OR1 22-NOV-2021

INDEX SELECTION PR. (continued)

query \ index	0	1	2	3	4	5
1	6200	1300	~~6200~~	~~6200~~	~~6200~~	~~6200~~
2	2000	900	~~700~~	~~2000~~	~~2000~~	~~2000~~
3	800	~~800~~	~~800~~	~~800~~	~~800~~	800
4	6700	~~6700~~	~~6700~~	1700	~~6700~~	2700
5	5000	~~5000~~	~~5000~~	2200	1200	4200
6	2000	~~2000~~	~~2000~~	~~2000~~	~~2000~~	750

$\gamma =$ (matrix above)

	0	1	2	3	4	5
$c =$	0	200	1200	400	2400	250
$d =$	0	10	5	10	8	6

ILP model

$S \subseteq \{1, \dots, n\}$ sol.

Variables

$$y_j = \begin{cases} 1 & \text{if } j \in S \\ 0 & \text{otherwise} \end{cases}, \quad j = 1, \dots, n$$

$$x_{ij} = \begin{cases} 1 & \text{if query } i \text{ uses index } j \\ 0 & \text{otherwise} \end{cases}, \quad i = 1, \dots, m, \; j = 0, \dots, n$$

$$\min \underbrace{\sum_{j=1}^{n} c_j y_j}_{\text{fixed cost: } \sum_{j \in S} c_j} + \underbrace{\sum_{i=1}^{m} \sum_{j=0}^{n} \gamma_{ij} x_{ij}}_{\text{cost for the queries}}$$

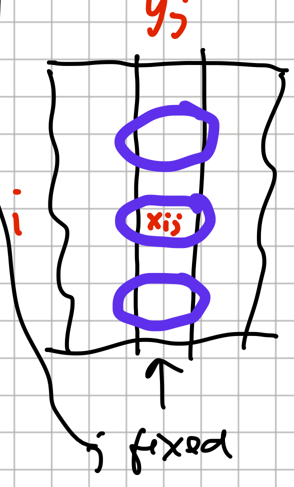
$$\underbrace{\sum_{j=1}^{n} d_j y_j}_{\sum_{j \in S} d_j} \le D$$

$$0 \le y_j \le 1 \quad \text{integer}, \quad \forall j = 1, \dots, n$$

$$0 \le x_{ij} \le 1 \quad \text{integer}, \quad \forall i = 1, \dots, m, \; j = 0, \dots, n$$

$$\boxed{\underbrace{\sum_{j=0}^{n} x_{ij}}_{\#\text{ of circles on row } i} = 1, \quad \forall i = 1, \dots, m}$$

" $x_{ij} = 0 \Leftarrow y_j = 0$ "



$$\underbrace{\sum_{i=1}^{m} x_{ij}}_{\#\text{ of circles in col. } j} \le m\, y_j, \quad \forall j = 1, \dots, n$$

↳ $y_j = 0 \Rightarrow$ all $x_{ij} = 0$

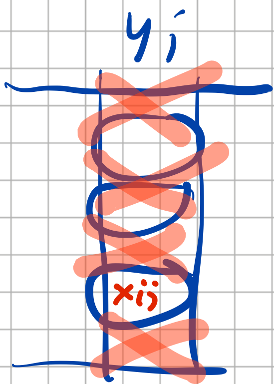
↳ $y_j = 1 \Rightarrow$ n. of circles $\le m$: OK

PRE-PROCESSING

$$x_{ij} = 0 \Leftarrow \gamma_{i0} \le \gamma_{ij}$$

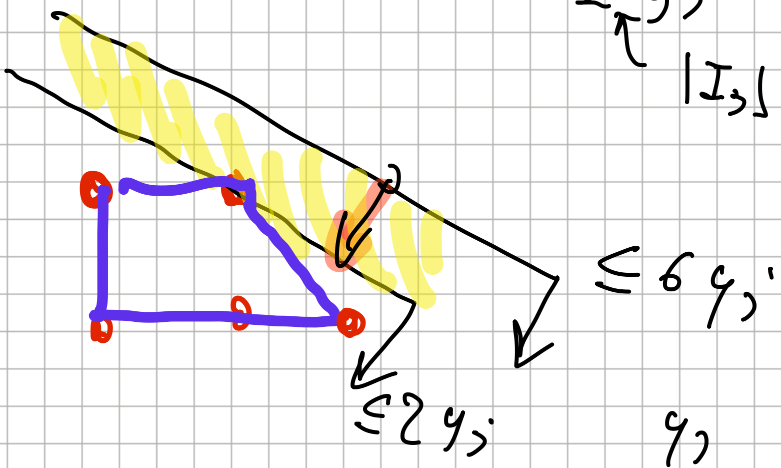
$$\boxed{I_j := \{ i \in \{1,\dots,m\} : \gamma_{ij} < \gamma_{i0} \}}$$

set of the "active entries" in col. j



$$\sum_{i \in I_j} x_{ij} \le \overset{m}{|I_j|}\, y_j \quad , \forall j = 1,\dots,n$$

$$xxxx \le \begin{cases} \overset{m}{6}\, y_j & \text{WEAK} \\ \underset{|I_j|}{2}\, y_j & \text{STRONGER} \end{cases}$$



✓ BETTER FORMULATION !!

