OR1   12-OCT-2021

Th: A point $x \in P$ is a vertex of $P = \{x \geq 0 : Ax = b\} \neq \emptyset$ if and only if x is basic feasible sol. w.r.t. some B.

Proof

"x is BFS $\Longrightarrow$ x is a vertex"

Let
$$x = [\underbrace{x_1, \dots, x_k}_{\text{strictly positive}}, 0, 0, \dots 0]$$

where $k \geq 0$. Then $A_1, \dots, A_k$ are LINEAR INDEPENDENT. By contradiction, assume that x is NOT a vertex.

Then $\exists$ two points

$$y = [y_1, \dots, y_k, 0, 0, \dots, 0] \in P$$
$$z = [z_1, \dots, z_k, 0, 0, \dots, 0] \in P$$

with $y \neq z$, $\exists \lambda \in]0,1[$ s.t.

$$x = \lambda y + (1-\lambda) z$$

($\Rightarrow k \geq 1$ otherwise $y = z = 0$)

$(\#)\quad y \in P \Rightarrow Ay = b \Rightarrow A_1 y_1 + \cdots + A_k y_k = b$

$(\#\#)\quad z \in P \Rightarrow Az = b \Rightarrow A_1 z_1 + \cdots + A_k z_k = b$

$(\#) - (\#\#) \Rightarrow \boxed{A_1 \underbrace{(y_1 - z_1)}_{=:\alpha_1} + \cdots + A_k \underbrace{(y_k - z_k)}_{=:\alpha_k} = 0}$

$\alpha_1, \dots, \alpha_k$ cannot be all zero $(\Leftarrow y \neq z)$

$\Rightarrow A_1, \dots, A_k$ are $\boxed{\text{L/N. DEPENDENT}}$!

$\Rightarrow$ CONTRADICTION

$\boxed{\text{“ } x \text{ is a vertex} \Rightarrow x \text{ is B}\cancel{\text{F}}\text{S ”}}$

$x = [\boxed{x_1, \dots, x_k}, \boxed{0, 0, \dots, 0, 0}]$

$\underbrace{x_1, \dots, x_k}_{\text{strictly positive}}$

with $k \geq 0$.

$x \in P \Rightarrow \boxed{A_1 x_1 + \cdots + A_k x_k = b}\ (*)$

Two cases:

1. $A_1, \dots, A_k$ are lin. independent.

$$x = [\underbrace{x_1, \dots, x_k}, \overbrace{0, 0, 0}^{k+1 \dots m}, \overbrace{0, \dots, 0}^{n-m}]$$

$$A = \left[\underbrace{A_1 \mid \dots \mid A_k \mid A_{k+1} \mid \dots \mid A_m} \mid A_{m+1} \dots A_n\right]$$

$$\underbrace{}_{=: \boxed{B}} \quad m \times m$$

Now

i) $x = [\underbrace{* * * * * * * *}_{B^{-1}b} \mid \underbrace{0, 0, \dots, 0}_{\substack{n-m \\ \text{non basic} \\ \text{var.s}}}]$

ii) $Ax = b$

$\Rightarrow$ x is $\boxed{\text{THE}}$ BS associated with the above B.

2. columns $A_1, \dots, A_k$ are lin. dependent $\Rightarrow \exists\ \alpha_1, \dots, \alpha_k$, not all zero, s.t.

$$\boxed{\alpha_1 A_1 + \dots + \alpha_k A_k = 0} \quad (**)$$

Take a "very small" $\varepsilon \underset{\neq}{>} 0$

$$(*) + \varepsilon(**) \Rightarrow \overbrace{(x_1 + \varepsilon\alpha_1)}^{z_1} A_1 + \dots + \overbrace{(x_k + \varepsilon\alpha_k)}^{z_k} A_k = b$$

$$(*) - \varepsilon(**) \Rightarrow \underbrace{(x_1 - \varepsilon\alpha_1)}_{y_1} A_1 + \dots + \underbrace{(x_k - \varepsilon\alpha_k)}_{y_k} A_k = b$$

$$y := [x_1 - \varepsilon\alpha_1, \dots, x_k - \varepsilon\alpha_k, 0, 0, \dots, 0]^T$$

$$z := [x_1 + \varepsilon\alpha_1, \dots, x_k + \varepsilon\alpha_k, 0, 0, \dots, 0]^T$$

