OR 1 10-NOV-2021

Exercise 9–11: Prove the total unimodularity of the following matrix:

$$
\begin{array}{c}
 \\ \boxed{\times} \\ \times \\ \boxed{o} \\ \\ o \\ o \\ \\ \times \\ \\
\end{array}
\begin{array}{c}
\begin{array}{cccccccc} \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & & \checkmark & \checkmark \end{array} \\
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 1 & -1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix} \\
\begin{array}{cccccccc} & & & & & & \text{ok} & \text{ok} \end{array}
\end{array}
$$

$\times \in I_1$

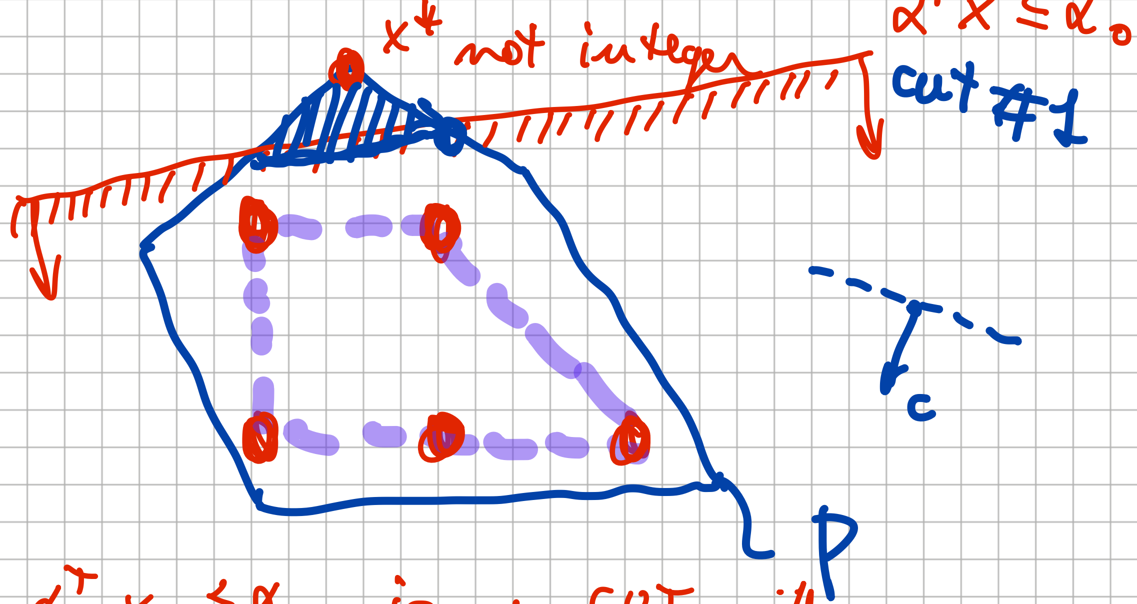
$o \in I_2$

$\min \{ c^T x : Ax = b, \ x \geq 0 \}$

$\rightarrow$ FBS (optimal) x^*

but x^* is not integer ?

