OR 1 9-NOV-2021

Hp: A, b integer

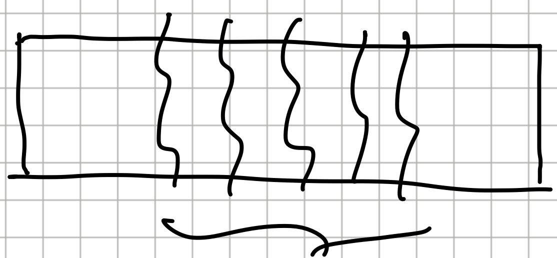
A is UM $\Rightarrow$ $\begin{cases} \min c^T x \\ Ax = b \\ x \geq 0 \quad \cancel{\text{integer}} \end{cases}$

IDEAL, i.e.,

$P = \{ x \geq 0 : Ax = b \}$ has ONLY integer vertices

A $m \times n$ is UM $\overset{def}{\Longleftrightarrow}$

A



M $m \times m$ $\det(M) \in \{-1, 0, +1\}$

$$\begin{cases} \min c^T x \\ Ax \geq b \\ x \geq 0 \text{ integer} \end{cases} \Rightarrow \quad Ax - Is = b \quad x, s \geq 0 \text{ integer}$$

CANONICAL — STANDARD FORM

$A' = [A, -I]$ is UM ?

$$A' = \begin{array}{c} 1 \\ \vdots \\ m \end{array}\left[\begin{array}{c|c} \underbrace{\begin{matrix} \times & \times & \times \\ \vdots & \vdots & \vdots \end{matrix}}_{A} & \underbrace{\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{matrix}}_{-I} \end{array}\right]$$

$$M = \left[\begin{array}{ccc|ccc} -1 & & 0 & \times & \times & \times \\ & -1 & & \times & \times & \times \\ 0 & & -1 & \times & \times & \times \\ \hline & 0 & & & \underbrace{Q}_{k} & \end{array}\right] \begin{array}{l} \Big\} h \\ \\ \\ \Big\} k = m - h \end{array}$$

[note: permuting rows/cols. does not change $|\det(M)|$]

$$\det(M) = \pm \det(Q)$$

$$\det(M) \in \{0, \pm 1\} \iff \det(Q) \in \{0, \pm 1\}$$

A' is UM $\iff$ A is <u>TOTALLY UNIMODULAR</u>

i.e.

$\det(Q) \in \{0, \pm 1\}$ for every square submatrix Q of A of size k, $k = 1, 2, \dots, m$

If b is integer,
A is TUM, then

$$\begin{cases} \min\ c^T x \\ Ax \geq b \\ x \geq 0 \text{ integer} \end{cases}$$

is IDEAL, i.e., all the vertices
of

$$P' = \{x \geq 0 : Ax \geq b\}$$

are integer

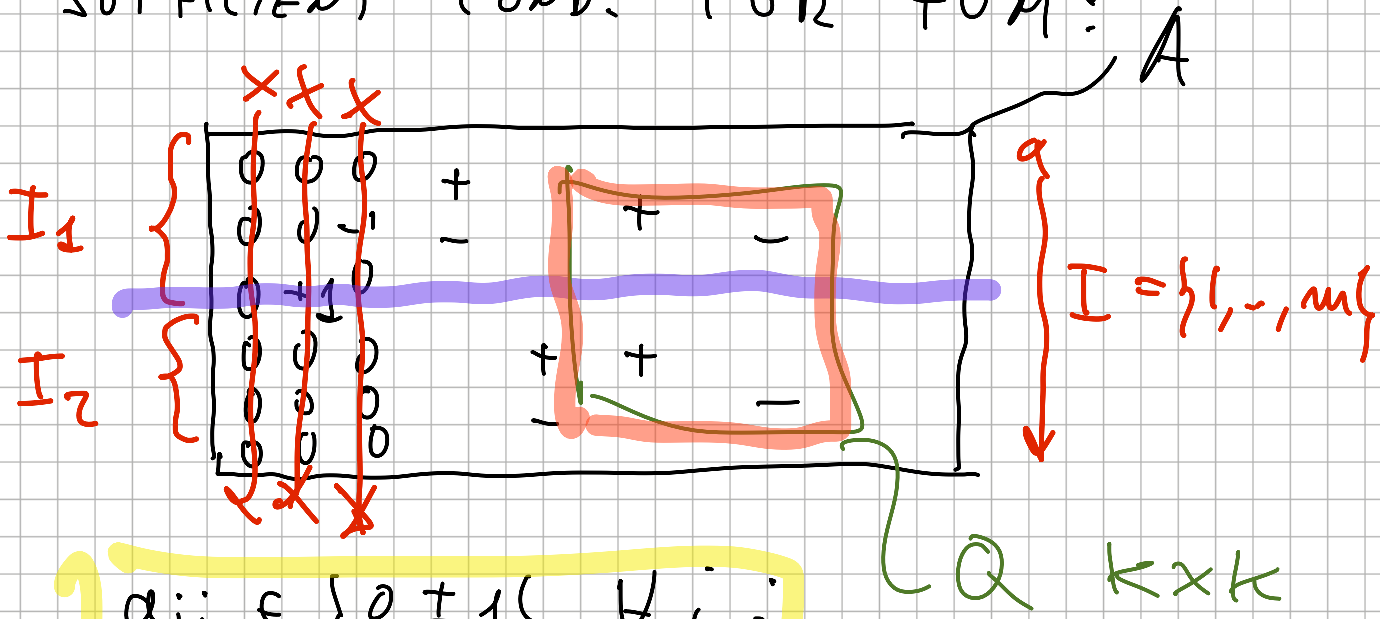
- $\boxed{a_{ij} \in \{0, \pm 1\} \ \forall i,j}$ NECESSARY COND. for TUM

