OR1 19-Oct-2021

2-PHASE METHOD

w.l.o.g.

$$\boxed{A x = b \geq 0}, \quad x \geq 0$$

$$\begin{cases} A x + I y = b \\ x, \; y \geq 0 \end{cases}$$

where $y \in \mathbb{R}^m$ of ARTIFICIAL VAR.S.

$$\begin{array}{r|c|ccc|ccc|c}
 & & \leftarrow & x & \rightarrow & \leftarrow & y & \rightarrow & w \\
\hline
-w = & 0 & 0 & \cdots & 0 & 1 & 1 \cdots & 1 & -1 \\
\hline
y_1 = & & & & & & & & \\
y_2 = & b \geq 0 & & A & & & I & & \\
\vdots & & & & & & & & \\
y_m = & & & & & & & & \\
\hline
\end{array}$$

BFS: $y_1 = b_1, \dots, y_m = b_m; \quad x = 0$

when $y = 0 \Rightarrow$ AUGMENTED MODEL coincides with the original model

Artificial obj. function:

$$\boxed{w = \sum_{i=1}^{m} y_i} \rightarrow \min.$$

$\rightarrow w^* = 0$

$y^* = 0$

$\Rightarrow$ get a feasible basic sol. x^* for the original problem.

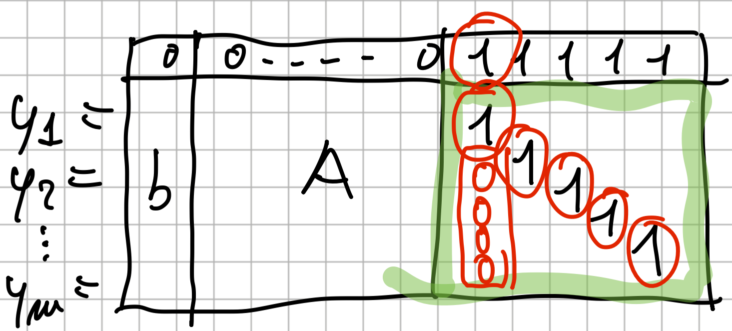
How to manipulate the obj. function

(1)

$$\boxed{w = \sum_{i=1}^{m} y_i = \sum_{i=1}^{m} \underbrace{\left(b_i - a_i^T x\right)}_{y_i}}$$

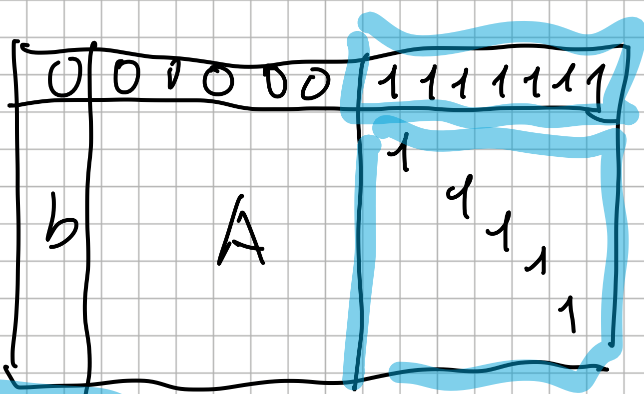
Canonical form of the obj. function

(2)



m mini-pivot on the 1's of I

(3)



← row 0

$$[\text{row } 0] - \sum_{i=1}^{m} [\text{row } i] \rightarrow [\text{new row } 0]$$

Ex :

$$\begin{cases} \min & z = x_1 + x_3 \\ & x_1 + 2x_2 + x_4 = 5 \\ & x_2 + 2x_3 = 6 \\ & x_1, x_2, x_3 \geq 0 \end{cases}$$

		x_1	x_2	x_3	x_4	y_1	y_2	w
$-w=$	0	0	0	0	0	1	1	-1
$y_1=$	5	1	2	0	1	1	0	0
$y_2=$	6	0	1	2	0	0	1	0

$\leftarrow w = y_1 + y_2$

$\rightarrow$ MIN

≥ 0

[row 0] − [row 1] − [row 2]

$\rightarrow$ [new row 0]

		x_1	x_2	x_3	x_4	y_1	y_2
$-w=$	-11	-1	-3	-2	-1	0	0
$y_1=$	5	1	2	0	1	1	0
$y_2=$	6	0	1	2	0	0	1

$\rightarrow$ SIMPLEX METHOD

$\downarrow$

...