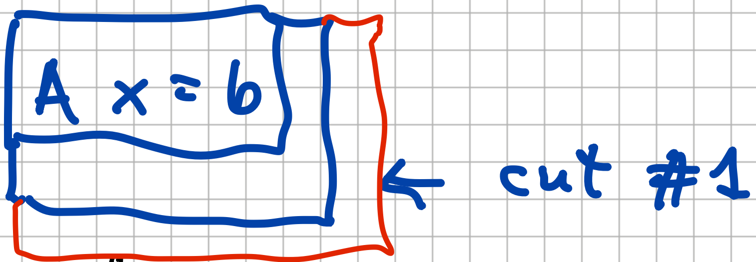
THE CUTTING PLANE METHOD

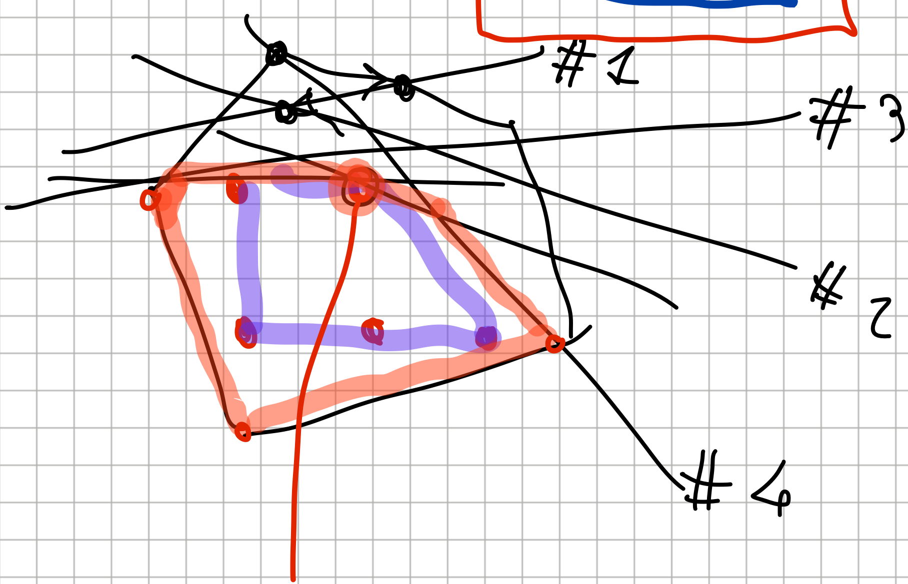


$\alpha^T x \le \alpha_0$ is a CUT iff:

1) $\alpha^T x \le \alpha_0 \quad \forall\, x \in X$ "valid for X"

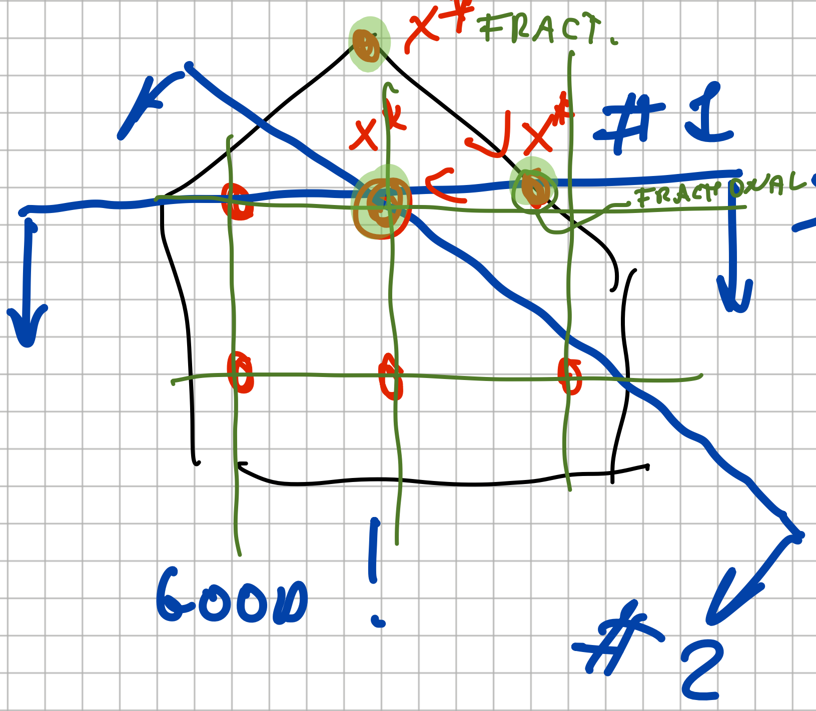
2) $\alpha^T x^* \underset{\neq}{\geq} \alpha_0$ "violated by x^*"

