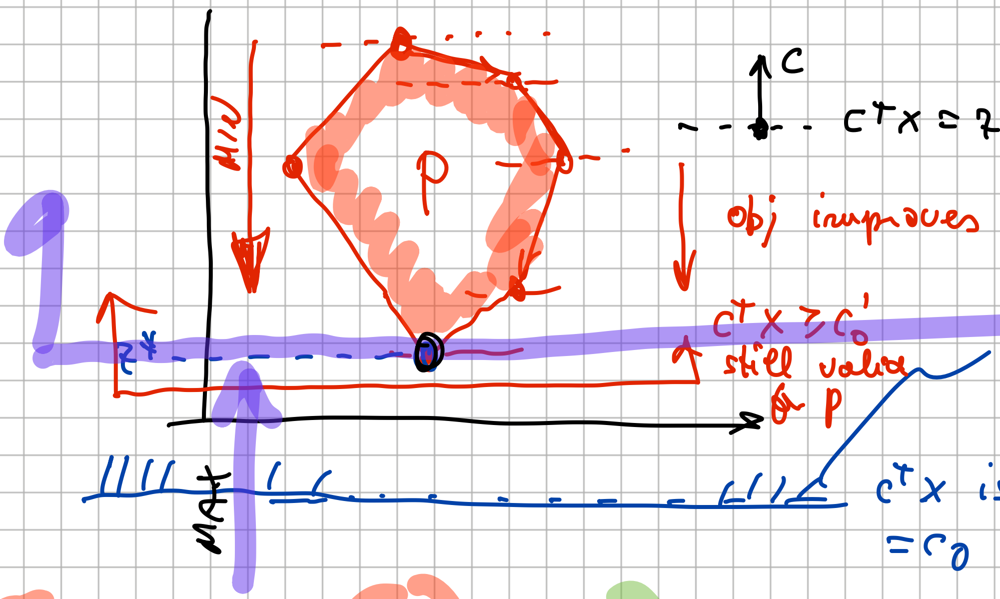
OR1 26-oct-2021

DUALITY

DEFINITION OF VALID INEQ.:

"$\alpha^T x \geq \alpha_0$ is valid for $P \subseteq \mathbb{R}^n$

$\Leftrightarrow$ $\alpha^T x \geq \alpha_0 \quad \forall x \in P$ "



$c^T x \geq c_0 = z^*$ valid

$c^T x \geq c_0$ is valid for P

$c^T x$ is "better than optimal" $= c_0$

$$z^* = \underbrace{\boxed{\min}\{c^T x : x \in P\}}_{\text{"PRIMAL PROBLEM"}} = \underbrace{\boxed{\max}\{c_0 : \text{"}c^T x \geq c_0 \text{ is VALID for } P\text{"}\}}_{\text{"DUAL PROBLEM"}}$$

"When a certain ineq. $c^T x \geq c_0$ is valid for P ? "

$\Rightarrow$ **FARKAS LEMMA**

EX. $P := \{x \geq 0 : Ax = b\}$ STANDARD FORM

$$\begin{cases} [3x_1 + 2x_2 - x_3 = 5] & *[u_1] \\ [-x_1 + 2x_2 + x_3 - x_4 = 6] & *[u_2] \end{cases}$$

$$\overline{(3u_1 - u_2)\underset{\geq 0}{x_1} + (2u_1 + 2u_2)x_2 + (-u_1 + u_2)x_3 - u_2 x_4 \geq 5u_1 + 6u_2}$$

∞^2 choices for $u \in \mathbb{R}^2$

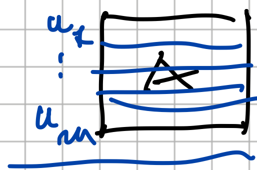
$x_1 \geq 0$ ...

$\infty^m * \infty^n = \infty^{m+n+1}$ valid ineq.s for P

In general, a valid ineq. can be obtained from

$$Ax = b, \quad x \geq 0$$

in 3 steps:



1) $u \in \mathbb{R}^m$: $u^T A x = u^T b$

valid eq.

2) $\Rightarrow \quad \underbrace{u^T A x}_{c^T \geq} \geq \underbrace{u^T b}_{\geq c_0}$ valid ineq.

