OR 1 23-NOV-2021

GRAPH THEORY

Seven Bridges of Königsberg

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Coordinates: 54°42′12″N 20°30′56″E

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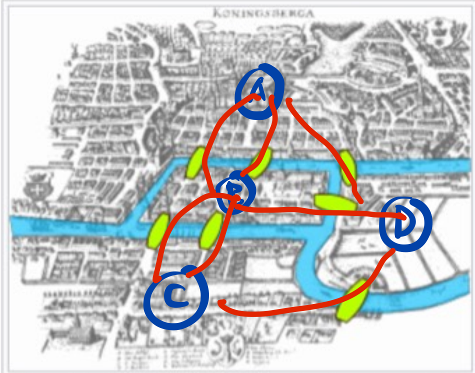
The **Seven Bridges of Königsberg** is a historically notable problem in mathematics. Its negative resolution by Leonhard Euler in 1736[1] laid the foundations of graph theory and prefigured the idea of topology.[2]

The city of Königsberg in Prussia (now Kaliningrad, Russia) was set on both sides of the Pregel River, and included two large islands—Kneiphof and Lomse—which were connected to each other, or to the two mainland portions of the city, by seven bridges. The problem was to devise a walk through the city that would cross each of those bridges once and only once.



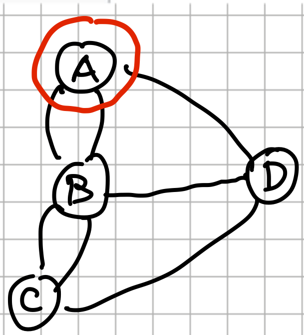
Map of Königsberg in Euler's time showing the actual layout of the seven bridges, highlighting the river Pregel and the bridges

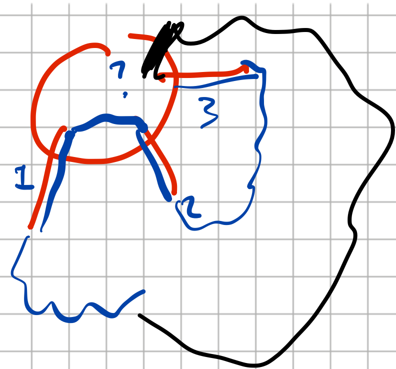
By way of specifying the logical task unambiguously, solutions involving either

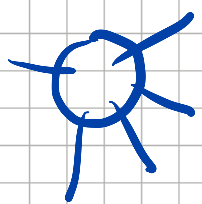
1. reaching an island or mainland bank other than via one of the bridges, or
2. accessing any bridge without crossing to its other end

are explicitly unacceptable.

Euler proved that the problem has no solution. The difficulty he faced was the development of a suitable technique of analysis, and of subsequent tests that established this assertion with mathematical rigor.

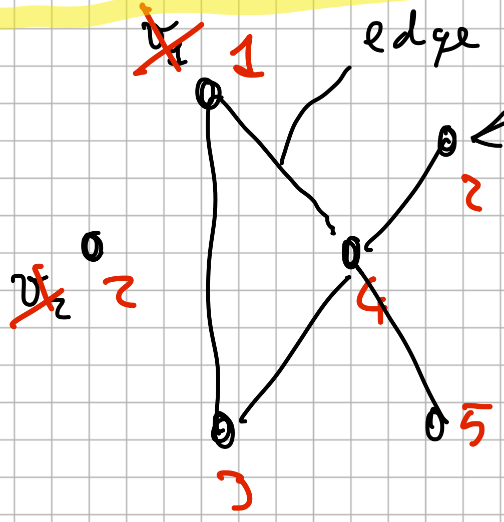






odd # of connection ⇒ ∄ path ...