not SUFFICIENT: e.g.

$$Q = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\det(Q) = 2 \Rightarrow$$ MATRIX is NOT TUM

SUFFICIENT COND. FOR TUM:



$a_{ij} \in \{0, \pm 1\} \quad \forall i, j$

NO MORE THAN 2 NONZEROS PER COLUMN

$\exists\ (I_1, I)$ of I s.t. ....

$\Rightarrow$ A is TUM

Proof: By induction on k:

- Base: $k = 1$

$$Q = [a_{ij}] \rightarrow \det(Q) = a_{ij} \in \{0, \pm 1\}$$

by assumption

- $\boxed{\text{Induction step:}}$

property holds for k'

$\Rightarrow$ it also holds for $k = k'+1$

Take any Q $k \times k$.

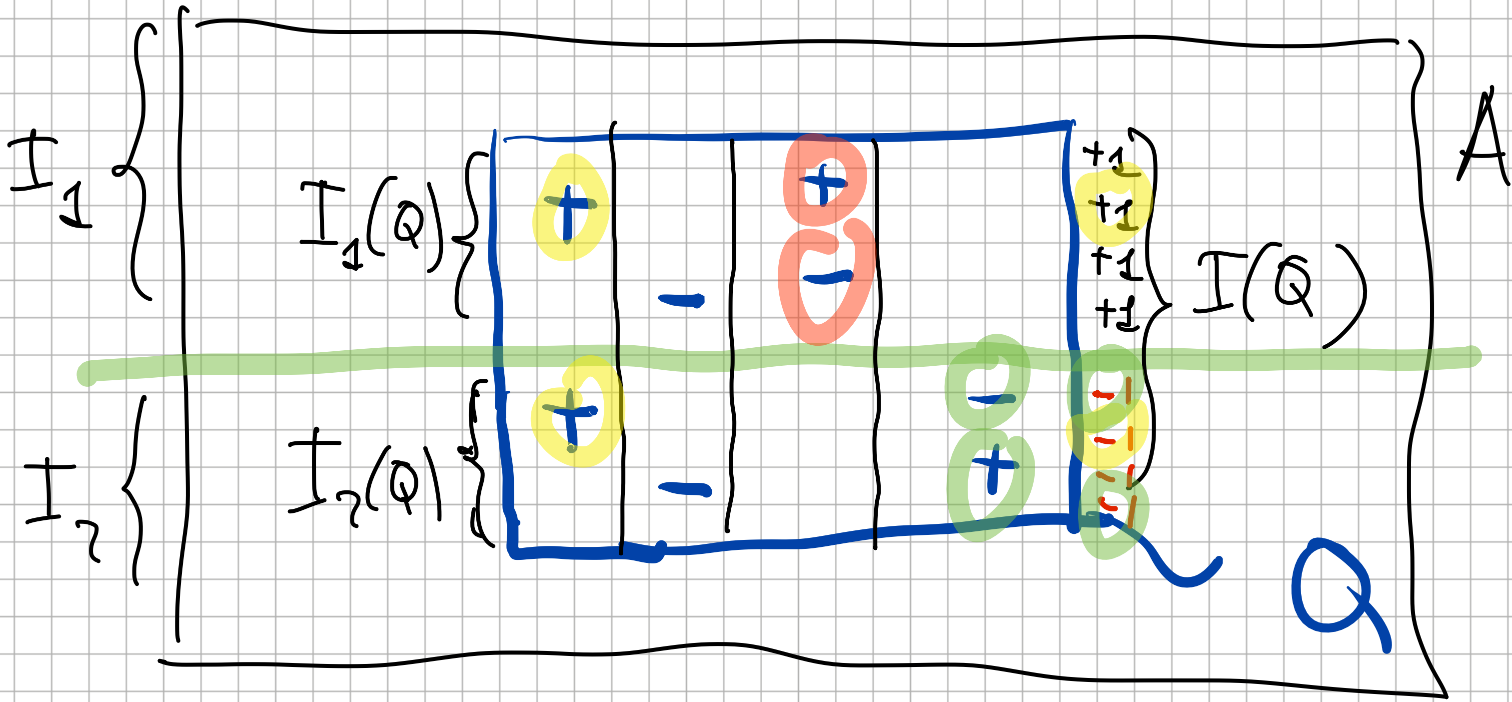
THREE cases can occur:

① $$Q = \begin{bmatrix} 0 & \\ 0 & \\ 0 & \\ 0 & \\ 0 & \end{bmatrix} \Rightarrow \det(Q) = 0 \quad \text{ok}$$

② $$Q = \begin{array}{c} \begin{bmatrix} \pm 1 & \text{x x x x} \\ 0 & \\ 0 & \\ 0 & Q' \\ 0 & \\ 0 & \end{bmatrix} \begin{array}{l} 1 \\ \\ \\ \Big\} \, k' = k-1 \\ \\ \\ \end{array} \\ \begin{array}{cc} 1 & \underbrace{\qquad\qquad}_{k'} \end{array} \end{array}$$

$$\det(Q) = \pm \underbrace{\det(Q')}_{\in \{0, \pm 1\} \text{ by inductive hyp.}} \in \{0, \pm 1\} \quad \text{ok}$$

③ ALL columns of Q have 2 nonzeros.



$$I_1(Q) := I(Q) \cap I_1$$

$$I_2(Q) := I(Q) \cap I_2$$

$$\sum_{i \in I_1(Q)} [\text{row } i \text{ of } Q] - \sum_{i \in I_2(Q)} [\text{row } i \text{ of } Q]$$

$$= [\text{row of zeros}]$$

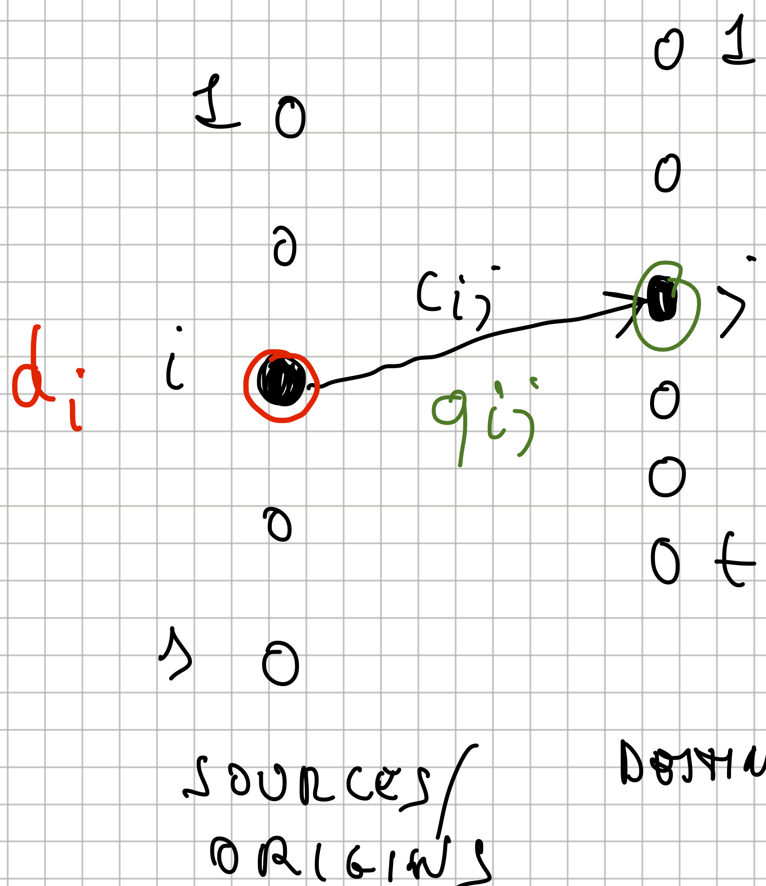
$\Rightarrow$ rows of Q are LIN. DEPENDENT