3) $\boxed{c^T x \geq c_0, \quad \text{where} \quad c^T \geq u^T A, \quad c_0 \leq u^T b}$

FARKAS LEMMA

The ineq. $c^T x \geq c_0$ is VALID for

$$P := \{x \geq 0 : Ax = b\} \neq \emptyset$$

if and only if $\exists\, u \in \mathbb{R}^m$ s.t.

$$c^T \geq u^T A \qquad (*)$$
$$c_0 \leq u^T b \qquad (**)$$

Proof The condition is SUFFICIENT:

$$\underbrace{c^T x}_{(*)\ \geq 0} \geq \underbrace{u^T \underbrace{Ax}_{= b}}_{(*)\ \geq 0} = u^T b \underbrace{\geq}_{(**)} c_0 \quad , \forall x \in P$$

The condition is necessary, i.e.

"$c^T x \geq c_0$ is valid for P" $\Rightarrow$ compute $u \in \mathbb{R}^m$: $(*)$ $(**)$ hold

$$c_0 \leq z^* := \underbrace{\min \{c^T x : Ax = b, x \geq 0\}}$$

PR. IS NOT UNBOUNDED !

$P \neq \emptyset$ $\Rightarrow$ PR. NOT INFEASIBLE

Solve the min. problem by SIMPLEX + Bland's rule

$\Rightarrow$ optimal basis B $\Rightarrow$ $\boxed{u^T := c_B^T B^{-1}}$

- B is optimal $\Rightarrow$

$$\bar{c}^T = c^T - u^T A \geq 0^T$$

$$\Leftrightarrow \quad c^T \geq u^T A \quad (*)$$

- $c_0 \leq z^* = [c_B^T, c_F^T] \begin{bmatrix} x_B \\ x_F \end{bmatrix}$

$$= c_B^T \underbrace{x_B}_{B^{-1}b} + c_F^T \underbrace{x_F}_{=0} = \underbrace{c_B^T B^{-1}}_{=:u^T} b = u^T b$$

$\Rightarrow$ $(**)$ holds $\square$

FROM MY PREVIOUS PART:

$$z^* = \underbrace{\min \{ c^T x : x \in P \}}_{\text{"PRIMAL PROBLEM"}} = \underbrace{\max \{ c_0 : \text{"} c^T x \geq c_0 \text{ is VALID for } P \text{"} \}}_{\text{"DUAL PROBLEM"}}$$

$\Updownarrow$ FARKAS' LEMMA

$$= \max_{u, c_0} \{ c_0 : c^T \geq u^T A, \boxed{c_0 \leq u^T b}, u \in \mathbb{R}^m \}$$

$c_0 = u^T b$

$$= \boxed{\max_u \{ u^T b : c^T \geq u^T A, u \gtrless 0 \}}$$

"DUAL PR."

If the primal problem has an optimal sol., then

$$\left\{\begin{array}{l} \min c^T x \\ Ax = b \\ x \geq 0 \end{array}\right. = \left\{\begin{array}{l} \max u^T b \\ c^T \geq u^T A \\ u \gtrless 0 \end{array}\right.$$

PRIMAL PR. DUAL PROBL.

PRIMAL	DUAL
MIN	MAX
$a_i^T x \geq b_i$	$u_i \geq 0$
$a_i^T x \leq b_i$	$u_i \leq 0$
$a_i^T x = b_i$	$u_i \gtrless 0$
$x_j \geq 0$	$\boxed{c_j} \geq \boxed{u^T A_j}$
$x_j \leq 0$	$c_j \leq u^T A_j$
$x_j \gtrless 0$	$c_j = u^T A_j$

For example :

$$
\left\{
\begin{array}{llcl}
 & \min\ 10x_1 + 20x_2 + 30x_3 & = & \max\ 1u_1 + 2u_2 + 3u_3 \\
{[u_1\ *]} & 2x_1 - x_2 \geq 1 & \rightsquigarrow & u_1 \geq 0 \\
{[u_2\ *]} & x_2 + x_3 \leq 7 & \rightsquigarrow & u_2 \leq 0 \\
{[u_3\ *]} & x_1 - x_3 = 3 & \rightsquigarrow & u_3 \gtrless 0 \\
 & x_1 \geq 0 & \rightsquigarrow & 2u_1 + u_3 \leq 10 \\
 & x_2 \leq 0 & \rightsquigarrow & -u_1 + u_2 \geq 20 \\
 & x_3 \gtrless 0 & \rightsquigarrow & u_2 - u_3 = 30
\end{array}
\right.
$$