UNDIRECTED GRAPHS

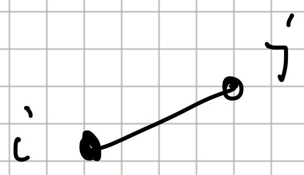


$$G = (V, E)$$

$$V = \{1, 2, \dots, n\}$$

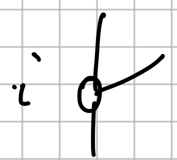
$$E \subseteq V \times V$$

where $n := |V|$



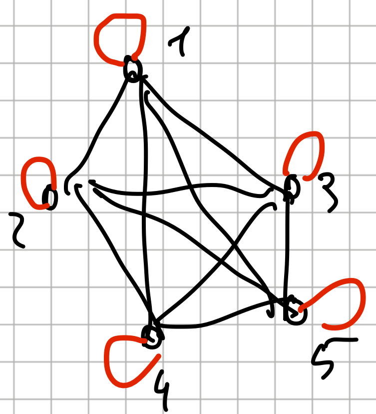
$\{i,j\} \in E$ $\quad$ $[i,j]$

$(\{j,i\})$ $\quad$ $i\ j$



degree of i = # edges connected to node i

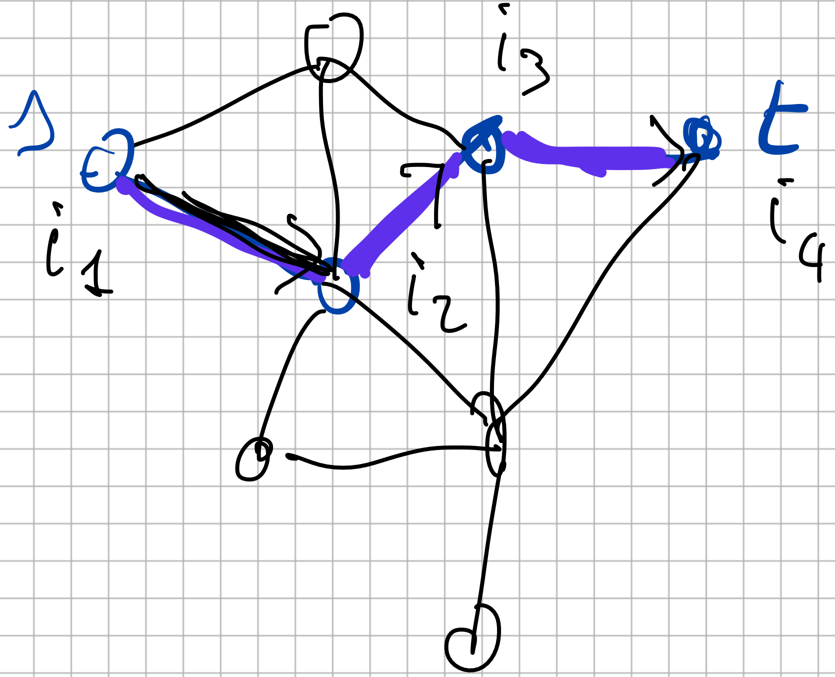
$\deg(i) = 3$



graph is COMPLETE

K_5

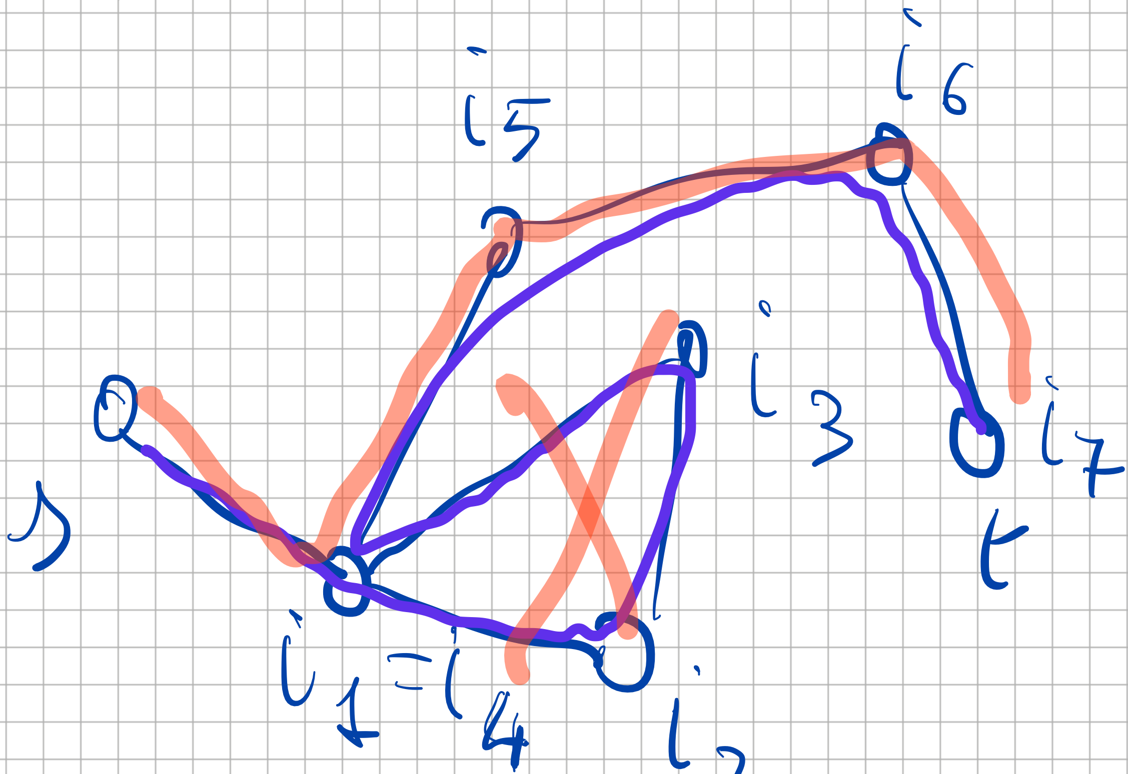
$[i,i]$ loop



path

$$(i_1, i_2), (i_2, i_3), \ldots, (i_{k-1}, i_k)$$

