OR1 13-Oct-2021 CODE: 074-376

Optimality test: current basis B

$$\boxed{\bar{c}^T := c^T - u^T A}, \text{ where } u^T = c_B^T B^{-1}$$

$1\times m \quad m\times m$

Assume $\bar{c}_h = c_h - u^T A_h < 0$

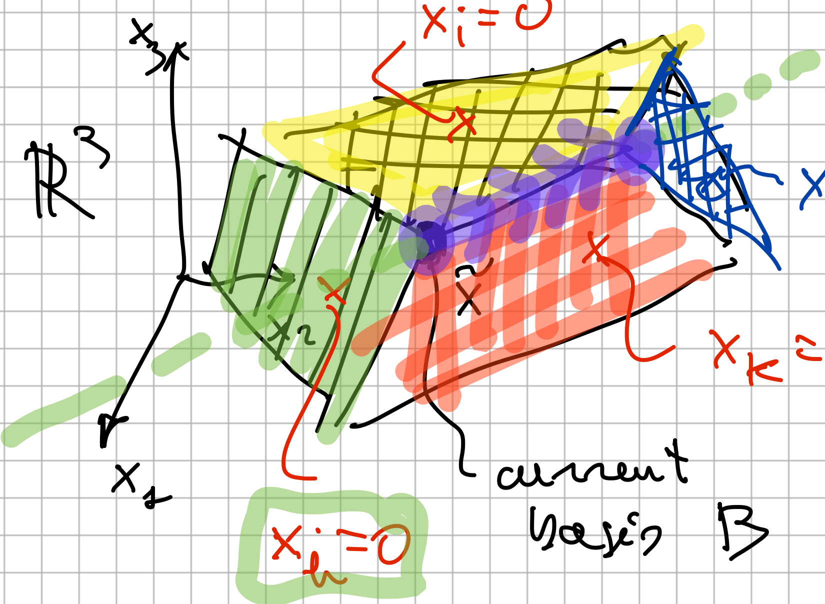
$$c^T x = \underbrace{c_0}_{u^T b} + \underbrace{\bar{c}_h}_{<0} x_h + \cdots$$

x_h nonbasic $\to$ $x_h = 0 \nearrow$ $\quad cx \downarrow$

(*) $$\boxed{x_B = \underbrace{B^{-1} b}_{=: \bar{b}} - B^{-1} A_h x_h}$$

" increase x_h as much as possible, without introducing negative basic var.s"

" max increase ϑ allowed "



$x_i, x_h, x_k = 0$

$\Rightarrow$ NON BASIC

$x_i = x_k = 0$

$x_h \nearrow$

$$\underbrace{\begin{bmatrix} x_{\beta[1]} \\ \vdots \\ x_{\beta[m]} \end{bmatrix}}_{x_B} = \underbrace{\begin{bmatrix} \bar{b}_1 \\ \vdots \\ \bar{b}_m \end{bmatrix}}_{\bar{b} := B^{-1} b \geq 0} - \underbrace{\begin{bmatrix} \bar{a}_{1h} \\ \vdots \\ \bar{a}_{mh} \end{bmatrix}}_{\bar{A}_h := B^{-1} A_h} x_h$$

where $\beta[1], \dots, \beta[m]$ identifies the index of the column in the basis

E.g.

$$B = \left[A_2 \middle| A_5 \middle| A_1 \middle| A_{10} \right]$$

$\beta[1] = 2 \quad \beta[2] = 5 \quad \beta[3] = 1 \quad \beta[4] = 10$

Then:

$$x_{\beta[i]} = \bar{b}_i - \bar{a}_{ih} x_h \geq 0$$

$i = 1, \dots, m$

Impose for each i:

$$\bar{b}_i - \bar{a}_{ih} x_h \geq 0 \quad , \text{i.e.}$$

$$\boxed{\underbrace{\bar{a}_{ih} x_h}_{\geq 0} \leq \bar{b}_i}$$

≥ 0

Two cases :

- $\bar{a}_{ik} \leq 0 \rightarrow$ ok
- $\boxed{\bar{a}_{ik} > 0} \rightarrow$

$\rightarrow \boxed{x_k \leq \bar{b}_i / \bar{a}_{ik}}$

$\bar{b}_i / \bar{a}_{ik}$ when $\bar{a}_{ik} > 0$

$$\boxed{\vartheta = \min \left\{ \frac{\bar{b}_i}{\bar{a}_{ik}} : \bar{a}_{ik} > 0 \right\}}$$

$$t := \operatorname{argmin} \left\{ \frac{\bar{b}_i}{\bar{a}_{ik}} : \bar{a}_{ik} > 0 \right\}$$

when

$$x_k = 0 \rightarrow \vartheta$$

$$x_{\beta[t]} = * \to 0$$

" x_h enters the basis,

$x_{\beta[t]}$ leaves the basis "

$$B = \left[A_{\beta[1]} \,\middle|\, \dots \,\middle|\, A_{\beta[t]} \,\middle|\, \dots \,\middle|\, A_{\beta[m]} \right]$$

