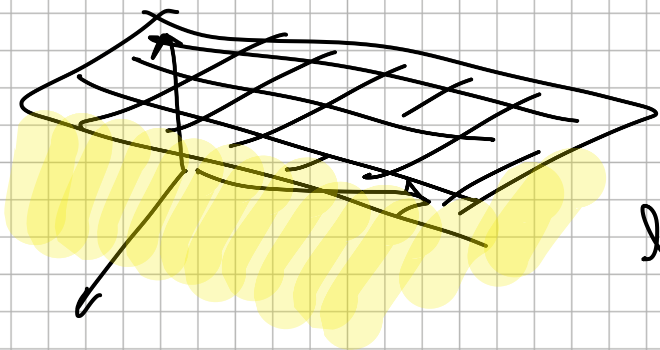
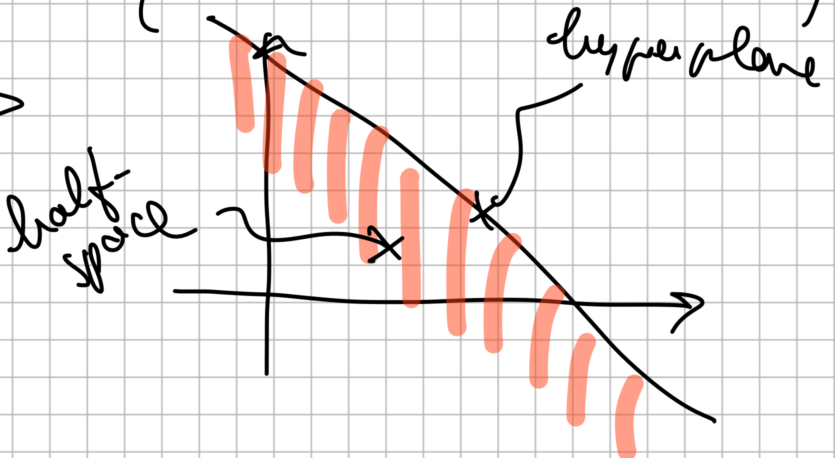
OR1 11-oct-2021

LP GEOMETRY

Def:

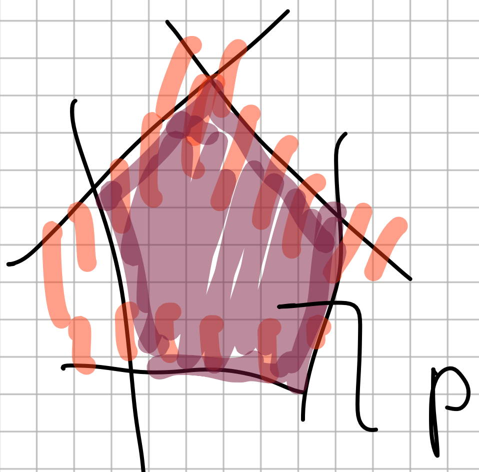
- HALF-SPACE $\{x \in \mathbb{R}^n : a^T x \leq \alpha_0\}$
- HYPERPLANE $\{x \in \mathbb{R}^n : a^T x = \alpha_0\}$