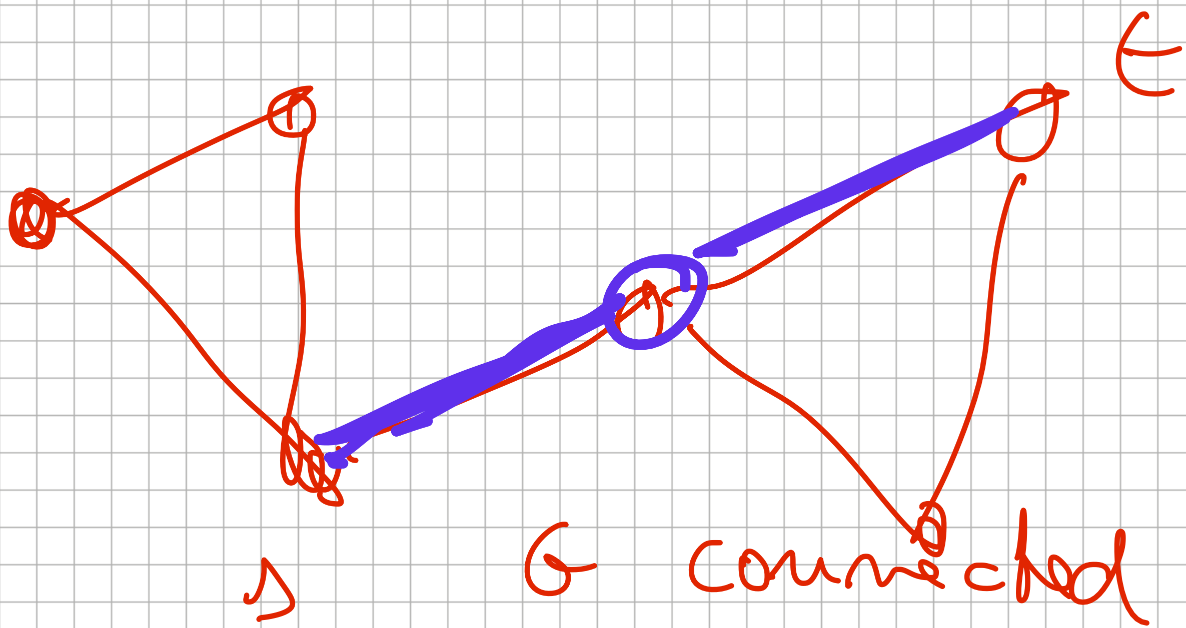
$i_1 = s$, $i_k = t$



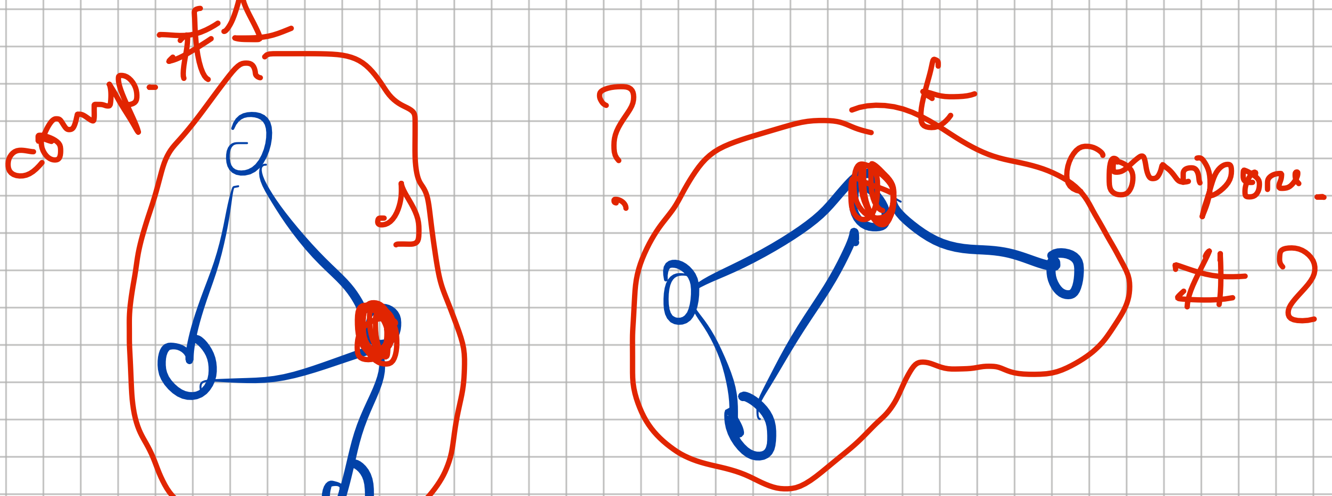
NON-SIMPLE PATH

G is <u>connected</u>

iff $\exists$ path s-t for every pair $s, t \in V$

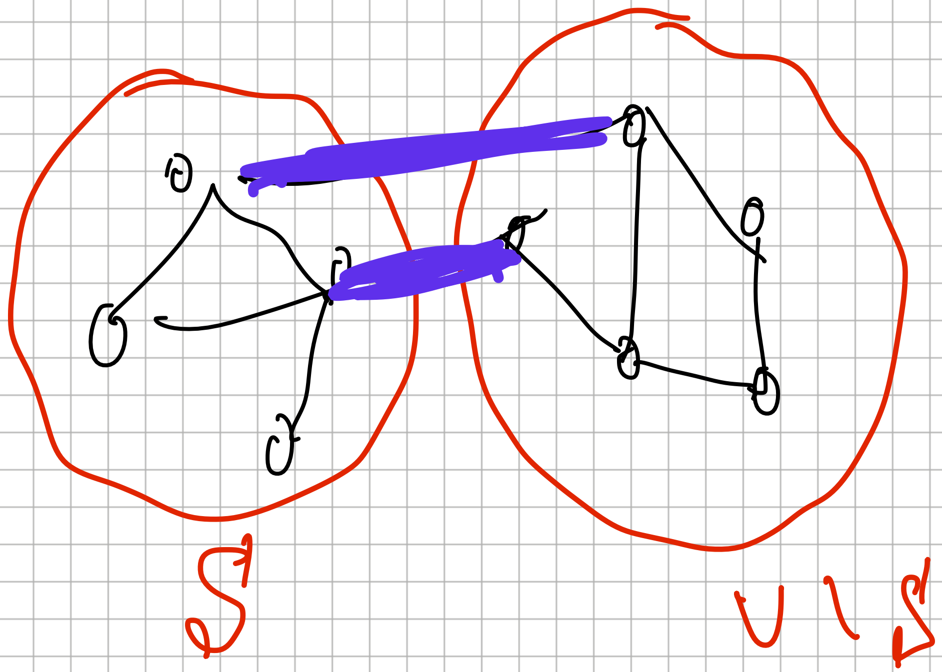


G connected



G is not connected

