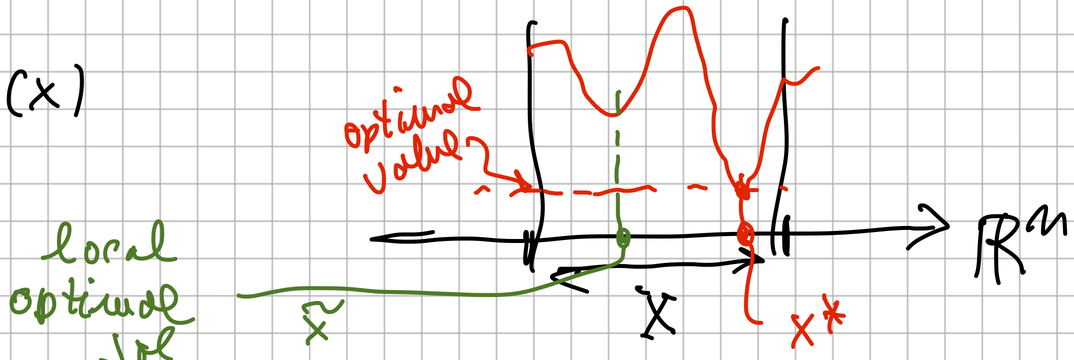
OR 1 5-OCT-2021

## MATHEMATICAL OPT. PROBLEM

$X \subseteq \mathbb{R}^n$ set of feasible sol.s

$f : X \to \mathbb{R}$ objective function

$$\min_{x \in X} f(x)$$




$x^*$ optimal sol.

$$\boxed{\max \varphi(x) = -\min \underbrace{-\varphi(x)}_{f(x)}}$$

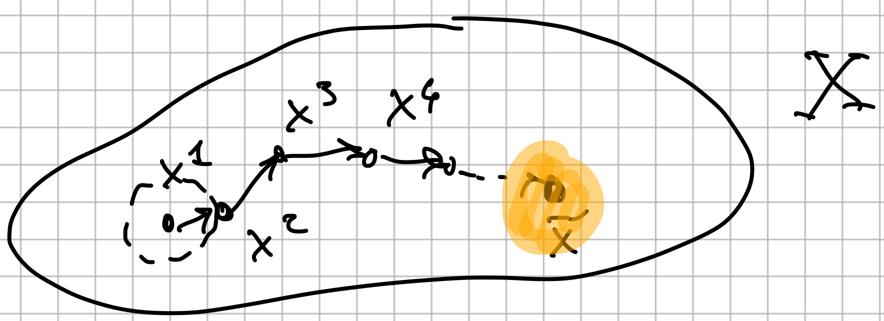
3 CASES:

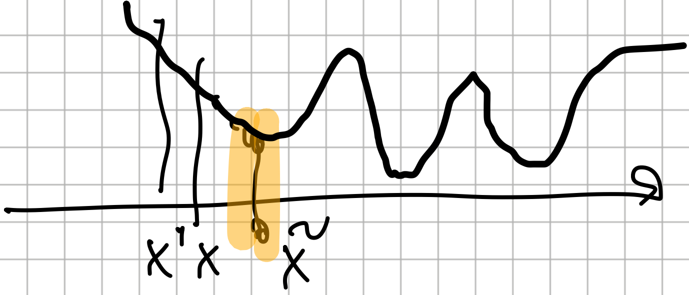
- $X = \emptyset$ PROBL. INFEASIBLE "min $= +\infty$"
- $f$ is not bounded from below on $X$ "min $= -\infty$"
- $x^* \in X$ : $f(x^*) \leq f(x)$, $\forall x \in X$

"GLOBAL optimal sol."

- $\tilde{x}$ is locally optimal $\Leftrightarrow \exists\, \varepsilon > 0$ :

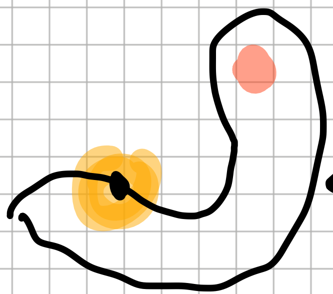
$$f(\tilde{x}) \leq f(x), \ \forall x \in X : \|x - \tilde{x}\| \leq \varepsilon$$





