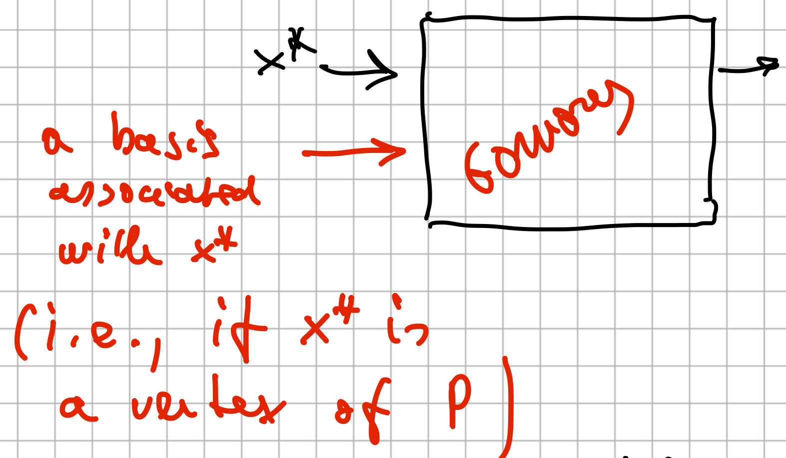
OR1 15-NOV-2021

GOMORY'S CUTS ~ 1958

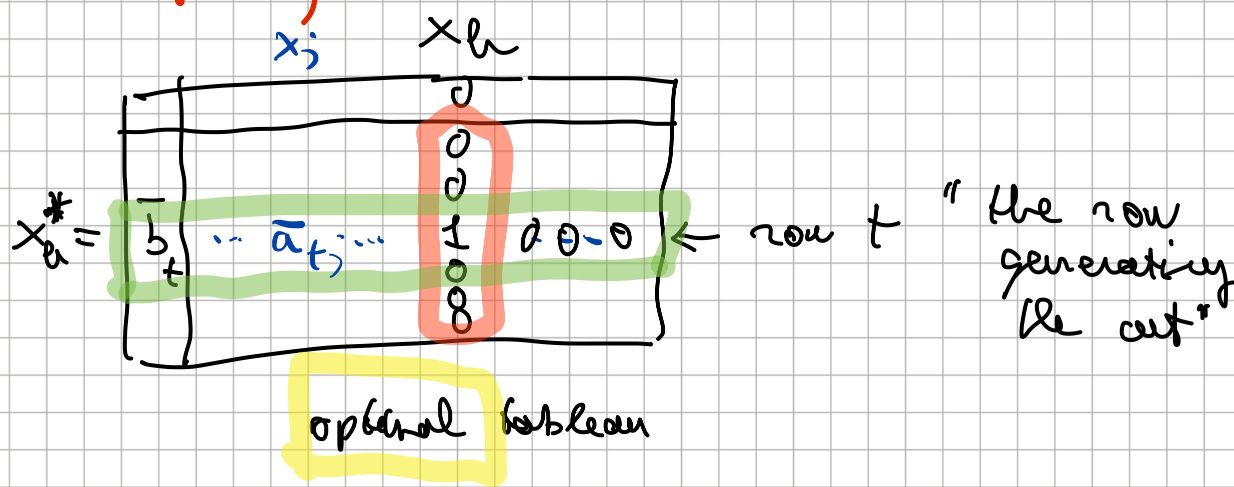
Separation for Chvátal cuts:



$u \in \mathbb{R}^m$:

$\lfloor u^T A \rfloor x \le \lfloor u^T b \rfloor$

is violated by x^* (if any)



x^*_h which is fractional $\Rightarrow$ x_h is a basic variable

row t s.t. $\bar{b}_h = x^*_h$ is fractional!

$$\boxed{\underbrace{1\, x_h}_{\text{only basic variable in row } t} + \sum_{j \in R}{}' \bar{a}_{tj} x_j = \bar{b}_t = x^*_h}$$

index set of the nonbasic var.s

Chvátal cut:

$$1\, x_h + \sum_{j \in R} \lfloor \bar{a}_{tj} \rfloor x_j \le \boxed{\lfloor \bar{b}_t \rfloor} \qquad (*)$$

row t

(note: row t of the optimal tableau $\equiv$

$$u^t a_t = u^t b_t$$

where u = t-th row of B^{-1})

(*) is valid for the integer sol.s (by construction) and violated by x^*:

$$\underbrace{\left(x_h + \sum_{j \in F} \lfloor \bar{a}_{tj} \rfloor x_j\right)}_{x = x^*} \le \lfloor \bar{b}_t \rfloor$$

$x^*_j = 0 \ \forall j \in F$

$$x^*_h \not\le \lfloor x^*_h \rfloor$$

(*) is called (CHVÁTAL-)GOMORY cut

$$(**) \quad x_h + {\sum_{j \in \mathcal{N}}}' \lfloor \bar{a}_{tj} \rfloor x_j + \sigma = \lfloor \bar{b}_t \rfloor$$

where $\sigma \geq 0$ is a slack variable

a <u>integer</u> variable

$$(***) \quad x_h + \sum_{j \in \mathcal{N}} \bar{a}_{tj} x_j = \bar{b}_t$$

$(**) - (***)$ produces:

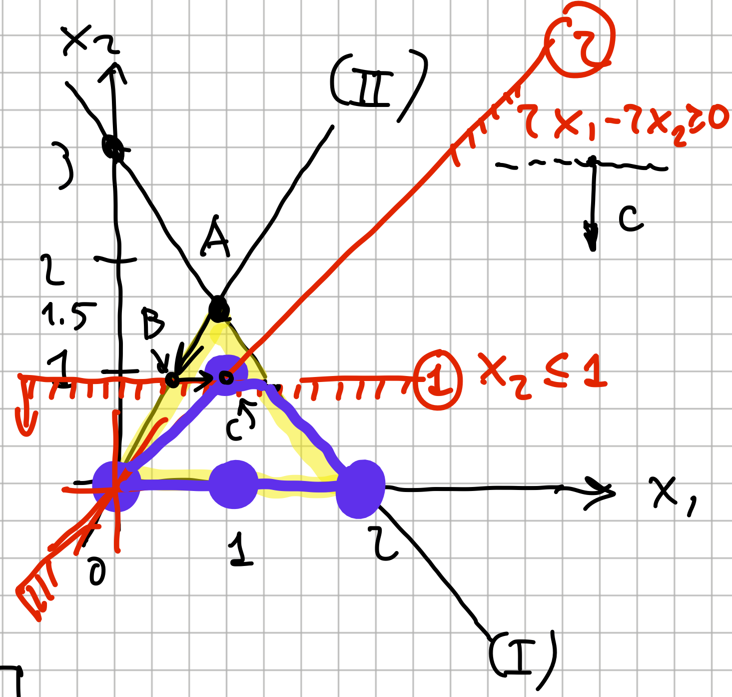
$$+ \sum_{j \in \mathcal{N}} -(\underbrace{\bar{a}_{tj} - \lfloor \bar{a}_{tj} \rfloor}_{\varphi(\bar{a}_{tj})}) x_j + \sigma = -(\underbrace{\bar{b}_t - \lfloor \bar{b}_t \rfloor}_{\varphi(\bar{b}_t)})$$

where $\varphi(z) = z - \lfloor z \rfloor \geq 0$

FRACTIONAL FORM OF THE GOMORY CUT

Ex:

$$\begin{cases} \min & -x_2 \\ & 3x_1 + 2x_2 \le 6 \quad (I) \\ & -3x_1 + 2x_2 \le 0 \quad (II) \\ & x_1, x_2 \ge 0 \\ & \text{integer} \end{cases}$$



		x_1	x_2	x_3	x_4	
$-z =$	0	0	-1	0	0	
$x_3 =$	6	3	2	1	0	$\leftarrow x_3 = 6 - 3x_1 - 2x_2$
$x_4 =$	0	-3	2	0	1	$\leftarrow x_4 = 0 + 3x_1 - 2x_2$

		x_1	x_2	x_3	x_4
$-z =$	0	$-3/2$	0	0	$1/2$
$x_3 =$	6	6	0	1	-1
$x_2 =$	0	$-3/2$	1	0	$1/2$

$x^* = (1, 3/2)$ vertex A

x_5 is integer var.

		x_1	x_2	x_3	x_4	x_5	
$-z =$	$3/2$	0	0	$1/4$	$1/4$	0	
$x_1 =$	1	1	0	$1/6$	$-1/6$	0	
$x_2 =$	$3/2$	0	1	$1/4$	$1/4$	0	$\leftarrow$ row t
$x_5 =$	$-1/2$	0	0	$-1/4$	$-1/4$	1	$\leftarrow x_5 = -1/2 + \frac{1}{4}x_3 + \frac{1}{4}x_4$

$$= -\frac{1}{2} + \frac{1}{4}\underbrace{(6 - 3x_1 - 2x_2)}_{x_3} + \frac{1}{4}\underbrace{(3x_1 - 2x_2)}_{x_4}$$

$$= \cdots = 1 - x_2$$

Gomory cut : $x_5 \geq 0 \rightarrow 1 - x_2 \geq 0 \rightarrow \boxed{x_2 \leq 1}$

DUAL simplex :