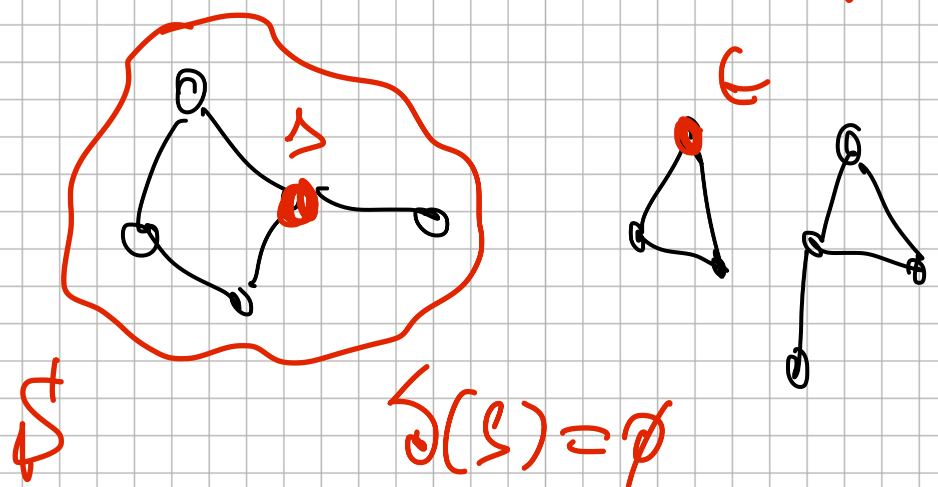
A cut in a graph:



$$\delta(S) = \{\{i,j\} \in E : i \in S, j \in V \setminus S\}$$

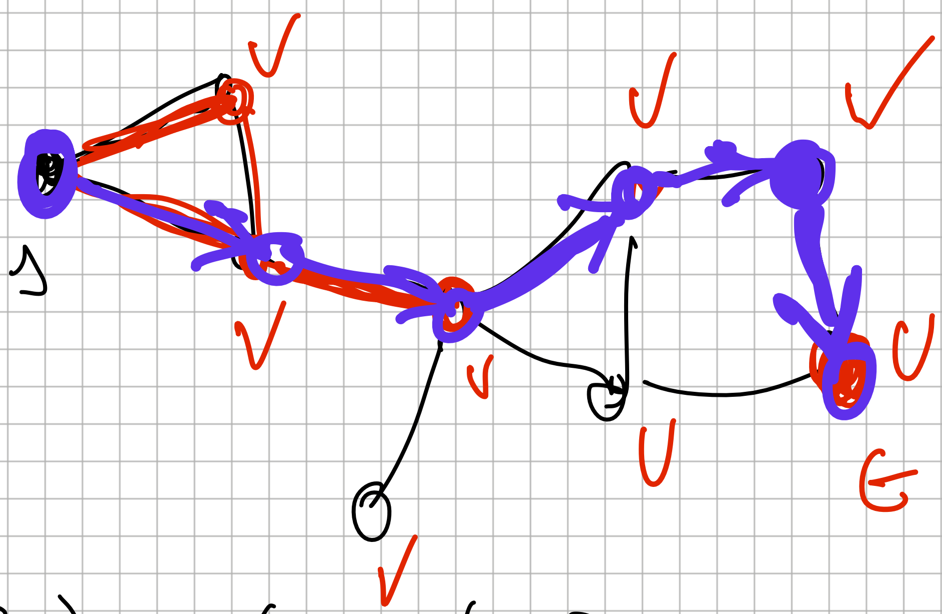
$$\boxed{G \text{ is connected} \iff \delta(S) \neq \emptyset \text{ for every } S \subsetneq V,\ S \neq \emptyset.}$$