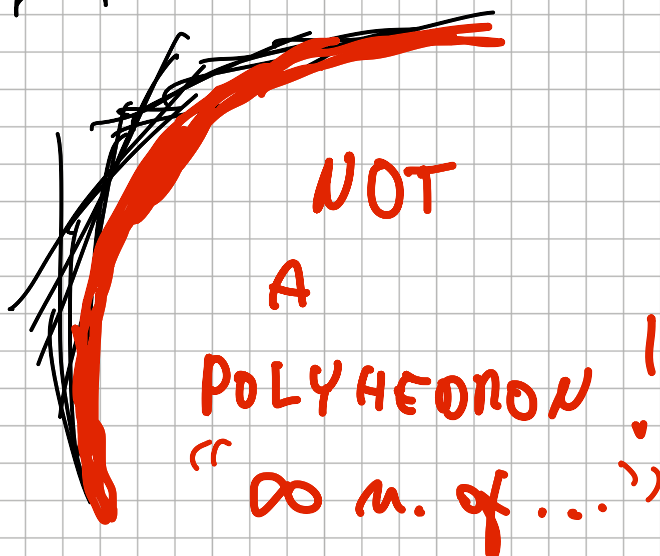


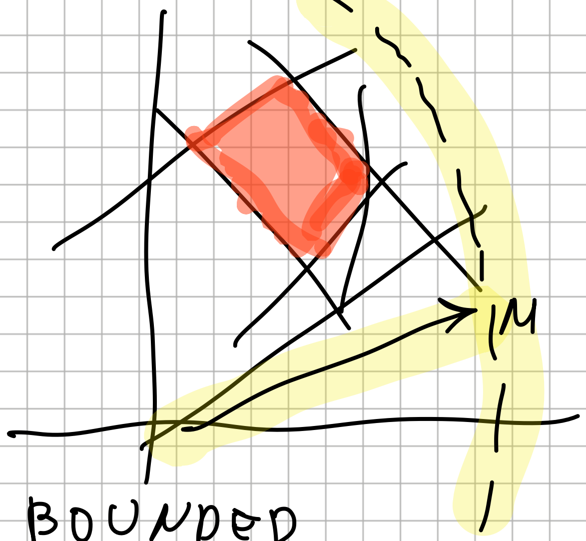


- POLYHEDRON

$\cap$ $\boxed{\text{finite}}$ n. of half-spaces & hyperplanes

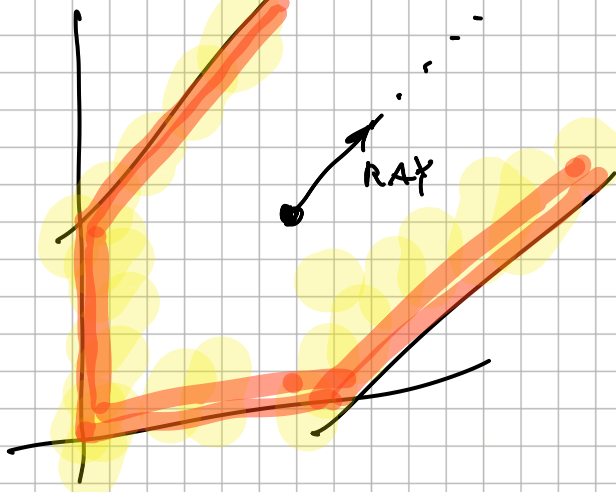






BOUNDED POLYHEDRON ≡ POLYTOPE

(POLÍTOPO, in Italian)

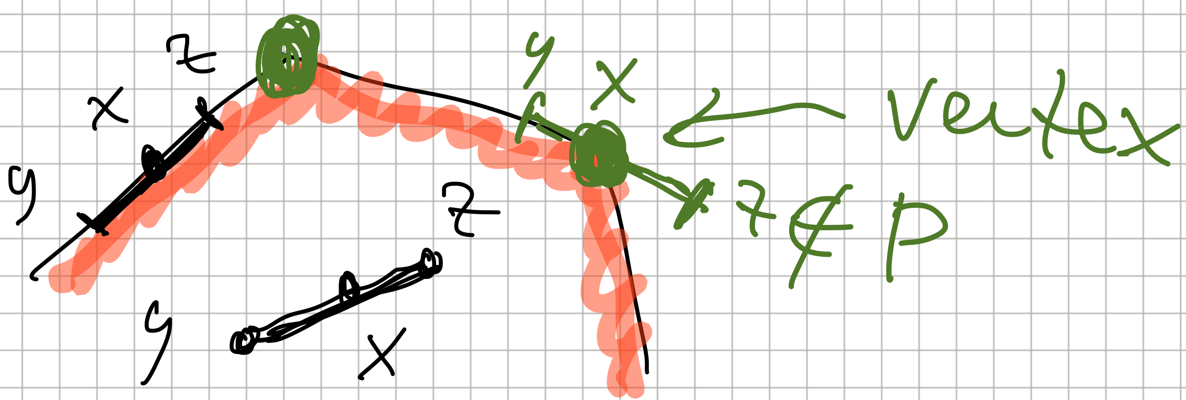


UNBOUNDED POLYHEDRON

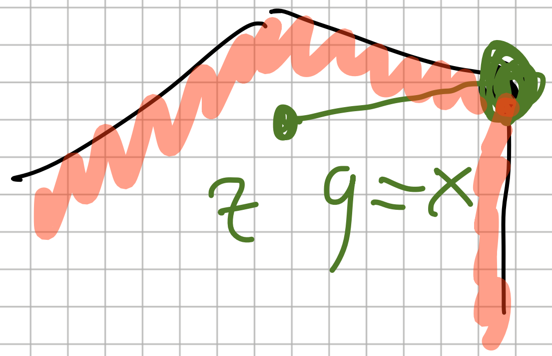
VERTEX

A point $x \in P$ is a VERTEX (or EXTREME POINT) of P iff it CANNOT be obtained as the STRICT convex combination of two distinct points in P, i.e., $\nexists\ y, z \in P$, $\boxed{y \neq z}$, and $\lambda \in]0,1[$:

$$x = \lambda y + (1-\lambda) z$$

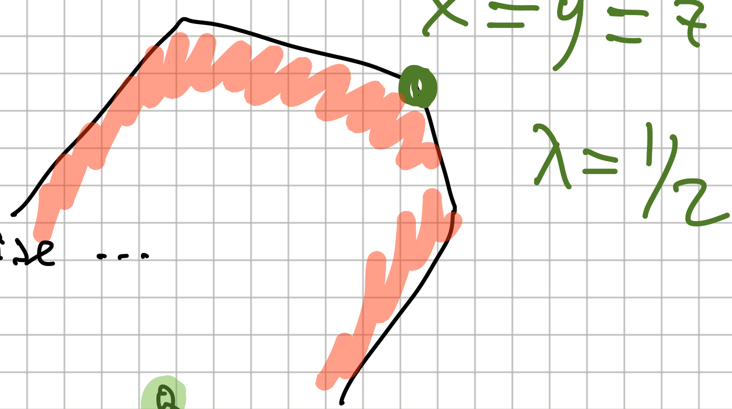


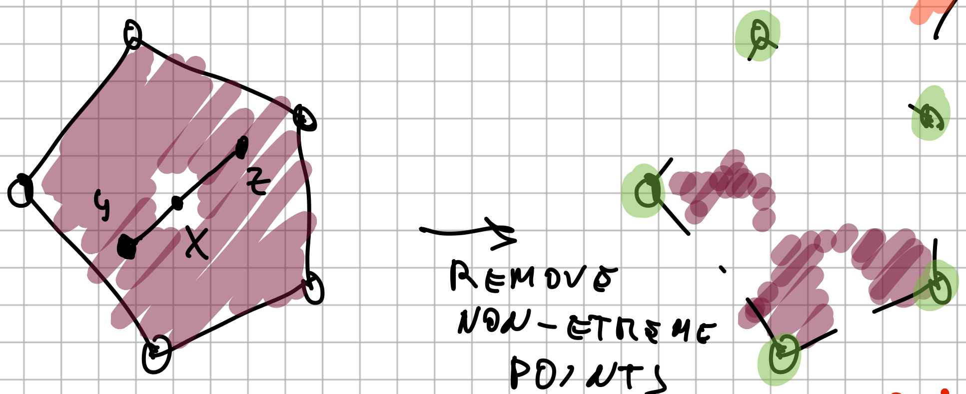
• $\lambda \in \{0, 1\}$ NOT allowed!



• $y \neq z$

otherwise ...





ONLY THE VERTICES REMAIN!

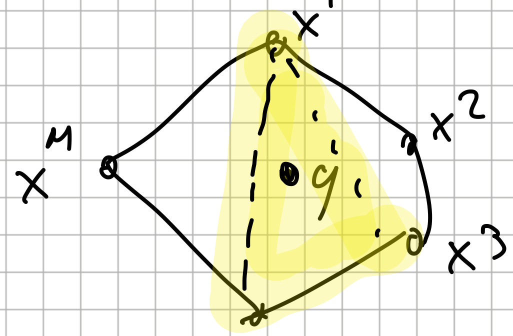
$$x = \lambda y + (1-\lambda) z$$

$$c^T x = \lambda \underbrace{c^T y} + (1-\lambda) \underbrace{c^T z}$$

$c^T x > c^T y$ if $c^T y < c^T z$ etc.

