OR1 27-OCT-2021

Duality :

$$\begin{cases}\min & c^T x\\ & Ax \geq b\\ & x \geq 0\end{cases} \rightsquigarrow \begin{cases}\max & u^T b\\ & c^T \geq u^T A\\ & u \geq 0\end{cases}$$

PRIMAL DUAL

$$\begin{tikzcd} P \arrow[r, "D"] & D \arrow[l, bend left, "D"] \arrow[r, "D"] & ? \arrow[r] & ? \arrow[r] & ? \end{tikzcd}$$

Th : The dual of the dual is the primal.

Proof :

$$\begin{cases}\min & c^T x\\ & Ax \geq b\\ & x \geq 0\end{cases} \overset{D}{\Longrightarrow} \begin{cases}\max & u^T b \; (= b^T u)\\ & c^T \geq u^T A\\ & u \geq 0\end{cases} = \begin{cases} -\min & (-b^T) u\\ & (-A^T) u \geq -c\\ & u \geq 0\end{cases}$$

$$(u^T A)^T = A^T u$$

$$\overset{D}{\Longrightarrow} \begin{cases} -\max & y^T (-c)\\ & -b^T \geq y^T (-A^T)\end{cases}$$

$$= \begin{cases} +\min & c^T y\\ & Ay \geq b\\ & y \geq 0\end{cases} = \text{PRIMAL PR.}$$

"$x \to y$"

$$\mathrm{PR} \xrightarrow{D} D \equiv D' \xrightarrow{D} (\;) \equiv \mathrm{PR}$$

What if PR is either INFEAS/UNBOUND?

TH "WEAK DUALITY"

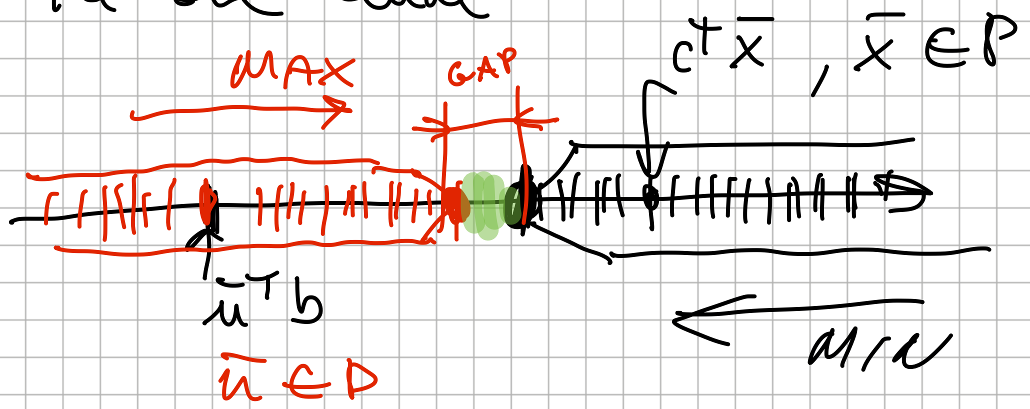
Let $P := \{\boxed{x \geq 0} : \boxed{Ax \geq b}\} \neq \emptyset$

$D := \{u \geq 0 : \boxed{c^T \geq u^T A}\} \neq \emptyset$

$\bar{x} \in P$, $\bar{u} \in D$ arbitrary points:

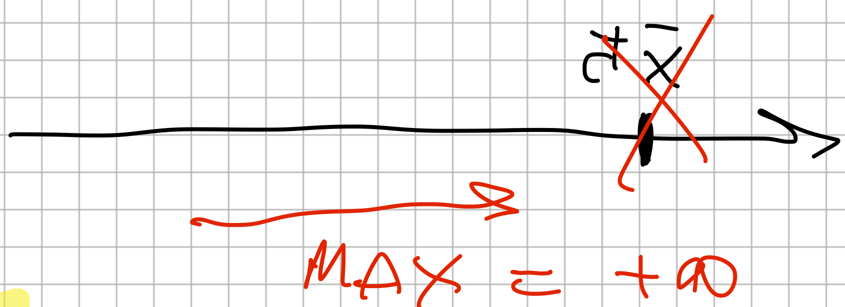
$$\underbrace{\bar{u}^T}_{\geq 0} \boxed{b} \;\boxed{\leq}\; \underbrace{\bar{u}^T}_{\geq 0} \boxed{A \bar{x}} \leq \boxed{c^T} \underbrace{\bar{x}}_{\geq 0}$$

obj. value of $\bar{u}$ in the dual



GAP is actually $= 0$ $\Leftarrow$ STRONG DUALITY!

