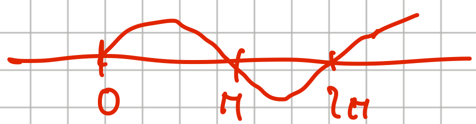
OR1 8-NOV-2021

Integer Linear Program (ILP):

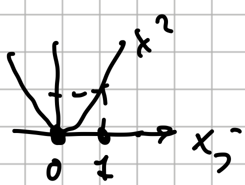
$$
\begin{cases}
z_{ILP} := \min \; c^T x \\
\quad \boxed{A x \ge b} \\
\quad x \ge 0 \quad \boxed{\text{"integer"}}
\end{cases}
$$

"x_j integer, $\forall j \in I$" MIXED-INTEGER LIN. PROG. (MILP)

x_j integer $\iff$ $\sin(\pi x_j) = 0$ $\rightarrow$ NON LINEAR (CONVEX)



"$\boxed{0 \le x \le 1}$ integer" $\iff$ $x_j \in \{0, 1\}$ BINARY

$\iff$ $x_j^2 = x_j$ ($x_j =$ )

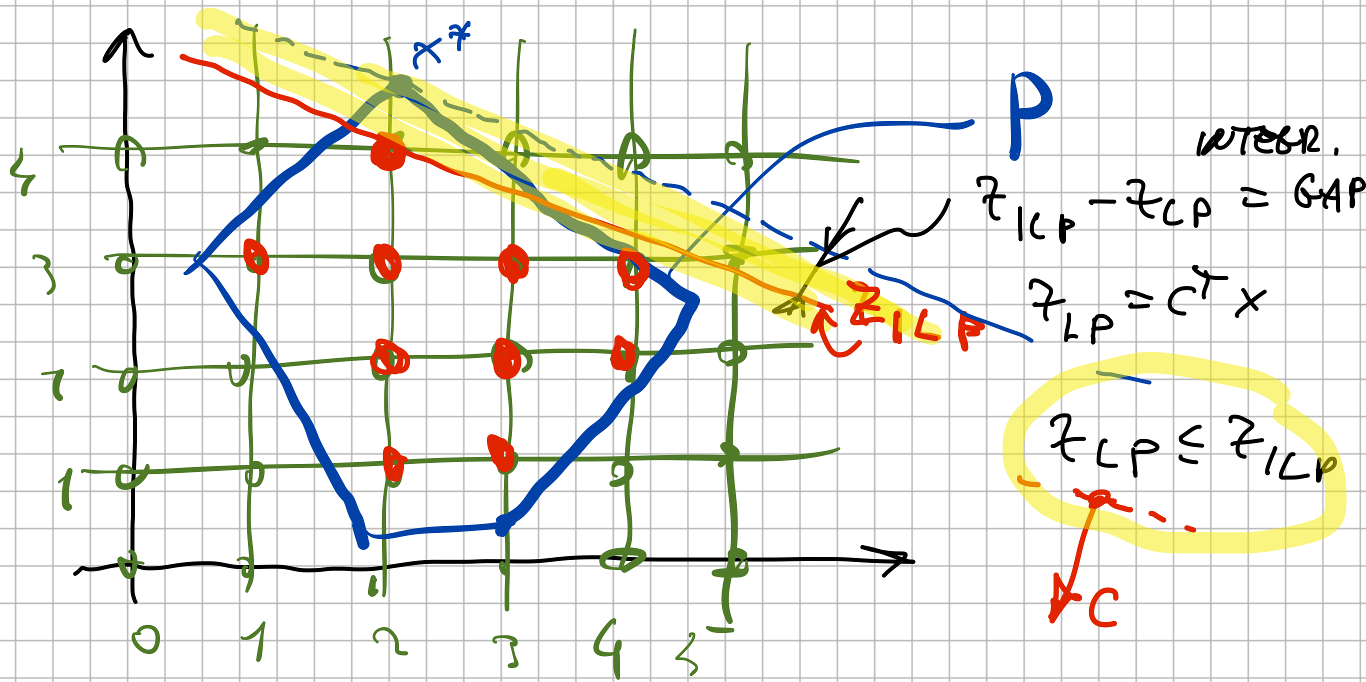
QUADRATIC NON-CONVEX

$P := \{ x \ge 0 : A x \ge b \}$ is BOUNDED - NONEMPTY

$X = P \cap \mathbb{Z}^n$ feasible sol. set

satisfy the linear conds.