$\boxed{\text{OBSERVATION:}}$ If $\boxed{P \text{ is bounded}}$, then every $y \in P$ can be expressed as the convex combination of the vertices $x^1, x^2, \dots, x^M$ of P $\iff$

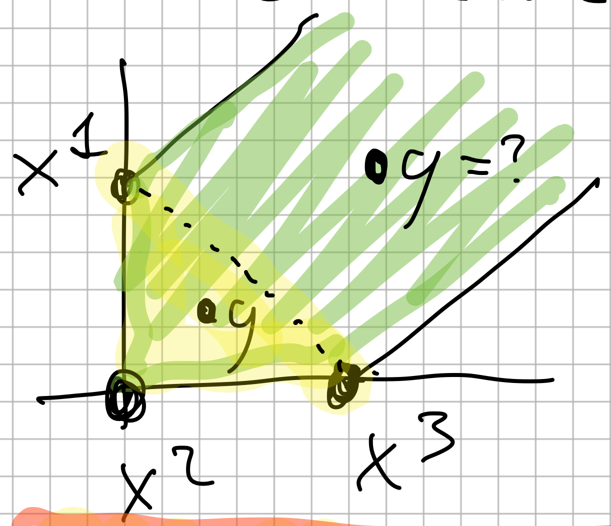
$$\exists\ \lambda_1, \dots, \lambda_M \geq 0, \quad \sum_{i=1}^{M} \lambda_i = 1 :$$



$$y = \sum_{i=1}^{M} \lambda_i x^i$$

"The MINKOWSKI – WEYL theorem"

Unbounded case: it does not apply !!



THEOREM If $P \neq \emptyset$ is the feasible set of LP problem

$$\min \{ c^T x : x \in P \}$$

and P is **BOUNDED**, then there exists at least one optimal solution that coincides with a vertex.

Proof: Let $x^1, x^2, \dots, x^k$ be the vertices of P, and y be any point in P. $\Rightarrow \exists$ $\lambda_1, \dots, \lambda_k \geq 0$, $\sum_i \lambda_i = 1$:

$$y = \sum_i \lambda_i x^i .$$

$$\boxed{c^T y} = c^T \underbrace{\left(\sum_i \lambda_i x^i \right)}_{y} = \sum_i \underbrace{\lambda_i}_{\geq 0} \underbrace{c^T x^i}_{\geq z^*}$$

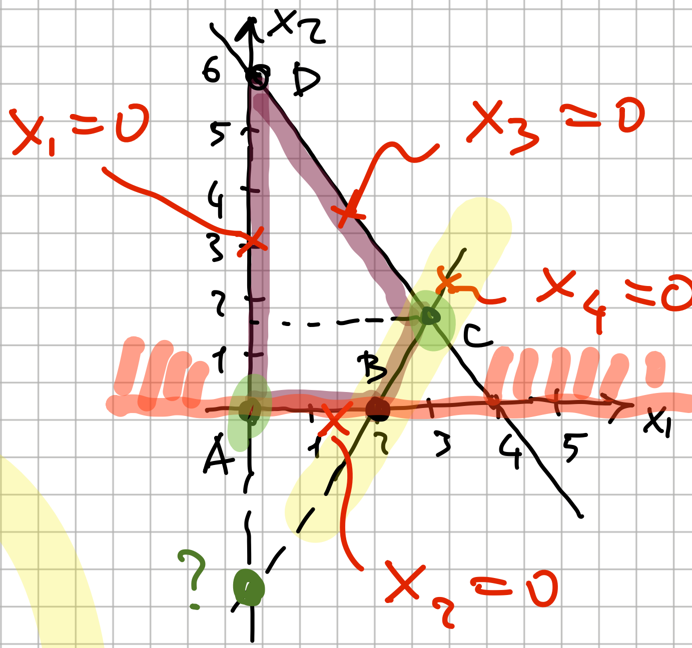
$$\boxed{\geq} z^* \underbrace{\left(\sum_i \lambda_i \right)}_{=1} = \boxed{z^*}$$

where $\boxed{z^* := \min \{ c^T x^i : i = 1, \dots, k \}}$ $\square$

$\boxed{\text{HOW TO COMPUTE (THE ACTUAL COORDINATES OF) THE VERTICES ?}}$

Example

$$
\begin{cases}
\min & -x_1 - x_2 \\
& 6x_1 + 4x_2 + x_3 = 24 \\
& 3x_1 - 2x_2 + x_4 = 6 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{cases}
$$