PRIM. \ DUAL	∃ opt. sol	INFEAS.	UNBOUND.
∃ opt. sol.	min = max		
INFEAS.		?	YES
UNBOUND		YES	



? = yes!

Indeed:

$$\begin{cases} \min -4x_1 - 2x_2 \\ -x_1 + x_2 \geq 2 \\ x_1 - x_2 \geq 1 \\ x_1, x_2 \geq 0 \end{cases} \rightarrow \begin{cases} \max 2u_1 + u_2 \\ -u_1 + u_2 \leq -4 \\ u_1 - u_2 \leq -2 \\ u_1, u_2 \geq 0 \end{cases}$$

PRIMAL infeas. "$-x_1 + x_2 \geq 2$ & ≤ -1"

DUAL infeas. "$-u_1 + u_2 \leq -4$ & ≥ 2"

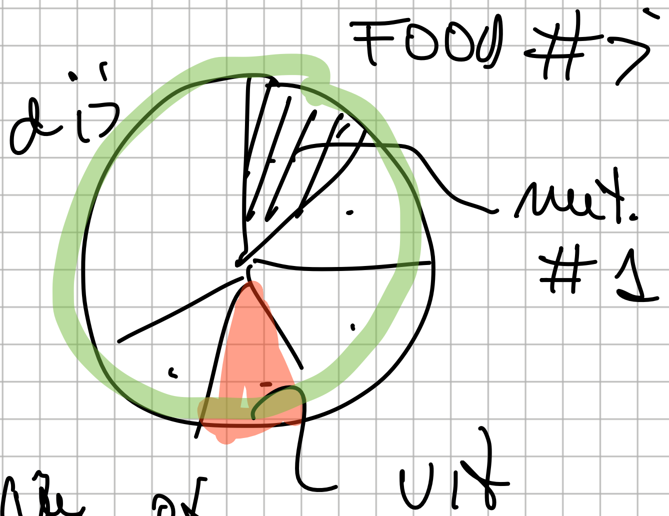
ECONOMICAL INTERP.

__Ex__: DIET problem

n foods

• c_j = unit cost of of the j-th food

m nutrients



b_i = min. quantity of NUTRIENT i in the diet

x_j = q.ty of food j in the diet

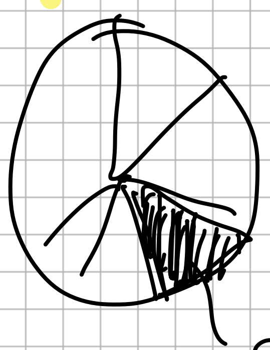
$$
\begin{cases}
\min \displaystyle\sum_{j=1}^{n} c_j x_j \\
\displaystyle\sum_{j=1}^{n} a_{ij} \cdot x_j \geq \boxed{b_i} & \forall i = 1, \dots, m \\
x_j \geq 0 & , \forall j = 1, \dots, n
\end{cases}
$$

$$\begin{cases} \min\ c^T x \\ Ax \ge b \\ x \ge 0 \end{cases}$$

$\boxed{\text{MYPROTEIN}}$

u_i = unit profit (for MYPROTEIN) for i-th nutrients

$$(*)\quad \begin{cases} \max \displaystyle\sum_{i=1}^{m} u_i \boxed{b_i} \\ \underbrace{\displaystyle\sum_{i=1}^{m} a_{ij} \boxed{u_i}}_{\text{synthetic cost of food } j} \le c_j, \quad \forall j = 1, \dots, n \\ u_i \ge 0, \quad \forall i = 1, \dots, m \end{cases}$$



$$\begin{cases} \max\ u^T b \\ c^T \ge u^T A \\ u \ge 0 \end{cases}$$

"STRONG DUALITY $\min = \max$"