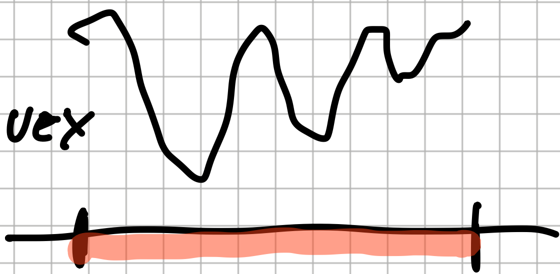

2 PROPERTIES :

- $X$ CONVEX

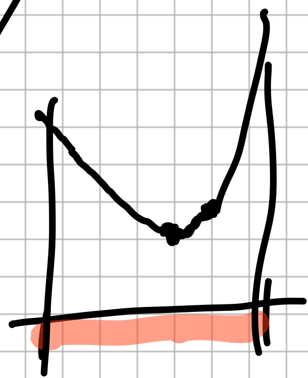


NOT CONVEX

- $f$ CONVEX



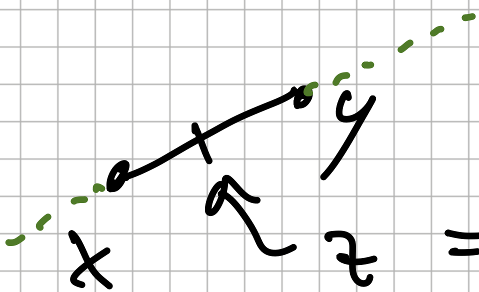

$f$ not convex



$f$ convex

<u>**Def**</u> :

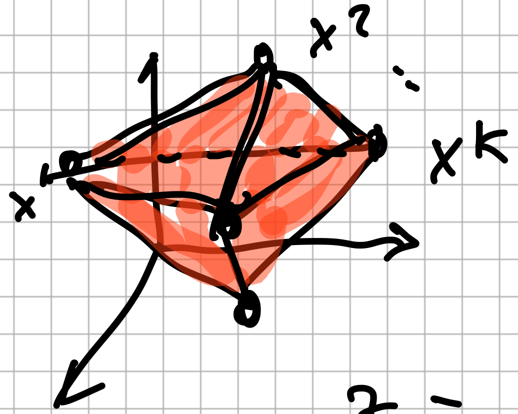
convex comb. of two points $x, y \in \mathbb{R}^n$




$$z = \lambda x + (1-\lambda) y$$

for $\lambda \in [0,1]$

$\lambda \notin \{0,1\} \Rightarrow$ STRICT conv. comb.

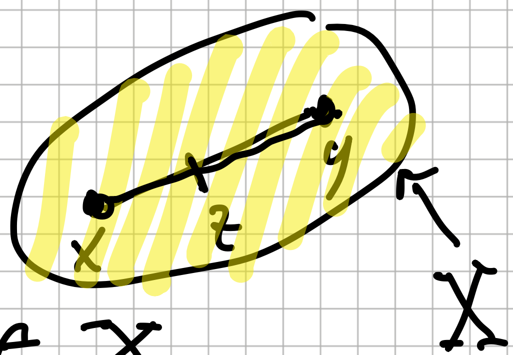



$$z = \sum_{i=1}^{k} \lambda_i x^i$$

$$\sum_{i=1}^{k} \lambda_i = 1 , \quad \lambda \geq 0$$

- CONVEX SET:

$\forall x, y \in X , \forall \lambda \in [0,1]$:

