1) Theory ($\geq 6/10$ to pass to exam)

1.a Define the total unimodularity of a matrix

1.b Prove the validity of Gomory cuts

1.c Write an ILP model for the traveling salesman problem

2) (Linear Programming) Consider the following LP

$$\begin{aligned} \min \quad & 1x_1 + 3x_2 + 2x_3 \\ & 2x_1 + x_2 \geq 8 \\ & 4x_1 + 2x_2 + x_3 \geq 12 \\ & 2x_1 + 1x_2 - x_3 \leq 6 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

2.1) Solve it using primal simplex method (two phase method bland's rule) Report all the tableaux and highlight with a circle each pivot element

2.2) Write the corresponding LP dual problem.

3) (Modeling) Write an ILP model for Steiner tree problem with the following additional constraints.

(a) At least half of the nodes of the graph must be covered (including the root)

(b) Given two distinct arcs a and b, if a selected then also b must be selected

POLATO ANNA

1.a) A matrix A of size $m \times n$, $m < n$, is TUM (totally unimodular) if and only if, for each squared submatrix Q of A, of any order, $\det(Q) = \{-1, 1, 0\}$

1.b) Gomory cuts: given the ILP problem suppose x^* to be the optimal solution for the continuous relaxation of the problem; x^* is fractional, then consider x_h the fractional component of x^*. We call generating row the following row i of the optimal tableau

$$x_h + \sum_{\substack{j=1 \\ j \neq h}}^{m} \bar{a}_{ij} x_j = \bar{b}_i \qquad (a)$$

We apply the chvàtal inequality

$$x_h + \sum \lfloor \bar{a}_{ij} \rfloor x_j \leq \lfloor \bar{b}_i \rfloor \qquad (b)$$

→ we obtain the cut valid $\forall x_j$ but violated by x_h

→ We transform in standard form adding the variable x_{m+1} (new variable)

$$x_h + \sum_{\substack{j=1 \\ j \neq h}}^{m} \lfloor \bar{a}_{ij} \rfloor x_j + x_{m+1} = \lfloor \bar{b}_i \rfloor \qquad (c)$$

The new constraint is obtained making (c) - (a) and can be added to the tableau.

With this procedure we iteratively add new constraints, valid for all x_j but x_h fractional chosen.

1.c) Given $G = (V, A)$, let

$$x_{ij} = \begin{cases} 1 & \text{if arc } (i,j) \text{ is considered in the Hamiltonian cycle} \\ 0 & \text{otherwise} \end{cases}$$

c_{ij} = cost of arc (i,j)

$$\begin{cases} \min \sum_{(i,j)\in A} c_{ij} x_{ij} \\ \sum_{(i,h)\in \delta^-(h)} x_{ih} = 1 & \forall h \in V \\ \sum_{(h,j)\in \delta^+(h)} x_{hj} = 1 & \forall h \in V \\ \sum_{(i,j)\in \delta^+(S)} x_{ij} \geq 1 & \forall S \subset V : 1 \in S \\ x_{ij} \geq 0, \text{ integer} & i = \{1,\dots,n\} \\ & j = \{1,\dots,n\} \end{cases}$$

2.1)

		x_1	x_2	x_3	x_4	x_5	x_6
$-z$	0	1	3	2	0	0	0
	8	2	1	0	-1	0	0
	12	4	2	1	0	-1	0
	6	2	1	-1	0	0	1

PHASE 1

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
$-w$	-26	-8	-4	0	1	1	-1	0	0	0
x_7	8	2	1	0	-1	0	0	1	0	0
x_8	12	4	2	1	0	-1	0	0	1	0
x_9	6	2	1	-1	0	0	1	0	0	1

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
$-w$	-2	0	0	2	1	-1	-1	0	2	0
x_7	2	0	0	$-\frac{1}{2}$	-1	$\frac{1}{2}$	0	1	$-\frac{1}{2}$	0
x_1	3	1	$\frac{1}{2}$	$\frac{1}{4}$	0	$-\frac{1}{4}$	0	0	$\frac{1}{4}$	0
x_9	0	0	0	$-\frac{3}{2}$	0	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	1

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
$-w$	-2	0	0	-1	1	0	1	0	1	2
x_7	2	0	0	1	-1	0	-1	1	0	-1
x_1	3	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$
x_5	0	0	0	-3	0	1	2	0	-1	2

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
$-w$	0	0	0	0	0	0	0	1	1	1
x_3	2	0	0	1	-1	0	-1	1	0	-1
x_1	4	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0
x_5	6	0	0	0	-3	1	-1	3	-1	-1

Phase 2 :

$$
\begin{aligned}
z &= x_1 + 3x_2 + 2x_3 \\
&\stackrel{!}{=} 4 - \tfrac{1}{2}x_2 + \tfrac{1}{2}x_4 + 3x_2 + 2\left(2 + x_4 + x_6\right) \\
&\stackrel{!}{=} 8 + \tfrac{5}{2}x_2 + \tfrac{5}{2}x_4 + 2x_6
\end{aligned}
$$

2 (continue)

		x_1	x_2	x_3	x_4	x_5	x_6
$-z$	-8	0	$\frac{5}{2}$	0	$\frac{5}{2}$	0	2
x_3	2	0	0	1	-1	0	-1
x_1	4	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	0
x_5	6	0	0	0	-3	1	-1

2.2)

$$
\begin{cases}
\max \quad 2u_1 + 12u_2 + 6u_3 \\
u_1 \geq 0 \\
u_2 \geq 0 \\
u_3 \leq 0 \\
2u_1 + 4u_2 + 2u_3 \leq 1 \\
u_1 + 2u_2 + u_3 \leq 3 \\
u_2 - u_3 \leq 2
\end{cases}
$$

3)

Given $G = (V, A)$, $T \subset V$

$$
\begin{cases}
\min \sum_{(i,j)\in A} c_{ij} x_{ij} \\
\sum_{(i,h)\in \delta^-(h)} x_{ih} \begin{cases} = 1 & \forall h \in T \quad (T \subset V) \\ = 0 & \text{if } h = r \quad (r = \text{root}) \\ \leq 1 & \forall h \in V \setminus (T \cup \{r\}) \end{cases} \\
\sum_{(i,j)\in \delta^+(S)} x_{ij} \geq \sum_{(i,t)\in \delta^-(t)} x_{it} \quad \forall S \subset V : r \in S \quad \forall t \in (V \setminus S) \\
\sum_{(i,j)\in A} x_{ij} \geq \left\lceil \frac{1}{2} |V| \right\rceil - 1 \\
x_a \leq x_b \\
x_{ij} \geq 0, \text{ integer } (i,j) \in A
\end{cases}
$$