$\Leftrightarrow$ $\det(Q) = 0$ OK

Simple observations:

- A is TUM $\Longleftrightarrow$ A^T is TUM
- A is TUM $\Longleftrightarrow$ A' is TUM, where A' is obtained from A by permuting rows and/or col.s
- A is TUM $\Longleftrightarrow$ $\left[\begin{array}{c|c} 0 & \\ 0 & \\ 0 & A \\ 0 & \\ 0 & \end{array}\right]$ is TUM $\left[\begin{array}{c|c} \pm 1 & \\ 0 & \\ 0 & A \\ 0 & \end{array}\right]$ is TUM

THE TRANSPORTATION PROBLEM

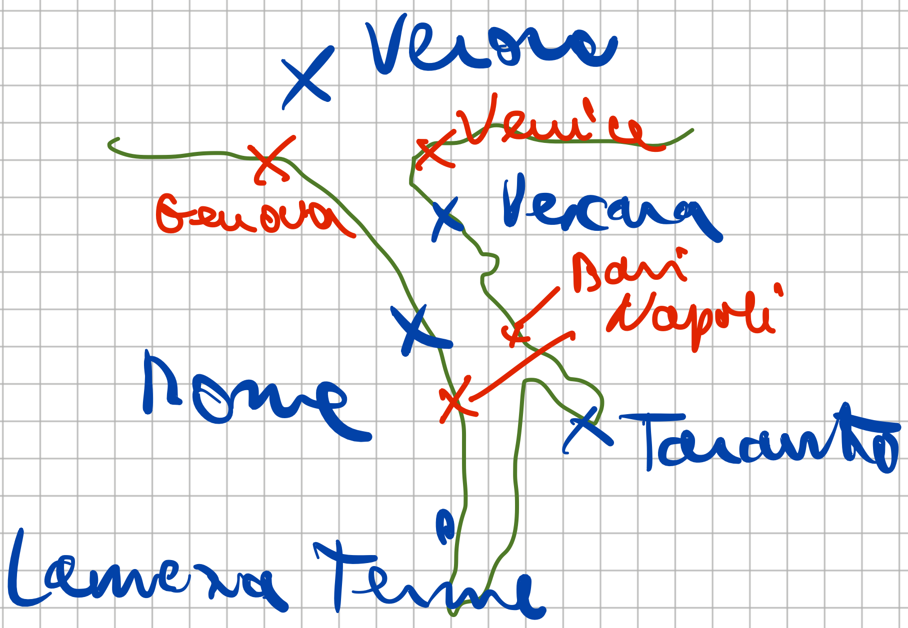


d_i = availability of the i-th origin

r_j = request of the j-th destination

q_{ij} = max allowed on $i \to j$

Es:



Verona ×		o	Genova
Pescara ×		o	Venezia
i ×	$\xrightarrow{c_{ij}}$	o j	⋮
×		o	
L. Terme ×		o	Bari
ORIGINS		DESTINATIONS	