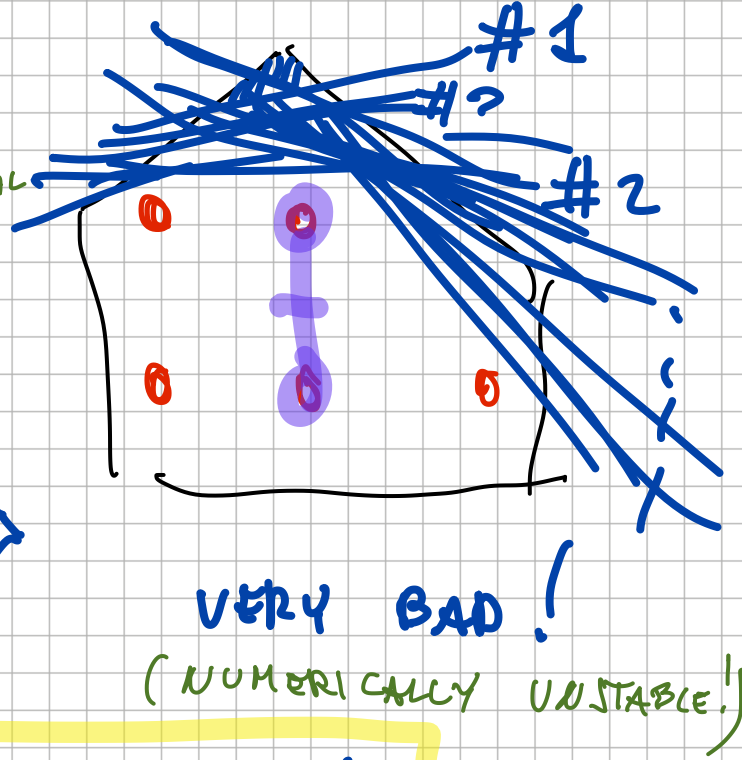




x^* integer $\Rightarrow$ STOP!

SIMPLEX DUAL 426.



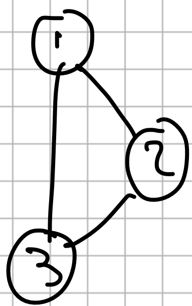


DEEP CUTS are needed!

$\boxed{\text{WHEN A GIVEN } \alpha^T x \le \alpha_0 \text{ IS VALID FOR } X \text{ ?!}}$

PARTIAL ANSWER (V. CHVÁTAL, ~1970)

EX:

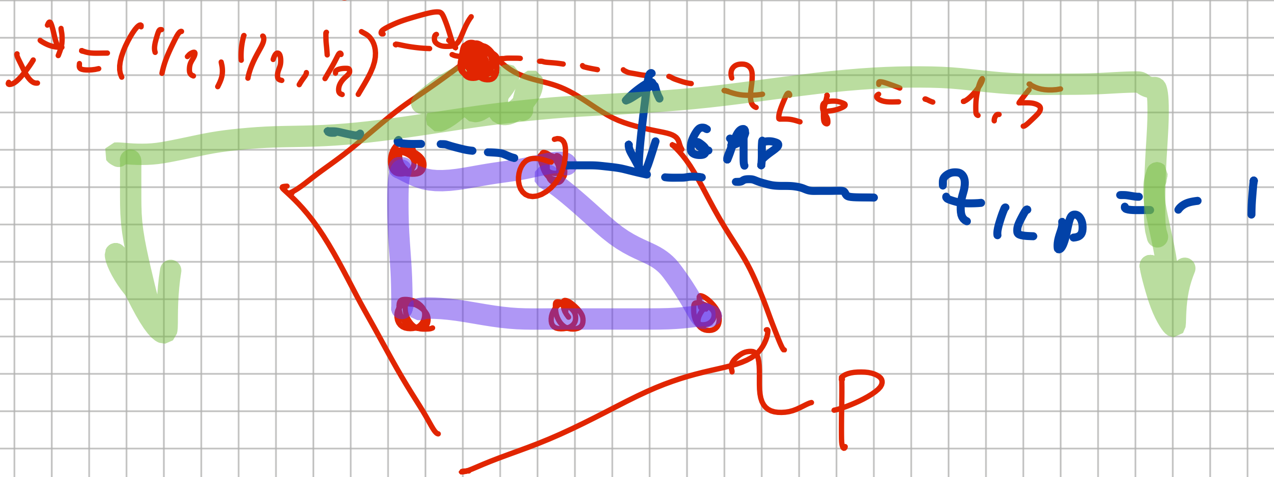


$$\begin{cases} \min \underbrace{-x_1 - x_2 - x_3}_{-(\#\text{ of loaded items})} \\ x_1 + x_2 \le 1 \quad \checkmark \\ x_2 + x_3 \le 1 \quad \checkmark \\ x_1 + x_3 \le 1 \quad \checkmark \\ x_1, x_2, x_3 \ge 0 \quad \cancel{\text{INTEGER}} \end{cases}$$

optimal LP sol $x^* = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$

$-z^* = -\frac{3}{2} = 1{,}5$

$z_{LP} = -1{,}5 \leq z_{ILP} = -1$



Standard form:

$$\begin{aligned}
1/2 * [\; x_1 + x_2 \quad\quad + x_4 \quad\quad &= 1 \;] \\
1/2 * [\; \quad x_2 + x_3 \quad\quad + x_5 \quad &= 1 \;] \\
1/2 * [\; x_1 \quad + x_3 \quad\quad\quad + x_6 &= 1 \;]
\end{aligned}$$