where $\mathbb{Z} = \{ \ldots -2, -1, 0, +1, +2, \ldots \}$



LINEAR PROGRAMM. RELAXATION:

$$z_{LP} := \min \{ c^T x : \underbrace{Ax \geq b, \ x \geq 0}_{x \in P} \}$$

→ NAIVE SOL. APPROACH

→ Solve LP relaxation:

$$x^* := \arg\min_x \{ c^T x : x \in P \}$$

→ if (x^* integer) then "x^* is OPTIMAL for ILP as well"

Indeed

$$\boxed{c^T x^*} \underset{x^* \text{ is optimal for LP}}{=} \boxed{\overbrace{z_{LP}}^{\min\{c^T x : x \in P\}} \leq \overbrace{z_{ILP}}^{\min\{c^T x : x \in \overbrace{P \cap \mathbb{Z}^n}^{X}\}}} \leq \boxed{c^T x_*}$$

x^* is feasible for ILP

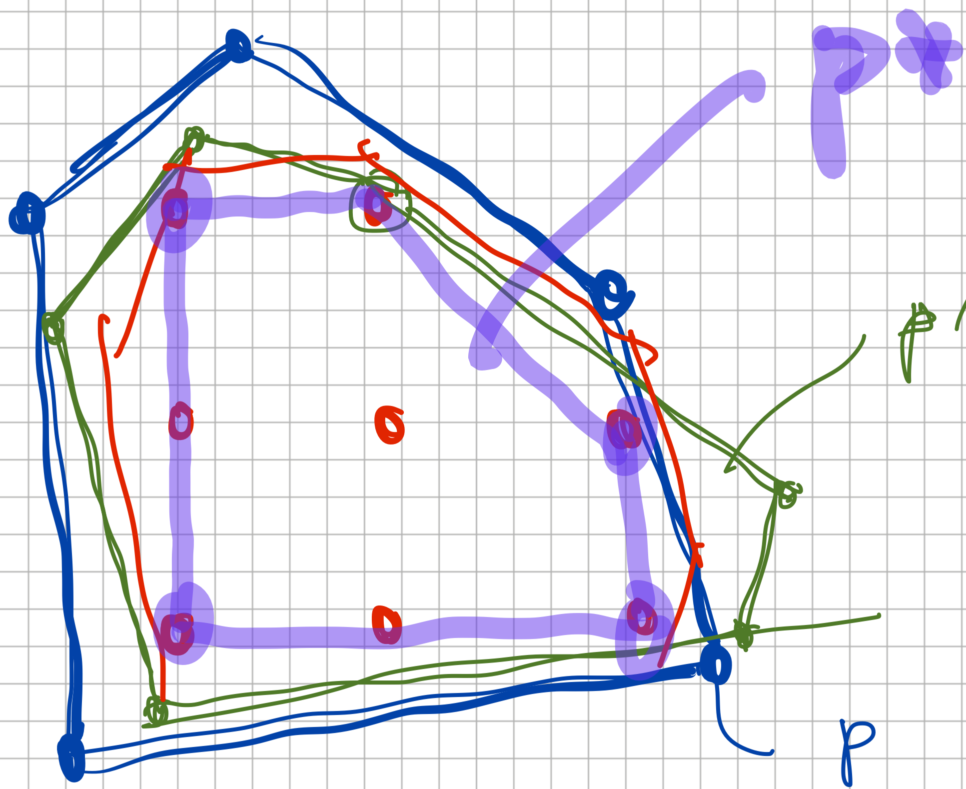
$z_{LP} \le z_{ILP}$ LOWER BOUND PROPERTY

But then $\le$ is in fact $=$

$$c^T x^* = z_{LP} = z_{ILP}$$

$\Longrightarrow$ x^* is OPTIMAL ILP

NOTE:

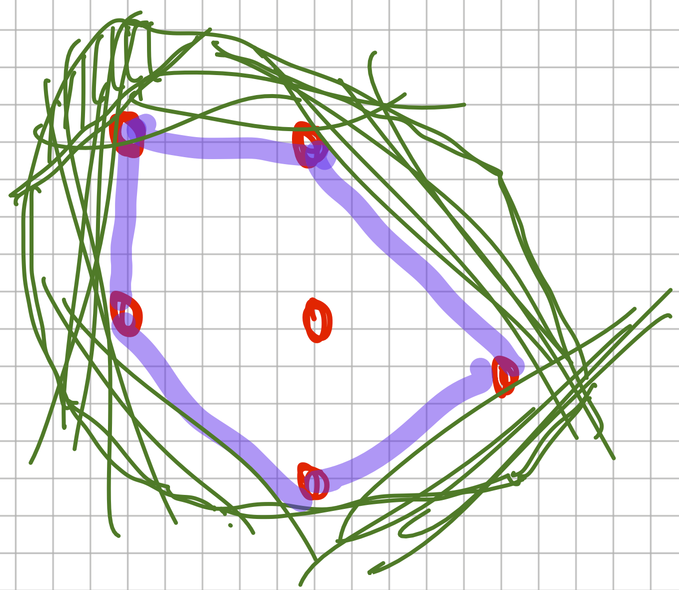


The FORMULATION ($Ax \ge b, x \ge 0$) is NOT UNIQUE !!

