **of the large LP** leading to a deeper cut w.r.t. the given x^*

  - Dash and Goycoolea (2009): heuristically look for a basis B **of the original LP** that is "close to B*" in the hope of cutting the given x^* with rank-1 GMI cuts associated with B

Back to Lagrangia

- Forget about Kelley: optimizing over the first GMI closure reads

$$\min c^T x$$
$$x \in P$$
$$< \text{all rank-1 GMI cuts} >$$

- Dualize (in a Lagrangian way) the GMI cuts, i.e. …

- … solve a sequence of Lagrangian subproblems

$$\min \{ c(\boldsymbol{\lambda})^T x : x \in P \}$$

on the **original LP** but using the Lagrangian cost vector $c(\boldsymbol{\lambda})$

- **Subgradient** $\boldsymbol{s}$ *at* $\boldsymbol{\lambda}$: s_i = violation of the *i*-th GMI cut w.r.t.

$$x^*(\lambda) := \text{argmin} \{ c(\lambda)^T x : x \in P \}$$

Back to Lagrangia

- During the Lagrangian dual optimization process, a large number of **bases of the original** LP is traced → round of rank-1 GMI cuts can easily be generated "on the fly" and stored

- Use of a **cut pool** to explicitly store the generated cuts, needed to compute (approx.) **subgradients** used by Lagrangian optimization

- **Warning**: new GMI cuts added on the fly → possible convergence issues due to the imperfect nature of the computed "subgradients"

- … as the separation oracle does not return the list of **all** violated GMI cuts, hence the subgradient is **truncated** somehow …

Lagrange + Gomory = LaGromory

- Generate cuts and immediately dualize them (don’t wait they become wild!)
- **No fractional point x* to cut:** separation used to find (approx.) subgrad.s
- Lagrangian optimization as a **stabilization filter** in the closed loop system → GMI cuts are loosely correlated with the underlying large LP (original+ previously generated GMI cuts) as they don’t even see the large tableau
- The method can add rank-1 GMI cuts on top of **nonlinear** constraints and of any other classes of cuts (Cplex cuts etc.), including branching conditions → just dualize them!
- **Key to success: resist to the “Kelley’s temptation” of reading the GMI from the optimal tableau of the large LP!**

Experiments with LaGromory cuts

- Three preliminary implementations:
  - **lagr**: naïve Held-Karp subgradient opt. scheme (10,000 iter.s)
  - **hybr:** as before, but every 1,000 subgr. iter.s we solve the “large LP” just to recompute the optimal Lagrangian multipliers for all the cuts in the current pool
  - **fast:** as before, but faster: only 10 large LPs solved, each followed by just 50 subgradient iterations with very small step size → 10 short walks around the Lagrangian dual optimum to collect bases of the original LP, each made by 50 small steps to perturb Lagrangian costs
- Comparision with **std** (one round of GMI cuts)

Preliminary computational results

Testbed: 32 instances from MIPLIB 3.0 and 2003

CPU seconds on a standard PC (1GB memory allowed)

	%gap Closed	Time (sec.s)
std	**24.4%**	**0.01**
lagr	**63.6%**	**46.69**
hybr	**69.6%**	**36.97**
fast	**59.1%**	**1.52**

Rank 1 GMI cuts

	%gap closed	Time (sec.s)
std	**33.1%**	**0.02**
lagr	**68.9%**	**72.64**
hybr	**73.4%**	**108.00**
fast	**69.3%**	**4.79**

Rank 2 GMI cuts

Fast with different parameters

		1 walk	5 walks	10 walks	15 walks	20 walks	25 walks
100 steps per walk	gap	37.00%	56.60%	60.60%	62.00%	62.60%	63.20%
	time	0.01	1.26	2.84	4.29	5.71	7.11
50 steps per walk	gap	36.50%	55.60%	59.10%	61.00%	61.80%	62.40%
	time	0.01	0.69	1.52	2.33	3.03	3.72

Rank 1 GMI cuts (**std** = 24.4% in 0.01 sec.s)

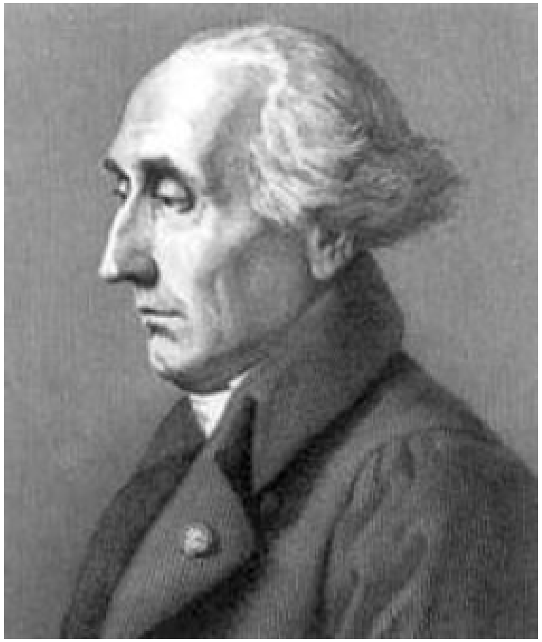
… thank you for your attention!

Joseph Louis Lagrange

From Wikipedia, the free encyclopedia
(Redirected from Lagrange)

Joseph-Louis Lagrange (25 January 1736 – 10 April 1813), born **Giuseppe Lodovico Lagrangia**, was an Italian-born mathematician and astronomer, who lived part of his life in Prussia and part in France, making significant contributions to all fields of analysis, to number theory, and to classical and celestial mechanics. On the recommendation of Euler and D'Alembert, in 1766 Lagrange succeeded Euler as the director of mathematics at the Prussian Academy of Sciences in Berlin, where he stayed for over twenty years, producing a large body of work and winning several prizes of the French Academy of Sciences. Lagrange's treatise on analytical mechanics (*Mécanique Analytique*, 4. ed., 2 vols. Paris: Gauthier-Villars et fils, 1888-89), written in Berlin and first published in 1788, offered the most comprehensive treatment of classical mechanics since Newton and formed a basis for the development of mathematical physics in the

Joseph-Louis Lagrange



Joseph-Louis (Giuseppe Lodovico), comte de Lagrange

Born	25 January 1736 Turin, Piedmont
Died	10 April 1813 (aged 77)

Lagrange + Gomory ?

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