(all x's are integer)

$x_1 + x_2 + x_3 + \frac{1}{2} x_4 + \frac{1}{2} x_5 + 1/2\, x_6 \cancel{=} 3/2$

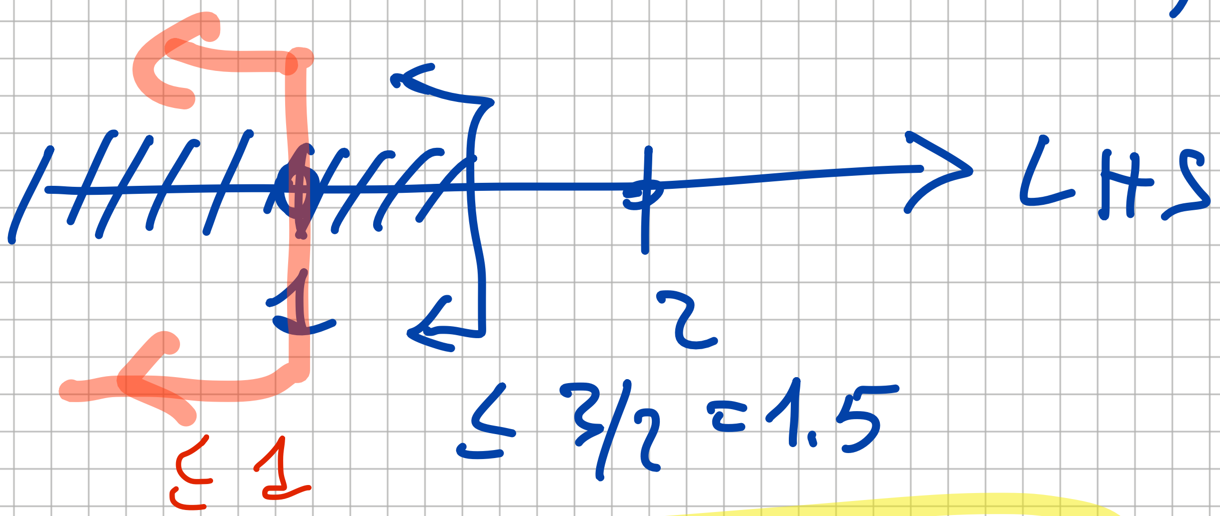
$\leq$

* ineq. valid for P

$$x_1 + x_2 + x_3 + 0\,x_4 + 0\,x_5 + 0\,x_6 \le 3/2 = 1.5$$

* ineq. valid for P

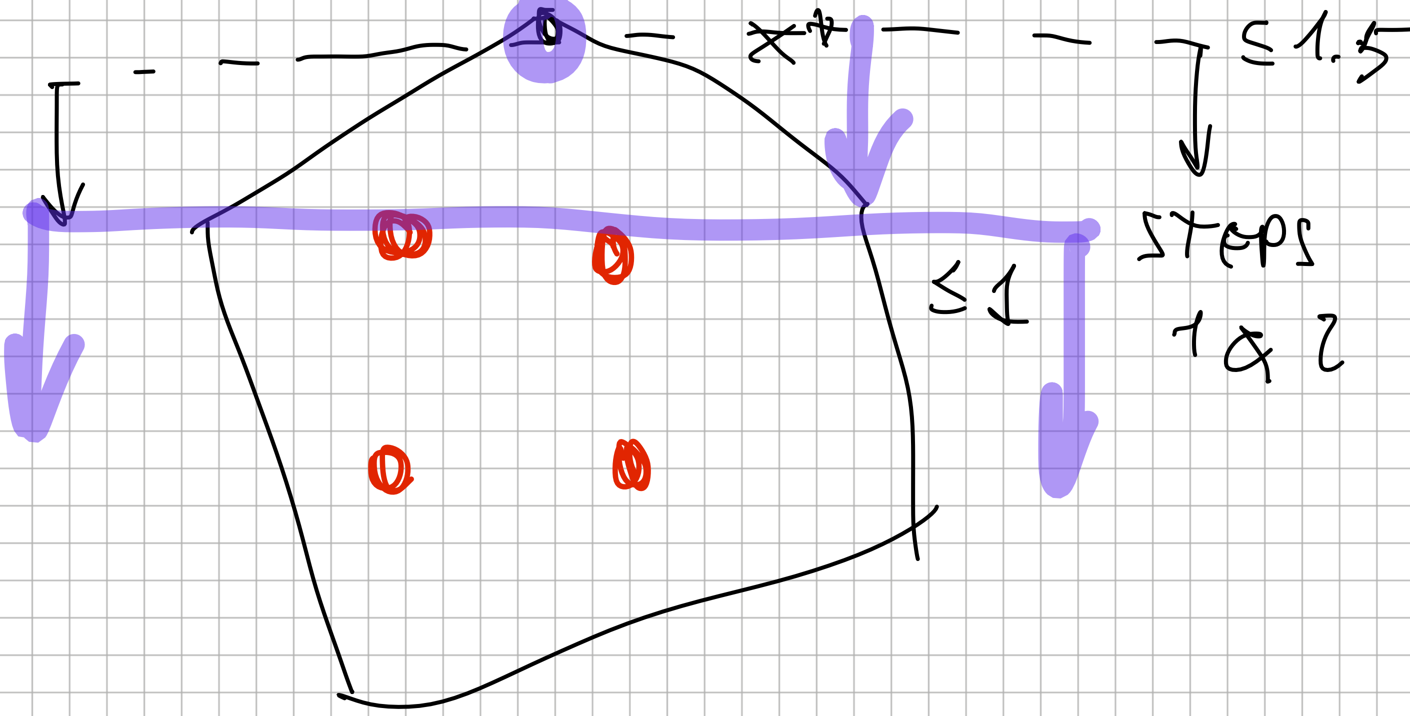
LHS must be integer (all coeff.s are integer)



$$x_1 + x_2 + x_3 \le \cancel{1.5}\ 1$$

STRONGER CONDITION!

ineq. valid for X but not necessarily for P



$x_1 + x_2 + x_3 \leq 1$ is a cut?

① valid for X by construction

② violated by $x^* = (1/2, 1/2, 1/2)$

$$(x_1 + x_2 + x_3)\Big|_{x = x^*} = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 3/2 \not\leq 1$$