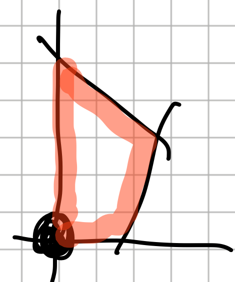
" $x_{\beta[t]}$ leaves the basis "

A_h

" x_h enters the basis "

$B \to B_{new}$ " ADJACENT BASIS "

THE SIMPLEX METHOD



1) Start with a basis $\boxed{B}$ that corresponds to a BFS (i.e., to a vertex): $B = [A_{\beta[1]}, \dots, A_{\beta[m]}]$

2) compute $\boxed{B^{-1}}$

$u^T := c_B^T B^{-1}$

if $\bar{c}^T := c^T - u^T A \geq 0^T$ then

STOP : an optimal sol. is $x = \begin{bmatrix} B^{-1}b \\ 0 \end{bmatrix} \geq 0$ and its cost is $u^T b$

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3) $\boxed{\text{Select}}$ $\bar{c}_h = c_h - u^T A_h < 0$ and

compute $\bar{b} := B^{-1} b$,

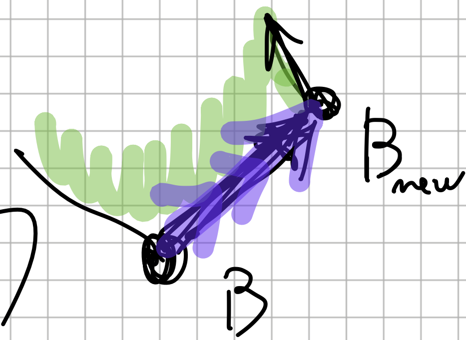
$\bar{A}_h := B^{-1} A_h$

(*) $t = \arg\min \left\{ \bar{b}_i / \bar{a}_{ih} : \bar{a}_{ih} > 0 \right\}$

replace $\boxed{A_{\beta[t]}}$ in B by $\boxed{A_h}$

"$\beta[t] := h$"

4) REPEAT from 2)



(*) if $\bar{a}_{ih} \leq 0 \quad \forall i$ then

ϑ has no limit

$\Rightarrow$ PROBLEM UNBOUNDED!

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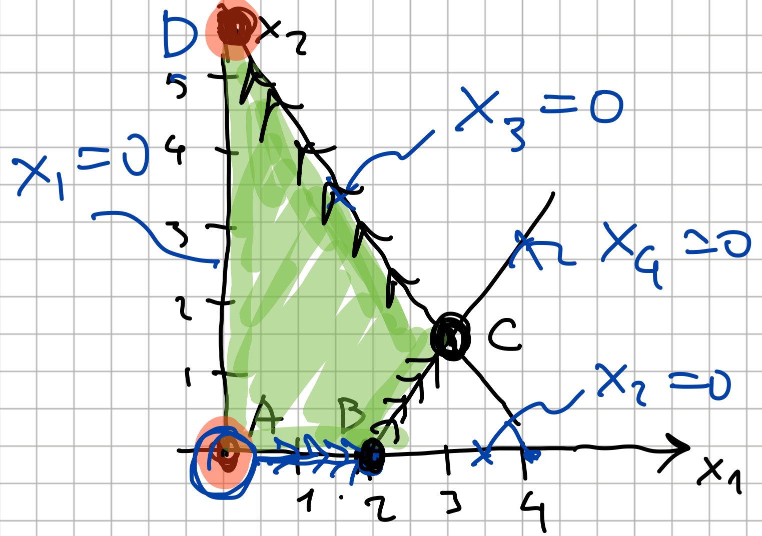
EXAMPLE

$$\begin{cases} \min & -x_1 - x_2 = z \\ & 6x_1 + 4x_2 + x_3 = 24 \\ & 3x_1 - 2x_2 + x_4 = 6 \\ & x_1, x_2, x_3, x_3 \geq 0 \end{cases}$$



• CANONICAL FORM w.r.t. current basis $\mathcal{B}$

$$\rightarrow \begin{cases} x_B = B^{-1} b - B^{-1} F x_F \\ z = c_B^T B^{-1} b + (c_F^T - c_B^T B^{-1}) F \end{cases}$$

where $z = c^T x$

1st iteration vertex A "$x_1 = x_2 = 0$ nonbasic, x_3, x_4 basic"

$$B = \left[A_{(3)} \mid A_{(4)} \right] \quad \text{where } \mathcal{B}[1],\ \mathcal{B}[2]$$

$$A = \begin{bmatrix} 6 & 4 & 1 & 0 \\ 3 & -2 & 0 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 24 \\ 6 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathcal{B}[1] = 3, \quad \mathcal{B}[2] = 4$$

$$\begin{cases} x_3 = 24 - 6x_1 - 4x_2 \\ x_4 = 6 - 3x_1 + 2x_2 \leftarrow t \end{cases}$$