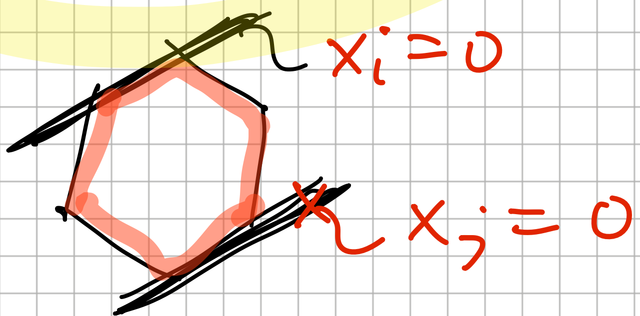
$x_1 = x_2 = 0 \Rightarrow A$

$x_2 = x_4 = 0 \Rightarrow B$

$x_3 = x_4 = 0 \Rightarrow C$

$x_1 = x_3 = 0 \Rightarrow D$

$x_1 = x_4 = 0 \Rightarrow ?$ ← NOT FEASIBLE BECAUSE $x_2 < 0$



$\Rightarrow$ DEGENERATE CASE TO BE AVOIDED!

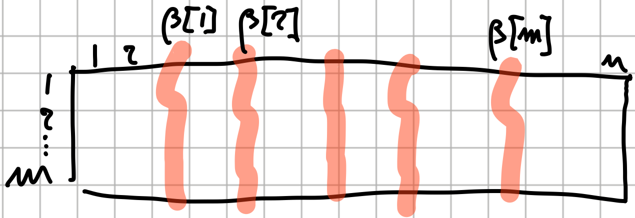
IN GENERAL

$$Ax = b, \quad x \geq 0$$

where A is $m \times n$ matrix, $m \leq n$, of full row rank, i.e., $\text{rank}(A) = m$

Def: BASIS of A:

$$\begin{array}{c} \\ 1 \\ \vdots \\ m \end{array}\left[\, A_1 \;\; A_2 \;\; \cdots \;\; A_n \,\right] \qquad B = \left[\, A_{\beta[1]} \;\; \cdots \;\; A_{\beta[m]} \,\right]$$



m linearly indep. A_j's

" B $m \times m$ NONSINGULAR matrix "

" $\det(B) \neq 0$ "

$$A = \big[\, \overset{x_1}{A_1} \;\; \overset{x_2}{A_2} \;\; \cdots \;\; \overset{x_n}{A_n} \,\big]$$

basis   not in the basis

$Ax = A_1 x_1 + \cdots + A_n x_n$

$$A \Rightarrow [\, \underbrace{B}_{X_B} \;,\; \underbrace{F}_{X_F} \,]$$

X_B BASIC var.s

X_F NONBASIC var.s

$$Ax = [B, F]\begin{bmatrix} x_B \\ x_F \end{bmatrix} =$$

$$= Bx_B + Fx_F = b \Rightarrow$$

$$B^{-1} * \left[Bx_B = b - Fx_F \right]$$

$$\boxed{x_B = B^{-1}b - B^{-1}Fx_F} \quad (*)$$

"equivalent to $Ax=b$ but in CANONICAL FORM w.r.t. B"

$x_F \to$ compute x_B by (*)

$\Rightarrow \begin{bmatrix} x_B \\ x_F \end{bmatrix}$ solution of $Ax=b$

n. of non basic var.s

"There are ∞^{n-m} feasible sol.s of the system $Ax=b$"

$x_F = 0 \implies x_B = B^{-1}b$

Sol. $\begin{bmatrix} x_B = B^{-1}b \\ x_F = 0 \end{bmatrix}$ is called

- $\boxed{\text{BASIC SOL.}}$ w.r.t. the chosen B
- $\boxed{\text{FEASIBLE BASIC SOL.}}$ when $B^{-1}b \geq 0$

there are $\dfrac{n!}{m!(n-m)!} = \begin{pmatrix} n \\ m \end{pmatrix}$ possible choices for the m col.s !

$\boxed{\text{NAÏVE PROCEDURE (enumerate all possible B):}}$

$B = \begin{bmatrix} | & | & | & | \end{bmatrix} \rightarrow$ check $\det(B) \neq 0$

$\rightarrow$ check $B^{-1}b \geq 0$

$\rightarrow$ evaluate $c^T \begin{bmatrix} x_B = B^{-1}b \\ x_F = 0 \end{bmatrix}$

$\rightarrow$ take the MINIMUM

huge n. of combinations!