$\Longrightarrow$ YES, it is a cut!

$\boxed{\text{GENERAL PROCEDURE:}}$

- Start with $Ax = b$
  $x \geq 0$ integer

1) Choose $u \in \mathbb{R}^m$:

$$\boxed{u^T A x = u^T b}$$

2) Replace

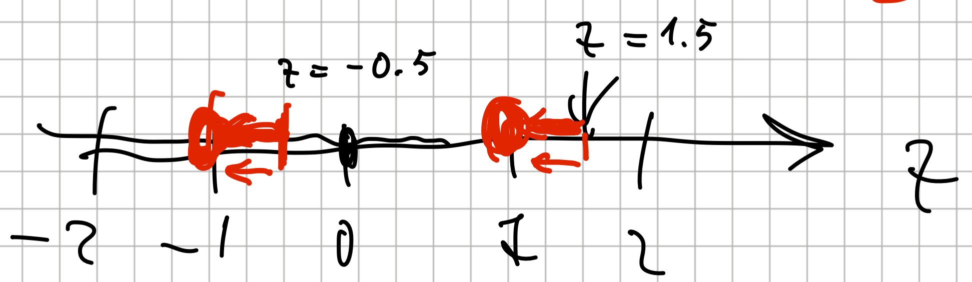
$$\lfloor u^T A \rfloor x \le u^T b$$

where $\lfloor \cdot \rfloor$ represents the "integer part of"

$\lfloor 1.5 \rfloor = 1$ $\qquad$ $\lfloor -0.5 \rfloor = -1$

$\lfloor 1.9 \rfloor = 1$ $\qquad$ $\lfloor -1.1 \rfloor = -2$

$\lfloor 2.0 \rfloor = 2$ $\qquad$ $\lfloor -3 \rfloor = -3$



"decrease z until it becomes integer"