✓ FEWER VAR.S

HOWEVER, B&BOUND STILL UNSATISFACTORY:

WE NEED AN IMPROVED FORMULATION (WITH MORE CONSTR.S) !

- Solve the LP relaxation ("root node") for the specific numerical instance above ....

$z^* = LB = 8940 \quad (= z_{LP} \leq \underbrace{z_{ILP} = 11100)}_{\text{GAP } 2160}$

(x^*, y^*) ... $x^*_{11} = 1$ ...

... $y^*_1 = 0.7$ ...

$\Rightarrow$ $\boxed{x_{11} \leq y_1}$ this is a valid ineq. for the integer sol.s

$\hookrightarrow y_1 = 1$

$\hookrightarrow y_1 = 0$

violated by (x^*, y^*)

$\Rightarrow$ CUT to be added to my formulation

CLASS OF VALID INEQ.S : $\mathcal{C}_1$

$$x_{ij} \leq y_j, \quad \forall i = 1, \dots, m;\ j = 1, \dots, n$$

SEPARATION FUNCTION ($\mathcal{C}_1$)

$$(x^*, y^*) \longrightarrow \boxed{\mathcal{C}_1} \longrightarrow$$ violated ineq. of class $\mathcal{C}_1$ (IF ANY)

INPUT: (x^*, y^*)

```
for j = 1 to n do
    for i = 1 to m do
        if x*_ij > y*_j then
            " ineq. x_ij ≤ y_j is violated "
        fi
    rof
rof
```

$\Rightarrow$ $z^* = 9'900$ (gap $= 1'200$)

(x^*, y^*) ... $y^*_1 = 1$ ($d_1 = 10$)

$y^*_2 = 3/4$ ($d_2 = 10$)

...

satisfies <u>all</u> ineq. in $\mathcal{C}_1$

" $y_1 + y_2 \le \cancel{2}\ 1$, because $d_1 + d_2 > 19$ "

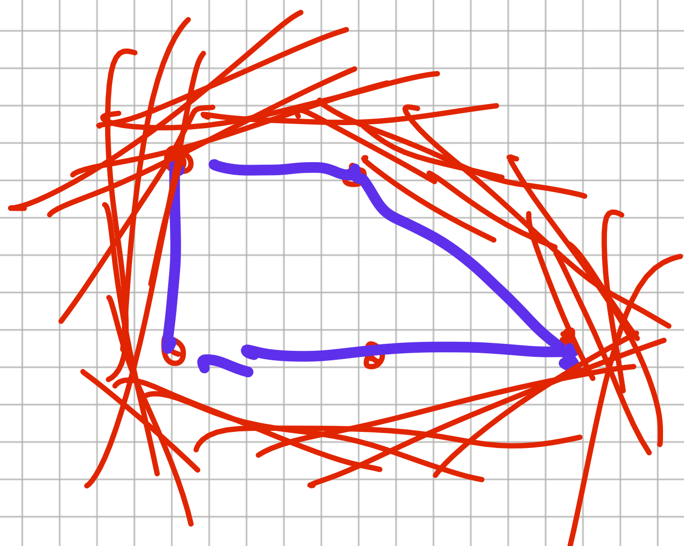
$y_i + y_j \le 1$, $\forall\ i<j : d_i + d_j > D$

$y_i + y_j + y_k \le 2$, $\forall\ i,j,k : d_i + d_j + d_k \ge D$

$|\{i,j,k\}| = 3$ ...

$\boxed{\text{CLASS } \mathcal{C}_2 :}$

$$\sum_{j \in S} y_j \le |S| - 1 \quad , \quad \forall\ S \subseteq \{1, \dots, n\} : \sum_{j \in S} d_j \gneq D$$



RICH CLASS OF INEQ.S !

SEPARATION FUNCTION $\mathcal{C}_2$:

$$y^* \rightarrow \boxed{\mathcal{C}_2} \rightarrow$$ violated ineq. in class $\mathcal{C}_2$

"HEURISTIC"
"EXACT"

$\boxed{\text{EXACT SEPARAT}}$ INPUT: $y^* \in [0,1]^n$

OUTPUT: set $S^* \subseteq \{1, \dots, n\}$ s.t.

(i) $\sum_{j \in S^*} d_j \underset{\neq}{\geq} D$ "VALID"

(ii) $\sum_{j \in S^*} y_j^* > |S^*| - 1$ "VIOLATED"

"$\underbrace{|S^*| - \sum_{j \in S^*} y_j^*}_{\text{objective function}} < 1$"

Var.s

$$z_j = \begin{cases} 1 & \text{if } j \in S^* \\ 0 & \text{otherwise} \end{cases}, \quad j = 1, \dots, n$$

"$\varepsilon \underset{\neq}{>} 0$ small"

$$\begin{cases} \min \sum_{j=1}^{n} (1 - y_j^*) z_j \\ \text{(i)} \quad \sum_{j=1}^{n} d_j z_j \geq D + \varepsilon \\ 0 \leq z_j \leq 1 \text{ integer}, \quad \forall j = 1, \dots, n \end{cases}$$

$\Longrightarrow$ VERY POWERFUL

PREPROCESSING !!