	d_i
VERONA	10
PERUGIA	12
⋮	⋮
L. TERME	40

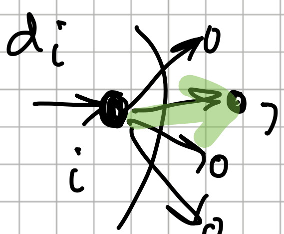
q_j	
20	GENOVA
15	VENEZIA
⋮	
21	BARI

Decision var. 1

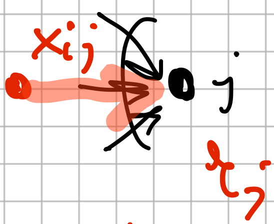
x_{ij} = q.ty moved from $i \to j$

$\forall \; i = 1, \dots, s,$
$j = 1, \dots, t$

$$\min \sum_{i=1}^{s} \sum_{j=1}^{t} c_{ij} \cdot x_{ij}$$



$$\boxed{-\sum_{j=1}^{t} x_{ij} \geq -d_i \;, \quad \forall i = 1, \dots, s}$$

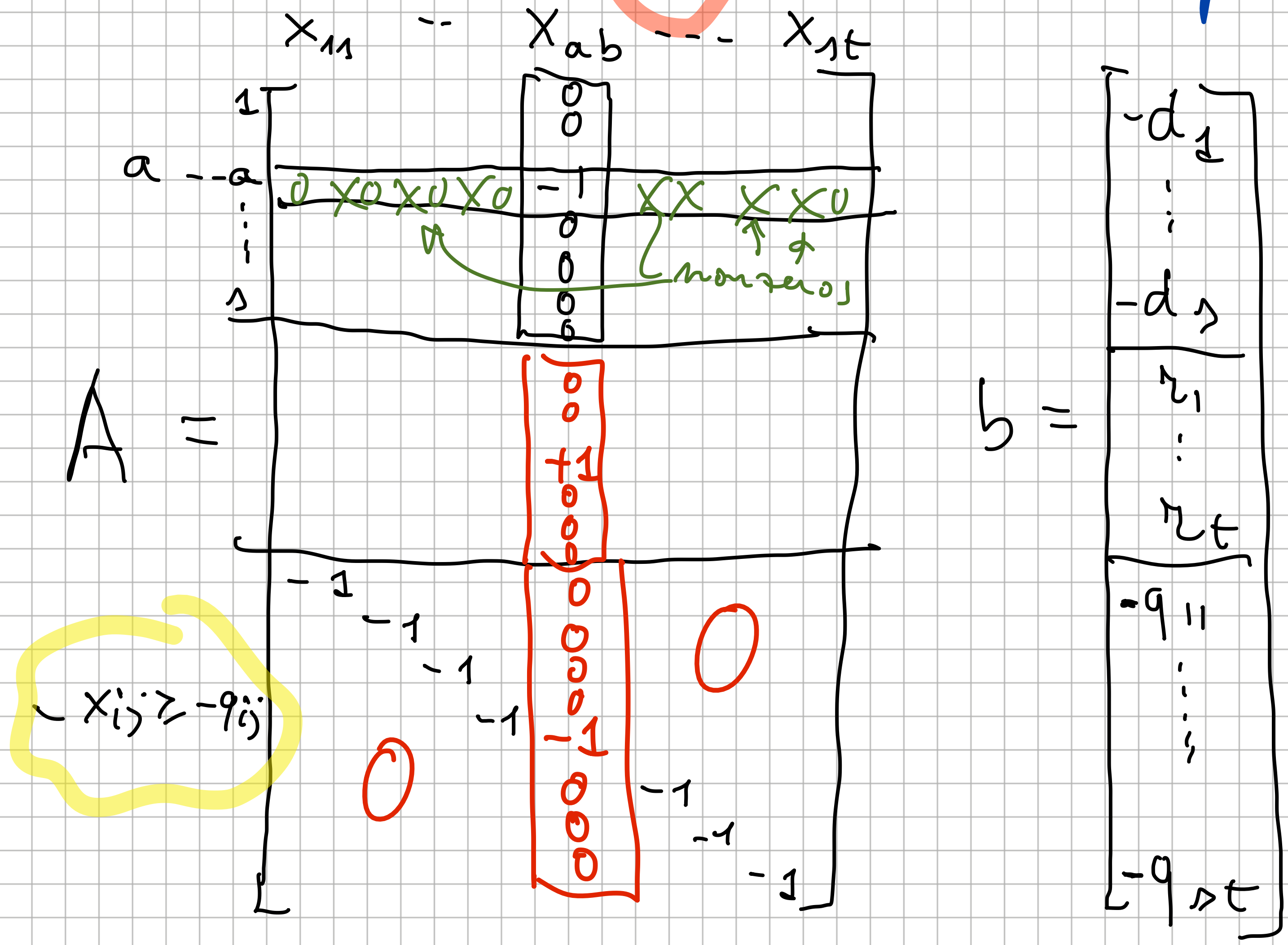


$$\sum_{i=1}^{s} x_{ij} \geq r_j \quad \forall j = 1, \dots, t$$

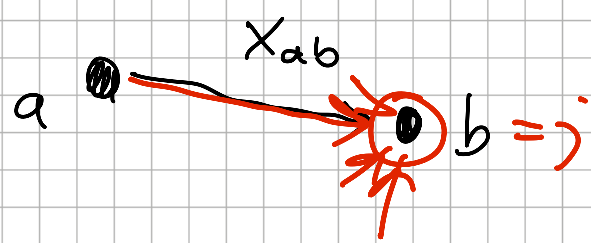
$$0 \leq x_{ij} \leq q_{ij} \;, \quad \forall i, j$$