PIVOT row

$$\big(z = 0 - x_1 - x_2$$

Compute red. costs.

$$\bar{c}_1 = -1 \quad , \quad \bar{c}_2 = -1$$

Select $x_h = x_1$

(while $x_2 = 0$)

$$\begin{cases} x_3 = 24 - 6x_1 \geq 0 \\ x_4 = 6 - 3x_1 \geq 0 \end{cases}$$

$$\begin{cases} 24 \geq 6x_1 \rightsquigarrow x_1 \leq 24/6 = 4 \\ 6 \geq 3x_1 \rightsquigarrow x_1 \leq 6/3 = 2 \end{cases}$$

$$\min \left\{ \frac{24}{6}, \frac{6}{3} \right\} = 2 = \vartheta$$

"$x_1 : 0 \to \vartheta = 2$

$x_4 : 6 \to 0$ "

$t = \arg\min \{ \dots \} = 2$

$\boxed{\text{2nd ITERATION}}$

- x_3, x_1 are basic
- x_2, x_4 are non basic

$$\begin{cases} x_3 = 12 - 8x_2 + 2x_4 \\ \boxed{x_1 = 2 + \frac{2}{3}x_2 - \frac{1}{3}x_4} \\ z = -2 - \frac{5}{3}x_2 + \frac{1}{3}x_4 \end{cases}$$

Pivot row in the previous iteration

$$x_4 = 6 - 3x_1 + 2x_2$$

$$\frac{3}{3}x_1 = \frac{6}{3} + \frac{2}{3}x_2 - \frac{1}{3}x_4$$

$$\boxed{x_1 = 2 + \frac{2}{3}x_2 - \frac{1}{3}x_4}$$

From the previous iter.

$$x_3 = 24 - 6\boxed{x_1} - 4x_2$$

$$x_3 = 24 - 6\Big(\underbrace{2 + \frac{2}{3}x_2 - \frac{1}{3}x_4}_{x_2}\Big) - 4x_2$$

$$\Rightarrow \boxed{x_3 = 12 - 8x_2 + 2x_4}$$

Similarly:

$$z = \boxed{-x_1} - x_2$$

$$z = -\underbrace{\left(2 + \tfrac{2}{3}x_2 - \tfrac{1}{3}x_4\right)}_{x_1} - x_2$$

$$\boxed{z = -2 - \tfrac{5}{3}x_2 + 1/3\, x_4}$$

3rd ITERATION

Therefore we have the new canonical system:

$$\begin{cases} x_3 = 12 - 8x_2 + 2x_4 \\ x_1 = 2 + 2/3\, x_2 - 1/3\, x_4 \\ z = -2 \boxed{-5/3}\, x_2 \boxed{+1/3}\, x_4 \end{cases}$$

?

BFS $x_2 = x_4 = 0 \rightarrow x_3 = 12,\ x_1 = 2$

cost $= -2$

Reduced costs: $\bar{c}_1 = \bar{c}_3 = 0$

$\bar{c}_2 = \boxed{-5/3}$ $\quad \bar{c}_4 = \boxed{+1/3}$

Select $\bar{c}_h = \bar{c}_2 = -5/3 < 0$

"x_2 will enter the basis"

$$\begin{cases} x_3 = 12 - 8x_2 \geq 0 & \rightarrow \boxed{x_2 \leq \frac{12}{8} = \frac{3}{2}} \\ x_1 = 2 + 1/3\, x_2 \geq 0 & \rightarrow \text{OK} \end{cases}$$

$\vartheta = 3/2 \Rightarrow$ PIVOT now $t = 1$

"x_3 leaves the basis"

By substitution ...

$$\begin{cases} x_2 = 3/2 - 1/8\, x_3 + 1/4\, x_4 \\ x_1 = 3 - 1/12\, x_3 - 1/6\, x_4 \\ z = -9/2 + \frac{5}{24} x_3 - \frac{1}{12} x_4 \end{cases}$$

$\boxed{\text{BFS}}$ $x_3 = x_4 = 0 \rightarrow x_2 = 3/2,\ x_1 = 3$

$z = -9/2$

4th ITERATION

"x_4 enters the basis"

$\Leftarrow \bar{c}_4 < 0$

.... "x_1 leaves the basis"

....

$$\begin{cases} x_2 = 6 - \frac{3}{2} x_1 - \frac{1}{4} x_3 \\ x_4 = 18 - 6 x_1 - \frac{1}{2} x_3 \\ \boxed{z = -6 + \frac{1}{2} x_1 + \frac{1}{4} x_3} \leftarrow \end{cases}$$

$\boxed{\text{BFS}}$ $\quad x_1 = x_3 = 0 \rightarrow x_2 = 6, \; x_4 = 18$

$$z = -6$$

$$\tilde{c} = \left[\frac{1}{2}, 0, \frac{1}{4}, 0\right]^T \geq 0$$

$\Longrightarrow$ STOP "OPTIMAL BFS!"