$\downarrow$

$$
\begin{array}{r|c|ccccc|cc|}
 & & x_1 & x_2 & x_3 & x_4 & & y_1 & y_2 \\
\hline
-w = & 0 & 0 & 0 & 0 & 0 & & 1 & 1 \\
\hline
x_2 = & 5/2 & 1/2 & 1 & 0 & 1/2 & & 1/2 & 0 \\
x_3 = & 7/4 & -1/4 & 0 & 1 & -\frac{1}{4} & & -\frac{1}{4} & \frac{1}{2} \\
\hline
\end{array}
$$

optimal ($\bar{c} \geq 0$)

$w^* = 0 \rightarrow y_1^* = y_2^* = 0$

PHASE II "$z = c^t x$"

remove y var, continue with the simplex method

$$
\begin{array}{r|c|cccc|c|}
 & & x_1 & x_2 & x_3 & x_4 & z \\
\hline
 & 0 & 1 & 0 & 1 & 0 & -1 \\
\hline
x_2 = & 5/2 & 1/2 & 1 & 0 & 1/2 & 0 \\
x_3 = & 7/4 & -1/4 & 0 & 1 & -1/4 & 0 \\
\hline
\end{array}
$$

$z = x_1 + x_3$

① mini-pivot

$$-\frac{1}{4}x_1 + x_3 - 1/4\, x_4 = 7/4$$

$$x_3 = 7/4 + 1/4\, x_1 + 1/4\, x_4$$

$$z = x_1 + \underbrace{(7/4 + 1/4\, x_1 + 1/4\, x_4)}_{x_3}$$

$$\boxed{z = 7/4 + 5/4\, x_1 + 1/4\, x_4}$$

	x_1	x_2	x_3	x_4	$\cancel{z}$
$-7/4$	$5/4$	0	0	$1/4$	-1

FULL canonical form !!

$\Longrightarrow$ INITIAL basis is already optimal !

"STRANGE" cases

(A) At the end of phase I, $w^* > 0$

$\rightarrow$ some $y_i^* > 0$

$\rightarrow$ THERE IS NO FEASIBLE SOL.

$\rightarrow$ PR. INFEAS. !

EX:
$$
\begin{cases}
\min\ z = x_1 + x_3 \\
\boxed{x_1 + 2x_2} \boxed{\leq -5} \\
x_2 + 2x_3 = 6 \\
x_1, x_2, x_3 \geq 0
\end{cases}
$$

$$\begin{cases} y_1 + 2x_2 + x_4 = -5 \\ x_2 + 2x_3 = 6 \end{cases}$$

		x_1	x_2	x_3	x_4	y_1	y_2
$w =$	-11	1	1	-2	1	0	0
$y_1 =$	$+5$	-1	-2	0	-1	1	0
$y_2 =$	6	0	1	2	0	0	1