Proof: $\exists\, S \subsetneq V,\ S \neq \emptyset :$
$\delta(S) = \emptyset$



$\delta(S) = \emptyset$

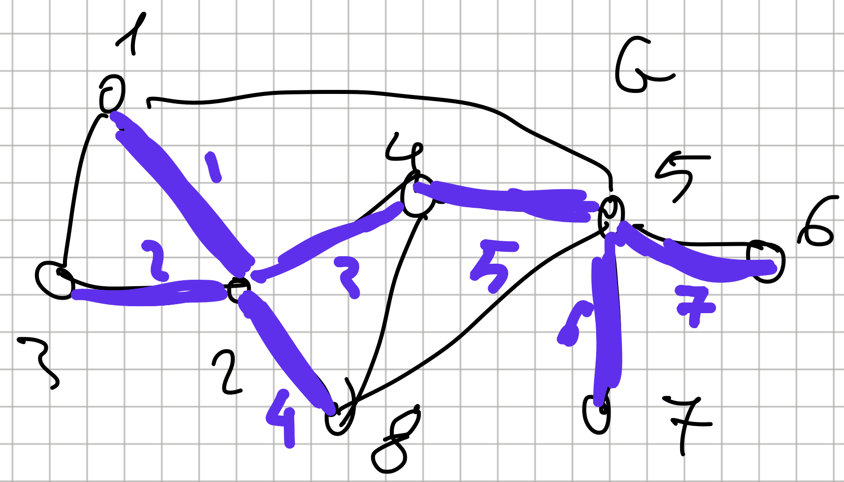
$\Rightarrow$ not connected



$\delta(S) \neq \emptyset \quad \forall\, S \ldots$

$\Rightarrow$ LABELING alg. $\Rightarrow$
found path $s - t$

TREE

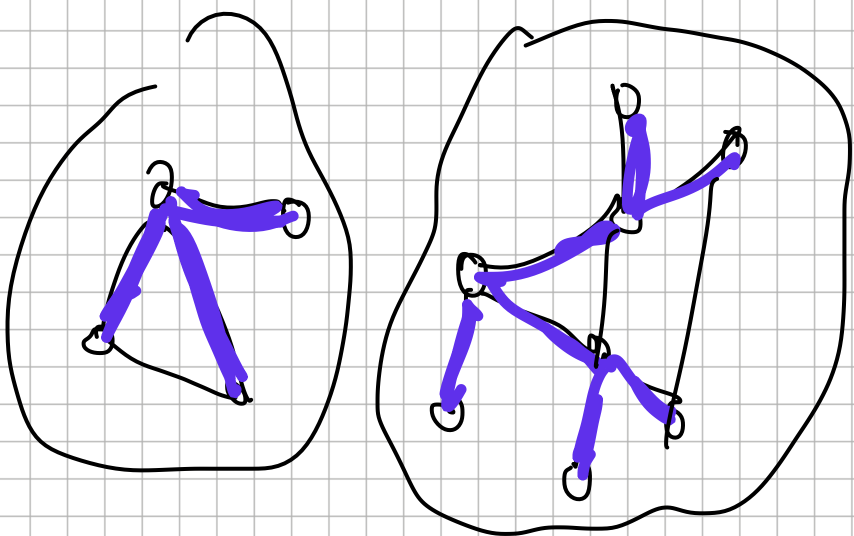


subgraph of G :