OPTIMALITY CONDITIONS

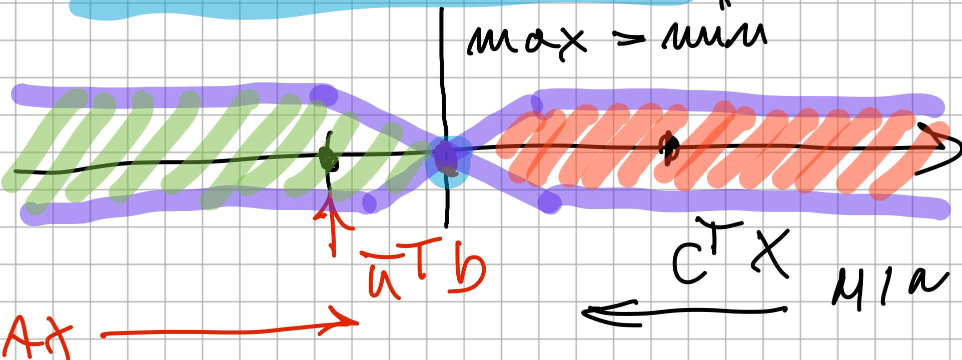
$$\begin{cases}\min\ c^T x\\ Ax \ge b\\ x \ge 0\end{cases}\qquad\qquad \begin{cases}\max\ u^T b\\ c^T \ge u^T A\\ u \ge 0\end{cases}$$

$\bar{x} \in \mathbb{R}^n$ $\qquad\qquad$ $\bar{u} \in \mathbb{R}^m$

(1) $A\bar{x} \ge b,\ \bar{x} \ge 0$ "PRIMAL FEAS"

(2) $c^T \ge \bar{u}^T A,\ \bar{u} \ge 0$ "DUAL FEAS"

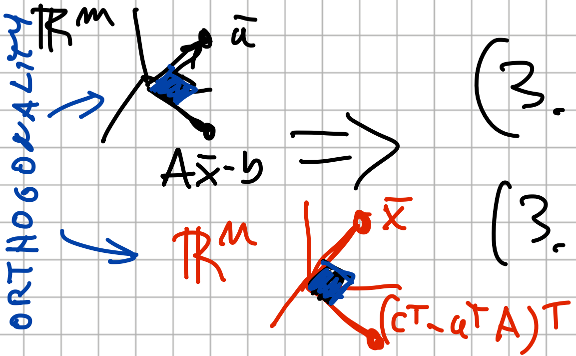
(3) $\boxed{c^T \bar{x} = \bar{u}^T b}$ "min = max"



Because (1) + (2) :

$$\underbrace{\bar{u}^T}_{\ge 0}\, b \ \overset{(3.\alpha)}{\le}\ \bar{u}^T \underbrace{A\bar{x}}_{\ge b} \ \overset{(3.\beta)}{\le}\ \underbrace{c^T}_{\ge u^T A}\ \underbrace{\bar{x}}_{\ge 0}$$

$=$ (3)



$$\Rightarrow \quad (3.\alpha) \quad \bar{u}^T \boxed{(A\bar{x}-b)} = 0$$

$$(3.\beta) \quad (c^T - \bar{u}^T A)\bar{x} = 0$$

$$(3.\alpha) \quad \sum_{i=1}^{m} \underbrace{\bar{u}_i}_{\geq 0} \underbrace{\boxed{(a_i^T \bar{x} - b_i)}}_{\geq 0} = 0$$

$$\boxed{A\bar{x} - b \geq 0}$$

$$\Rightarrow \bar{u}_i (a_i^T \bar{x} - b_i) = 0, \quad \forall i = 1, \dots, m$$

$$\bar{u}_i > 0 \Rightarrow (a_i^T \bar{x} - b_i) = 0$$

$$(3.\beta) \quad \sum_{j=1}^{n} \underbrace{(c_j - \bar{u}^T A_j)}_{\geq 0 \text{ because } c^T \geq \bar{u}^T A} \underbrace{\bar{x}_j}_{\geq 0} = 0$$

$$\Rightarrow \underbrace{(c_j - \bar{u}^T A_j)}\,\underbrace{\bar{x}_j} = 0, \quad \forall j = 1, \dots, n$$

"COMPLEMENTARITY COND.S"