≥ 0

$w = y_1 + y_2$

$\rightarrow$ SIMPLEX METHOD

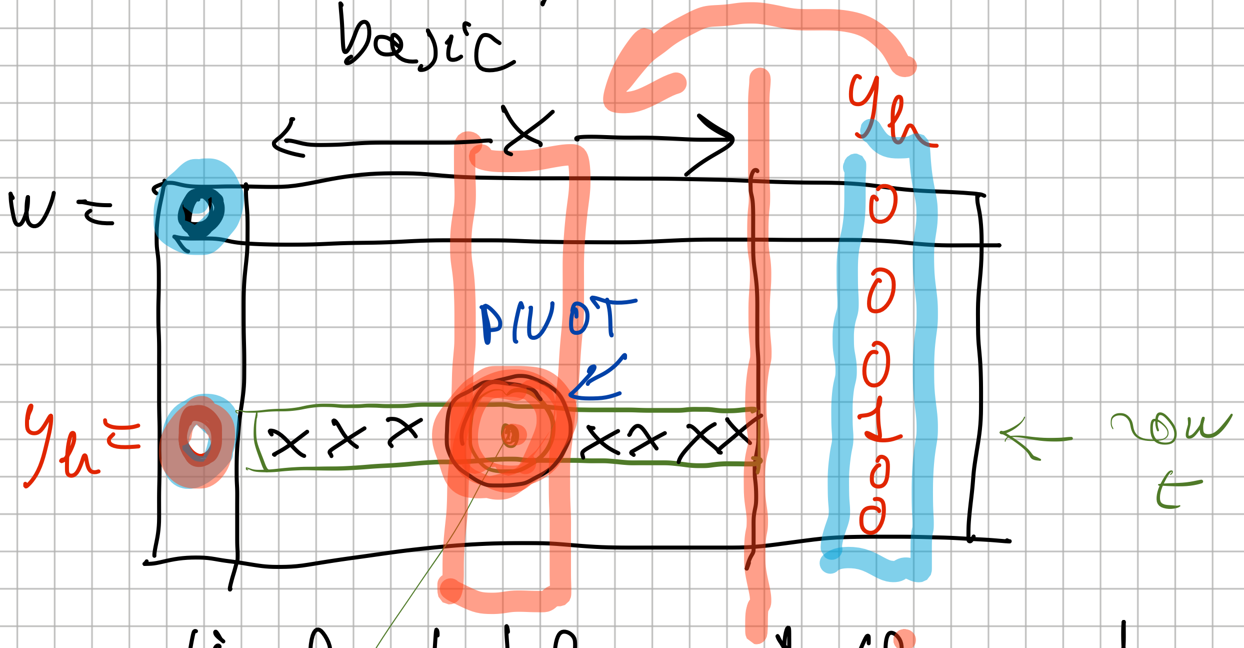
still a basic var.

....

		x_1	x_2	x_3	x_4	y_1	y_2
$-w =$	-5	1	2	0	1	0	1
$y_1 =$	5	-1	-2	0	-1	1	0
$x_3 =$	3	0	$1/2$	1	0	0	$1/2$

$w^* = 5 > 0 \implies$ ORIGINAL PR. INFEAS.