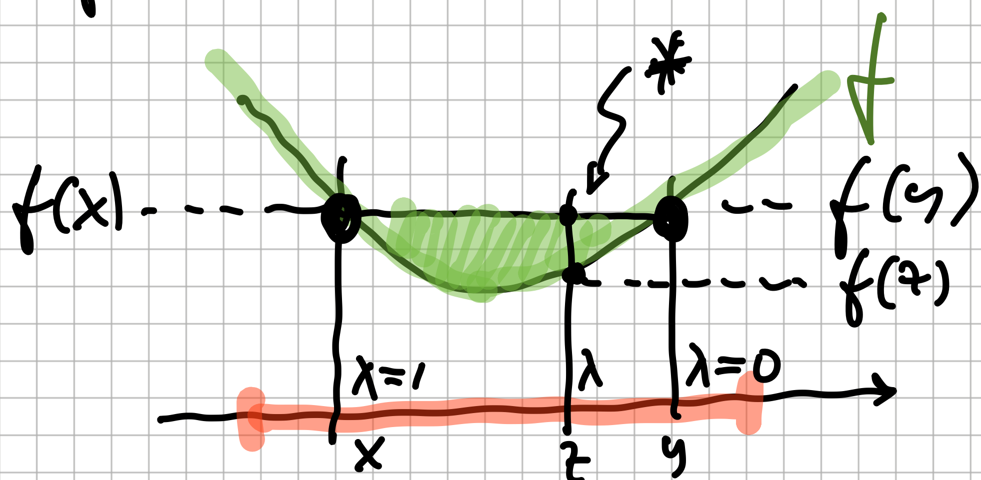



$$z = \lambda x + (1-\lambda) y \in X$$

- $\bigcap_{i=1}^{k} X_i = X$ is convex if all $X_i$'s are convex

CONVEX FUNCTION

$f : X \to \mathbb{R}$ with $X$ convex

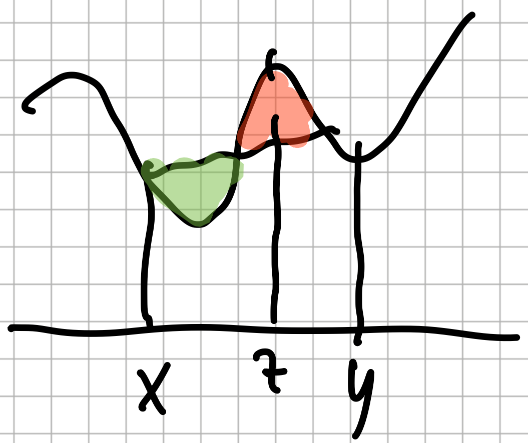


$$z = \lambda x + (1-\lambda) y \quad \text{for } \lambda \in [0,1]$$

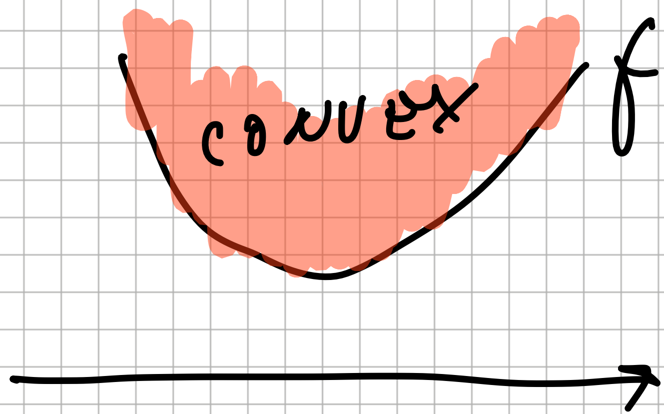
$$(*) = \lambda f(x) + (1-\lambda) f(y)$$

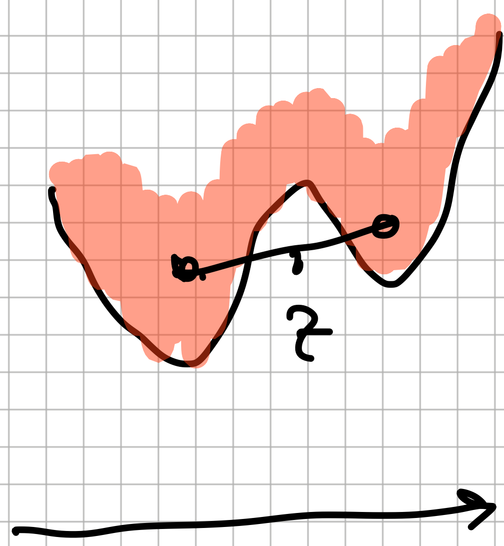
$$f(z) \overset{?}{\leq} (*) = \lambda f(x) + (1-\lambda) f(y)$$

$$\boxed{f(\underbrace{\lambda x + (1-\lambda) y}_{z}) \leq \lambda f(x) + (1-\lambda) f(y)}$$




NOT convex !




f is convex

f not convex

- $\boxed{X = \{x \in \mathbb{R}^n : g_i(x) \le 0,\ i = 1, \dots, m\}}$

$g_i : \mathbb{R}^n \to \mathbb{R}$ convex $\forall i$

$\Rightarrow X$ is convex

<u>Proof</u>: $X = \bigcap_{i=1}^{m} X_i$ with

$X_i = \{x \in \mathbb{R}^n : g_i(x) \le 0\}$

Take any $X_i$ and any two points $x, y \in X_i$ and any $\lambda \in [0,1]$:

$$z = \lambda x + (1-\lambda) y$$

by convexity of $g_i$

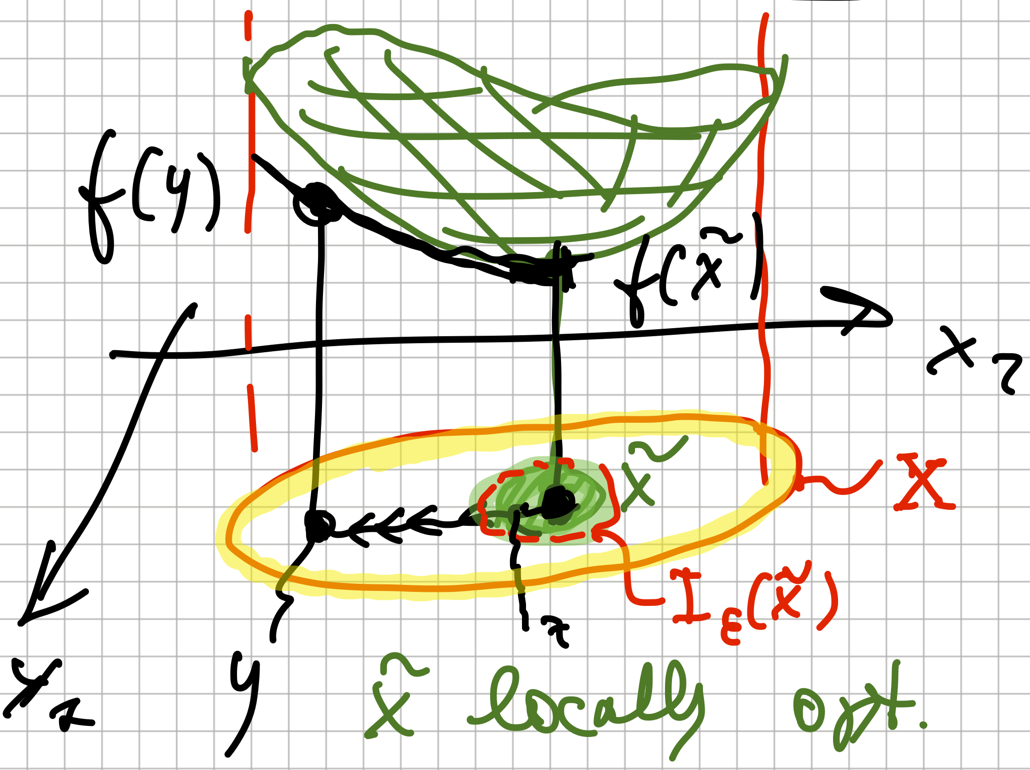
$$\boxed{g_i(z)} = g_i(\lambda x + (1-\lambda) y) \le$$

$$\underbrace{\lambda}_{\ge 0} \underbrace{g_i(x)}_{\le 0} + \underbrace{(1-\lambda)}_{\ge 0} \underbrace{g_i(y)}_{\le 0} \ \boxed{\le 0}$$

$\Rightarrow \quad z \in X_i \Rightarrow X_i$ CONVEX $\square$

$\boxed{\text{THEOREM}}$ $\min \{ f(x) : x \in X \}$

with $f$, $X$ are convex.
Every <u>local</u> optimal sol. is also <u>globally optimal</u>.




<u>Proof</u> : Let $\tilde{x} \in X$ LOCAL optimal sol. Then, $\exists\ \varepsilon > 0$ :

$$f(\tilde{x}) \leq f(z), \forall z \in I_\varepsilon(\tilde{x})$$

Take $\underline{\text{ANY}}$ $y \in \tilde{X}$ and consider

$$z = \lambda \tilde{x} + (1-\lambda) y$$

with $\lambda \lessapprox 1 \Rightarrow z \in I_\varepsilon(\tilde{x})$

$$\boxed{f(\tilde{x}) \leq} f(z) =$$

$z \in I_\varepsilon(\tilde{x})$

$f$ convex

$$= f(\lambda \tilde{x} + (1-\lambda) y) \leq$$

$$\boxed{\lambda f(\tilde{x}) + (1-\lambda) f(y)} \Rightarrow$$

$$\boxed{f(\tilde{x})} \cancel{\underbrace{(1-\lambda)}_{>0}} \leq \cancel{\underbrace{(1-\lambda)}_{>0}} \boxed{f(y)}$$

$\neq 0$ $\neq 0$

$\square$

## SPECIAL CASE:

- all $g_i(x)$'s and $f$ are LINEAR $\Longrightarrow$

$$\min f(x) \qquad g_i(x) \le 0 \quad \Longrightarrow \quad \begin{cases} \min \; c^T x \\ -a_i^T x + b_i \le 0 \\ -x_i \le 0, \\ i = 1, \dots, n \end{cases}$$

$$c^T x = \overbrace{[c_1, \dots, c_n]}^{c^T} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} =$$

$$= \sum_{i=1}^{n} c_i x_i$$

$$\Longrightarrow \min \{ c^T x : \boxed{Ax \ge b}, \boxed{x \ge 0} \}$$

LINEAR OPT. PROBL.