IDEAL formulation

" all the vertices of P^* are integer "

" P^* is called
CONVEX HULL of X "



$\operatorname{conv}(X)$ is the smallest
(w.r.t. set inclusion)
convex set that
contains X

$\Longrightarrow$ $\operatorname{conv}(X)$ is a polyhedron
namely P^* of
the ideal formulation

CASES IN WHICH ORIGINAL FORMULATION IS IDEAL

$$z_{LP} = \min \{ c^T x : Ax = b, x \geq 0 \}$$

$\Rightarrow$ optimal basis B

$\Rightarrow$ optimal FBS

$$x^* = \begin{bmatrix} x_B = B^{-1} b \\ x_F = 0 \end{bmatrix}$$

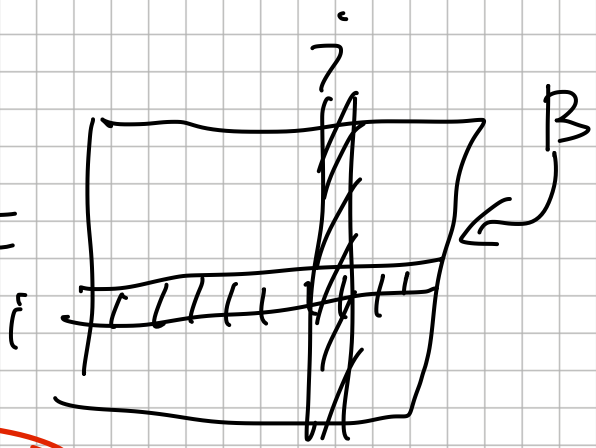
Assume (A, b) is integer

"Why x^* is fractional even if B and b are integer?"

$\Rightarrow$ B^{-1} can be fractional!

$$B^{-1} = \frac{1}{\det(B)} \begin{bmatrix} & \vdots & \\ \cdots & (-1)^{i+j}\det(M_{ij}) & \cdots \\ & \vdots & \end{bmatrix}^T$$

$(-1)^{i+j} \det(M_{ij})$

where $M_{ij} =$ 

IS INTEGER !!

$$\boxed{\det(B) = \pm 1} \Rightarrow$$

B^{-1} is integer $\Rightarrow$

$B^{-1}b$ is integer $\Rightarrow$ x^*

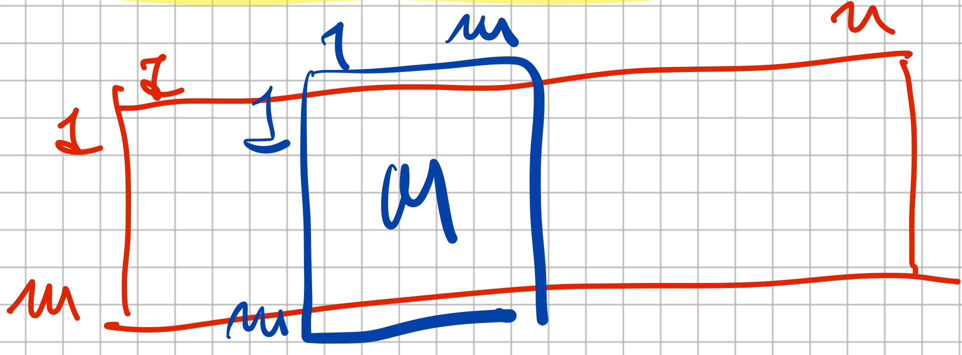
is integer

If $\det(B) \in \{+1, -1\}$ for all bases B then the formulation is IDEAL

UNIMODULARITY PROPERTY

An integer $m \times n$ matrix A is said to be UNIMODULAR $\iff$

$\det(M) \in \{-1, 0, +1\}$ for all $m \times m$ submatrices M of



$$\det(M) \in \{-1, 0, +1\}$$

WHY NOT JUST ROUNDING?