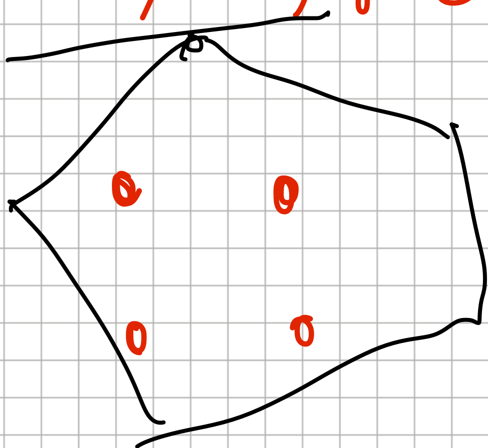
3) Decrease the RHS by exploiting integrality:

$$\lfloor u^T A \rfloor x \leq \lfloor u^T b \rfloor$$

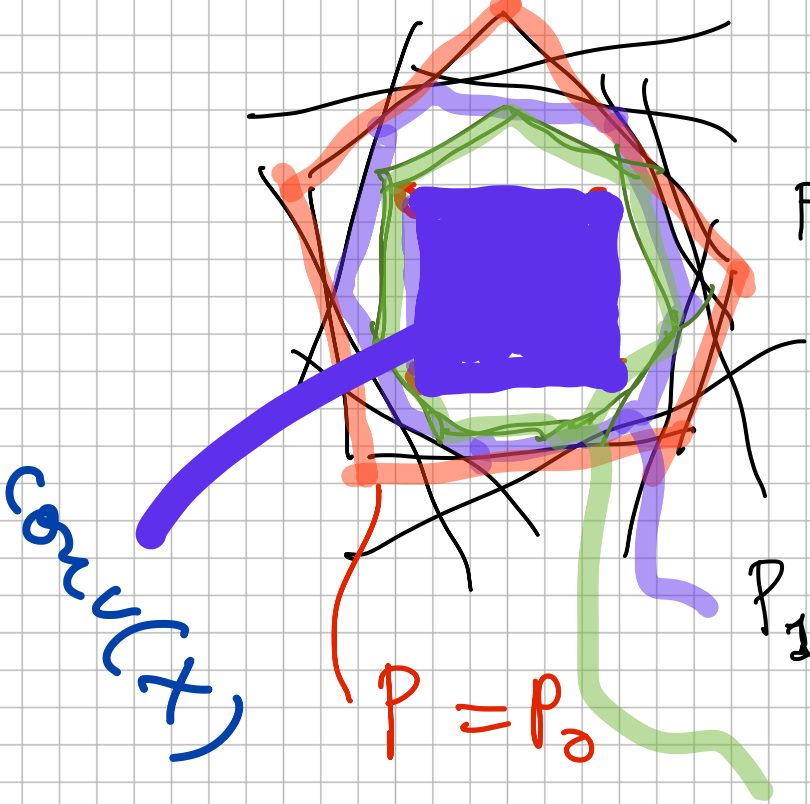
"CHVATAL CUT"

Produces ∞^m valid ineq.s for $\bar{X}$

$\Rightarrow$ only a FINITE n. of CUTS are "undominated" i.e. really useful



HIERARCHY OF RELAXATIONS P:



$P_1 \equiv \begin{cases} Ax = b, \; x \geq 0 \\ \text{all possible UNDOMINATED CHV. cuts} \end{cases}$

P_1 = 1st CHVATAL closure of P

P_2 = 2nd CHV. closure of P

$$P = P_0 \supsetneq P_1 \supsetneq P_2 \; \dots$$

$$\supseteq P_{(k)} \equiv P^* \text{ ideal } \boxed{= \mathrm{conv}(X)}$$

finite chvátal rank w.r.t. P

$\boxed{\text{In practice:}}$

Given x^* (optimal vertex of the current formulation) find $u \in \mathbb{R}^m$:

$$\lfloor u^T A \rfloor x \leq \lfloor u^T b \rfloor$$

is violated by x^*

SEPARATION ALGORITHM:

$x^* \to \boxed{u\,?} \to \lfloor u^T A \rfloor x \leq \lfloor u^T b \rfloor$ violated by x^*