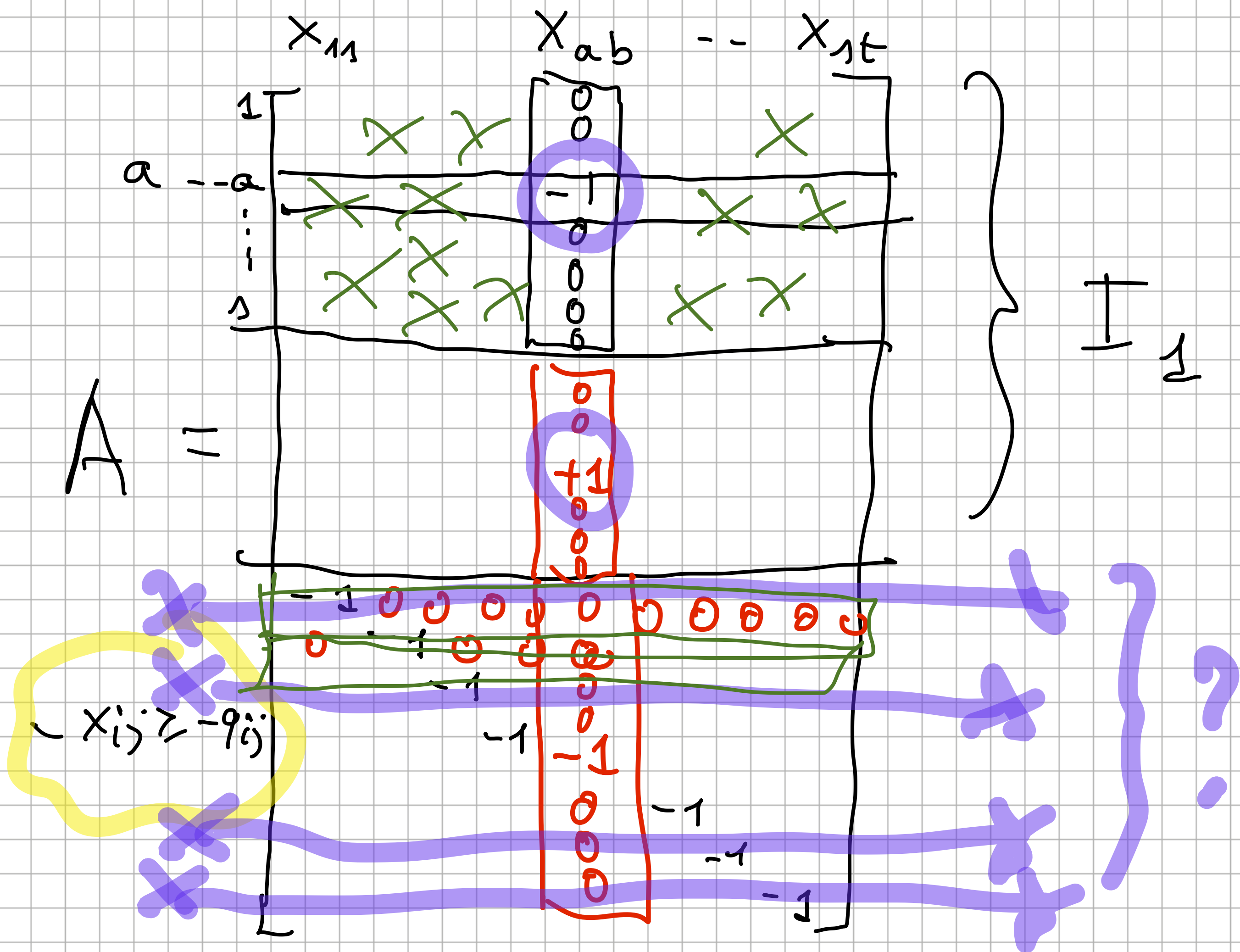
x_{ij} ~~INTEGER~~

$$\min\{c^T x : A x \geq b,\ x \geq 0\}$$



x_{ab}





I_1 = all rows

$I_2 = \emptyset$

$\Rightarrow$ SUFF. COND. APPLIES

$\Rightarrow$ A is TUM

$\Rightarrow$ FORMULATION IS IDEAL (assuming d_i, r_i, q_{ij} integer) □