$Ay = Az = b$, $\quad y, z \geq 0$ (if $\varepsilon \underset{\neq}{>} 0$ small enough)

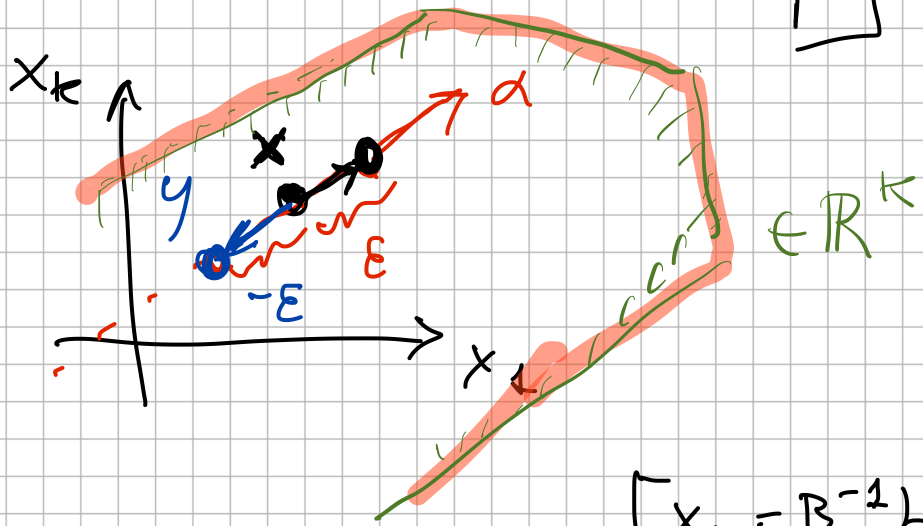
$\Rightarrow y, z \in P$ and $y \neq z$ ($\Leftarrow \alpha$ in nonzero). But

$$x = \frac{1}{2} y + \frac{1}{2} z \quad (\lambda = 1/2)$$

i.e., x is a STRICT convex comb. of two DISTINCT points $y, z \in P$

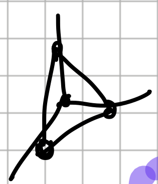
$\Rightarrow$ x is NOT a vertex

$\Rightarrow$ contradiction. $\square$

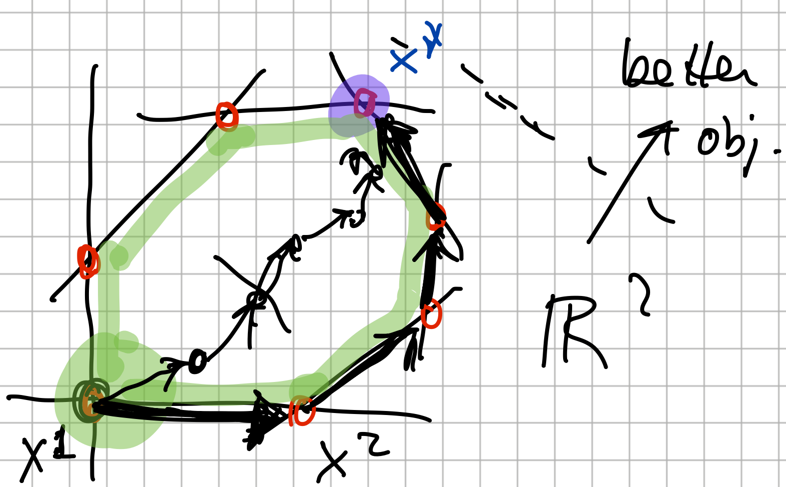


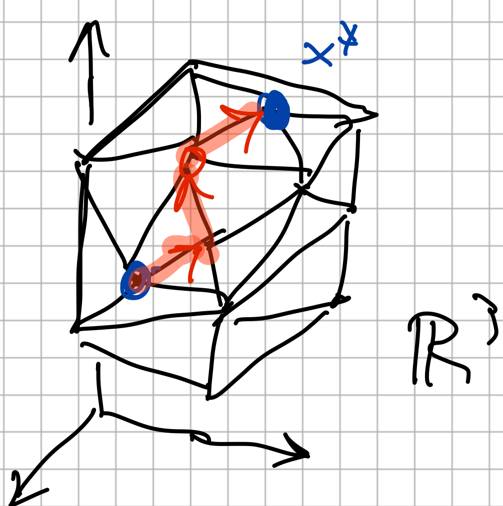
Note : $B \rightarrow x = \begin{bmatrix} x_B = B^{-1}b \\ x_F = 0 \end{bmatrix} \geq 0$