(B) $w^* = 0$ but some y_h is still basic

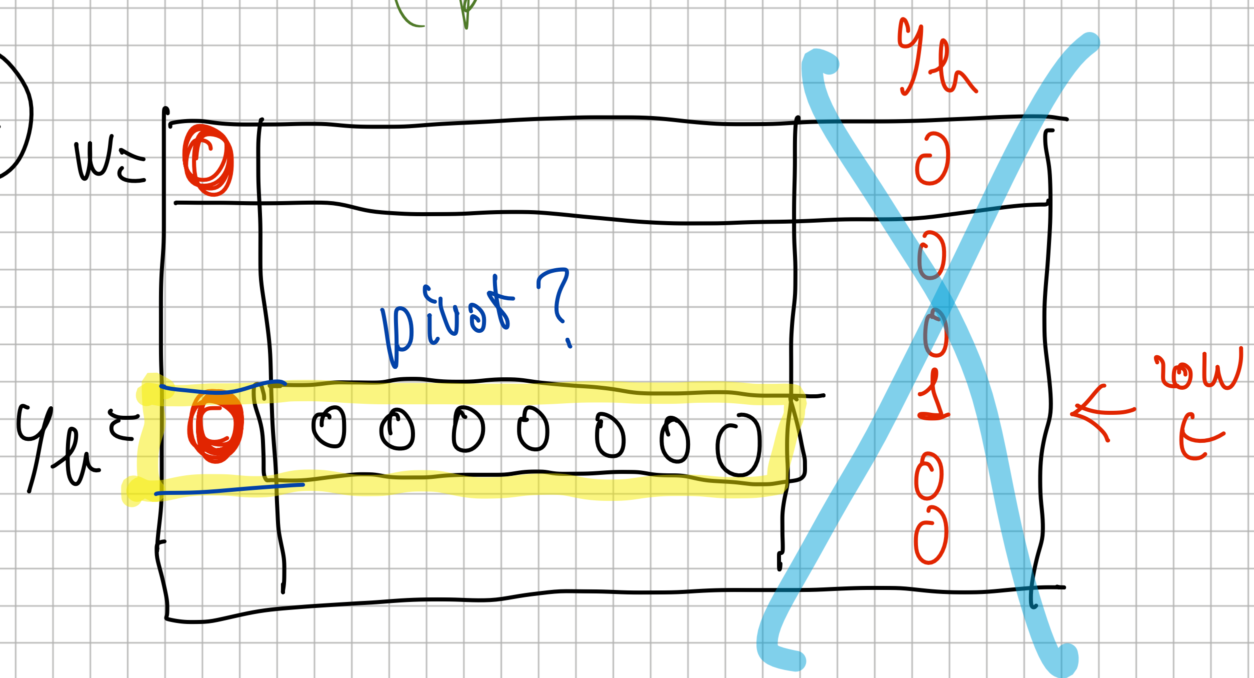


optimal tableau at the end of phase I.

$$w^* = 0 \to y_1^* = \dots = y_h^* = \dots y_m^* = 0$$

any nonzero (positive or negative)

(C)



Write down eq. for row t:

$$0x_1 + 0x_2 + \cdots + 0x_n = 0$$

" row t is a zero row"

$\Rightarrow$ row of zeros can be obtained from

$(Ax = b)$

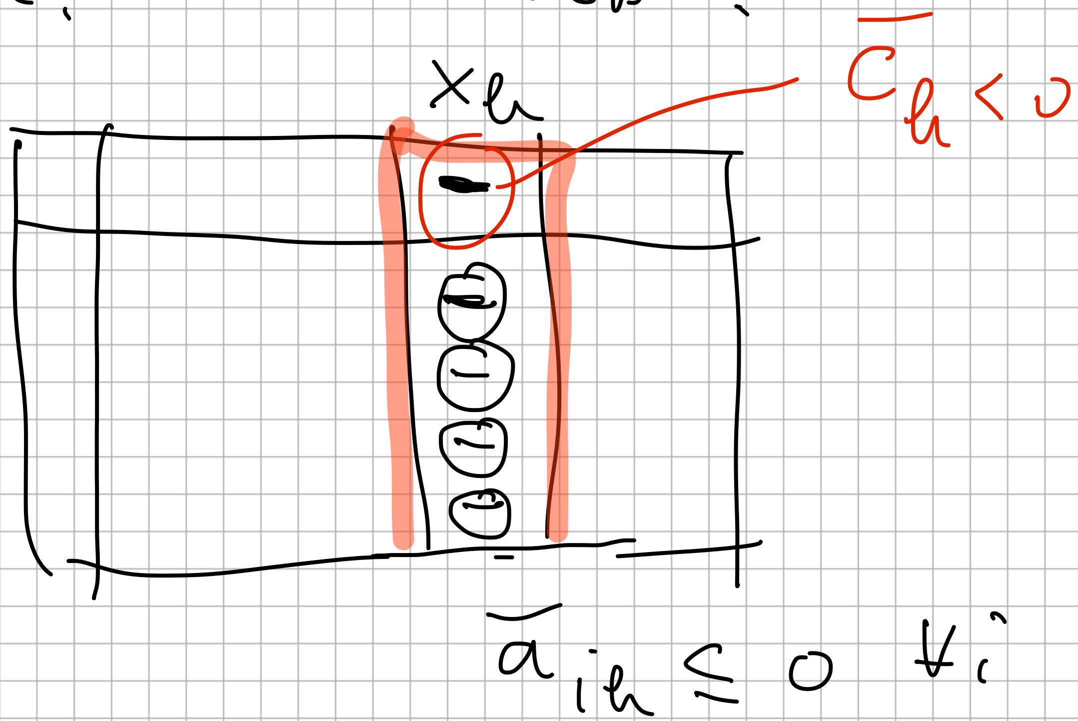
by combining the rows of (A, b)

$\Rightarrow$ row t is LIN. DEPENDENT on the other rows of $Ax = b$

$\Rightarrow$ SAFELY REMOVE row t from the tableau

• When I discover that

Pr. UNBOUNDED ?



$$\Rightarrow \quad \vartheta = +\infty \;\rightarrow\; \text{“}\min = -\infty\text{”}$$