$x^* = (2/3, 1)$

vertex B

		x_1	x_2	x_3	x_4	x_5	x_6
$-z =$	1	0	0	0	0	1	0
$x_1 =$	2/3	1	0	0	-1/3	2/3	0
$x_2 =$	1	0	1	0	0	1	0
$x_3 =$	2	0	0	1	1	-4	0
$x_6 =$	-2/3	0	0	0	-2/3	-2/3	1

row t

$\varphi(-1/3) = 2/3$

$$\varphi(z) := z - \lfloor z \rfloor$$

$$\varphi(1.9) = 1.9 - 1 = 0.9$$

$$\varphi(-1.1) = -1.1 - (-2) = 0.9$$

$$\text{"} \varphi(-|z|) = 1 - \varphi(|z|) \text{"}$$

$$x_6 = -2/3 + \frac{2}{3}\underbrace{(\ \cdots\ )}_{x_4} + \frac{2}{3}\underbrace{(\ \cdots\ )}_{x_5}$$

$$= \cdots = 2x_1 - 2x_2$$

cut $x_6 \geq 0 \Rightarrow 2x_1 - 2x_2 \geq 0$

$x_1 - x_2 \geq 0$

		x_1	x_2	x_3	x_4	x_5	x_6
$-z=$	1	0	0	0	0	1	0
$x_1=$	1	1	0	0	0	1	$-1/2$
$x_2=$	1	0	1	0	0	1	0
$x_3=$	1	0	0	1	0	-5	$3/2$
$x_4=$	1	0	0	0	1	1	$-3/2$

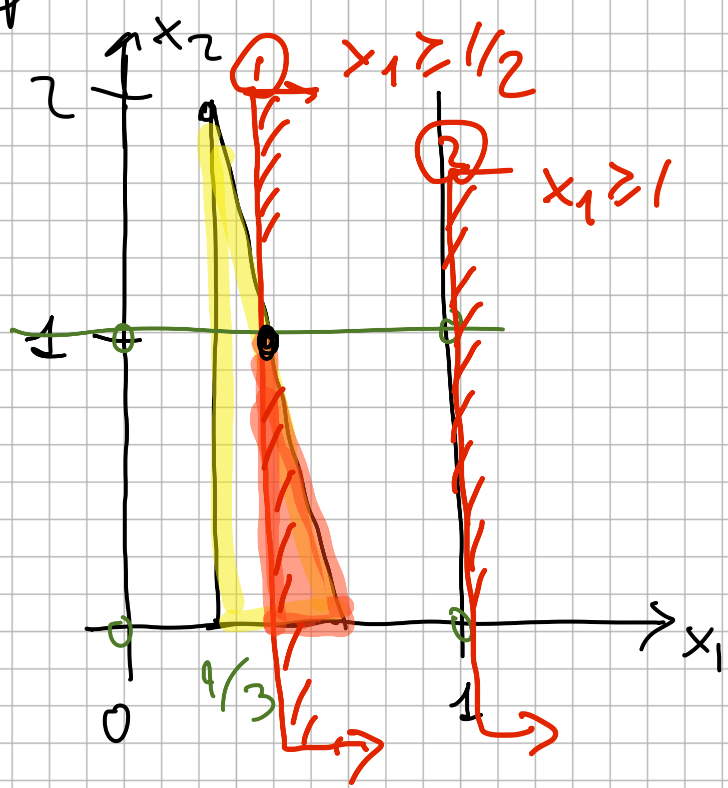
$x^* = (1,1)$ vertex G

x^* is integer (NO GOMORY cut can be produced)

$\Rightarrow$ STOP

EX

$$\begin{cases} \min x_1 + x_2 \\ 6x_1 + x_2 \leq 4 \\ 3x_1 \geq \cancel{1}\ 3 \\ x_1, x_2 \geq 0 \\ \text{integer} \end{cases}$$



Gomory cuts ...

$\Rightarrow$ after 2 cuts you realize infeasibility!

last tableau : INFEASIBLE

	x_1	x_2	x_3	x_4	x_5	x_6	
-2	0	1	1	0	0	6	$\leftarrow (*)$

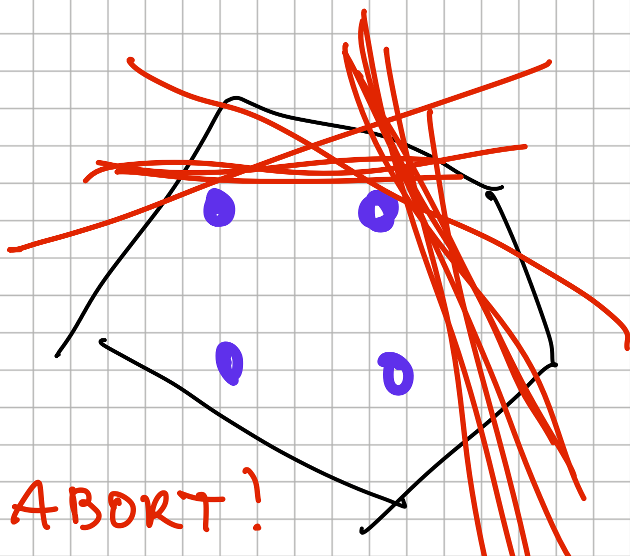
eq. (*) reads:

$$x_2 + x_3 + 6x_6 = -2$$

impossible as $x \geq 0$

$\Rightarrow$ NO FEASIBLE x exists

IN PRACTICE:



$\Rightarrow$ NUMERICAL ISSUES $\Rightarrow$ ABORT!