but $(B^{-1}b)_i = 0 \Rightarrow$ DEGENERATE BFS

 $\leftarrow$ simplex

THE SIMPLEX METHOD (George Dantzig)

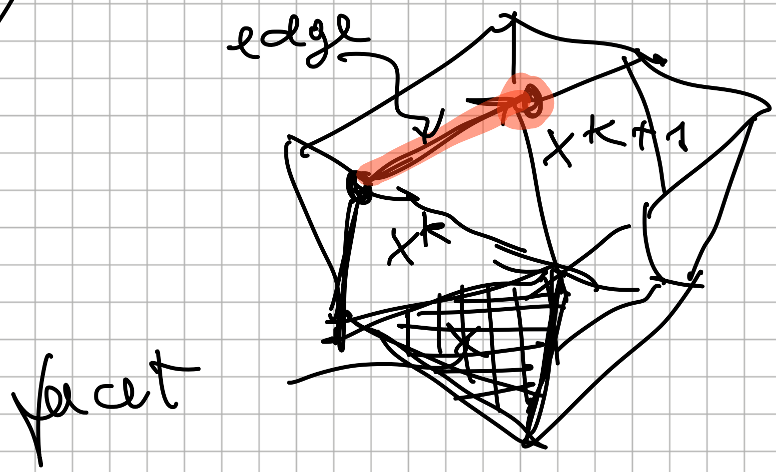




1) OPTIMALITY TEST

"is the current vertex/BFS optimal?"

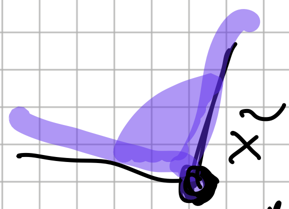
2) MOVE TO A "NEARBY" VERTEX



ADJECENT vertex along an EDGE

[1] OPTIMALITY TEST

$$c^T x = [c_B^T, c_F^T] \begin{bmatrix} x_B \\ x_F \end{bmatrix} =$$



current vertex $\tilde{x}$ corresp. to a basis B

$$= c_B^T x_B + c_F^T x_F$$

for any $x \in P$, $\Rightarrow Ax = b \Rightarrow$

$$x_B = B^{-1} b - B^{-1} F x_F$$

$$\Rightarrow c^T x = \dots = c_B^T B^{-1} b - c_B^T B^{-1} F x_F + c_F^T x_F =$$

$$= \underbrace{c_B^T B^{-1} b}_{\text{const.}} + 0^T x_B + \underbrace{\left(c_F^T - c_B^T B^{-1} F\right)}_{=:\ \bar{c}_F^T} \underbrace{x_F}_{\geq 0^T}$$

$=: \bar{c}_F^T$ reduced costs of nonbasic var.s

If $\bar{c}_F^T \geq 0^T$ then
the current basic sol. $\tilde{x} = \begin{bmatrix} B^{-1} b \\ 0 \end{bmatrix}$ is OPTIMAL.

$$c^T \tilde{x} = [c_B^T, c_F^T] \begin{bmatrix} B^{-1} b \\ 0 \end{bmatrix} = c_B^T B^{-1} b$$

cost of the current BFS $\tilde{x}$

Def: FULL REDUCED COST VECTOR

(*) $$\boxed{\bar{c}^T := c^T - c_B^T B^{-1} A}$$

$$\bar{c}^T = [\bar{c}_B^T, \bar{c}_F^T] = [\underbrace{c_B^T - c_B^T B^{-1} B}_{= 0^T}, \underbrace{c_F^T - c_B^T B^{-1} F}_{= \bar{c}_F^T}]$$

Test: compute $\bar{c}^T := c^T - c_B^T B^{-1} A$
and check $\bar{c} \geq 0$