- it is connected
- it contains n − 1 edges
- it contains no CYCLES

s = t

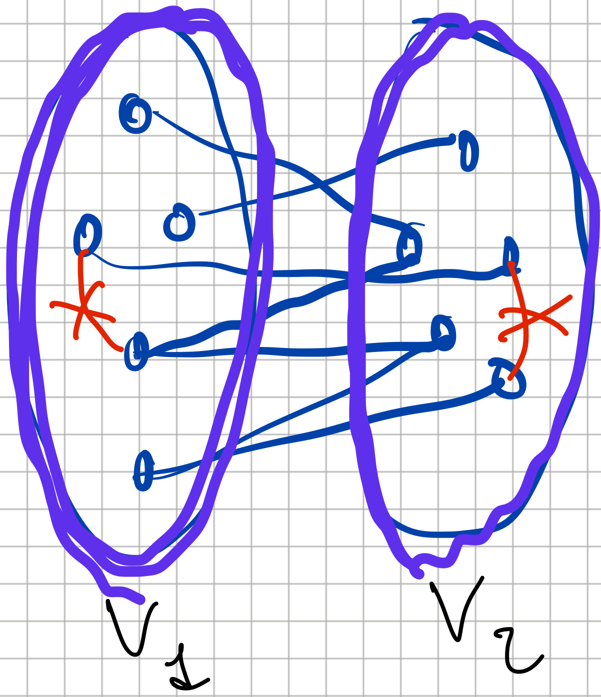
CYCLE = closed path



not connected G

⟺ FOREST

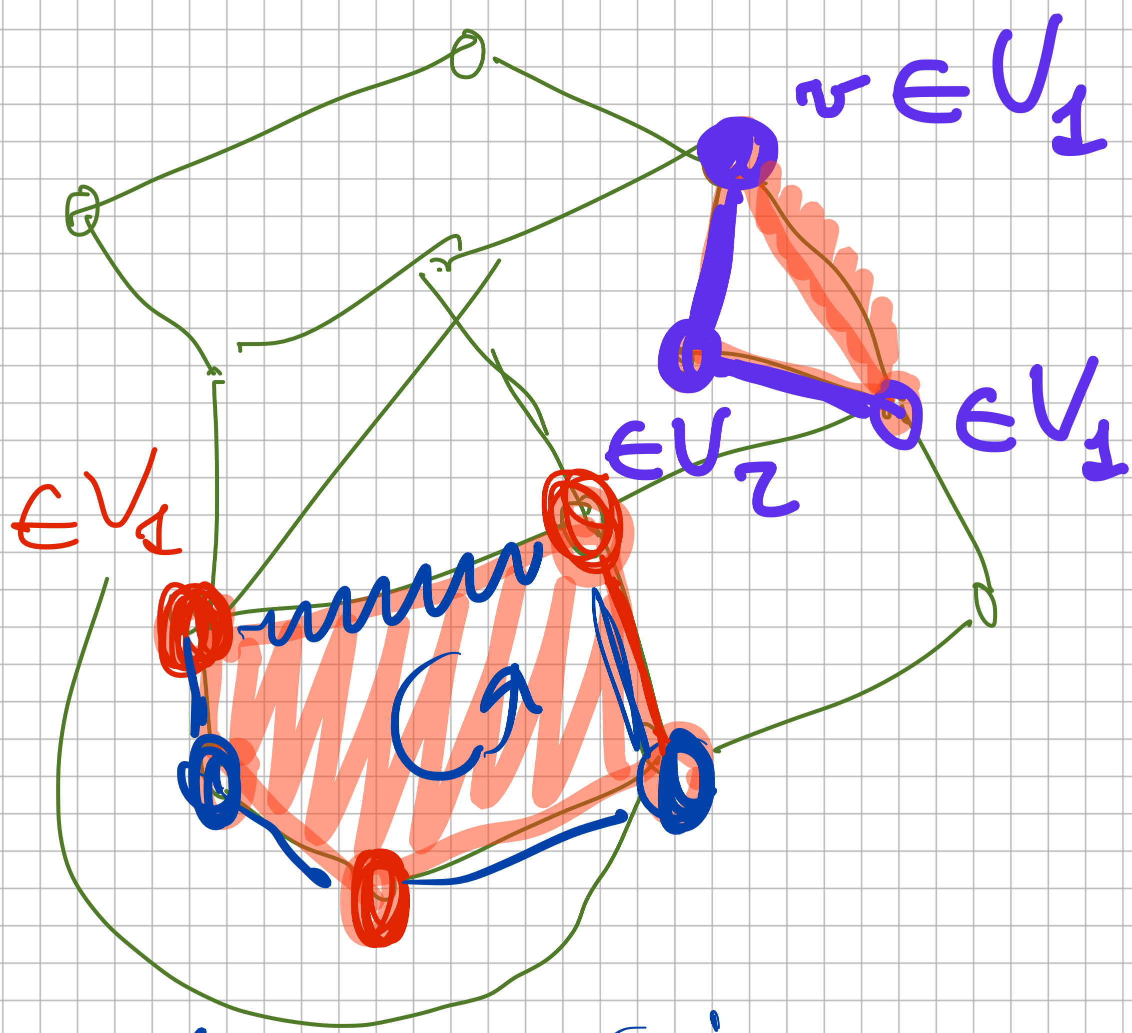
BIPARTITE GRAPH



$G = (V, E)$

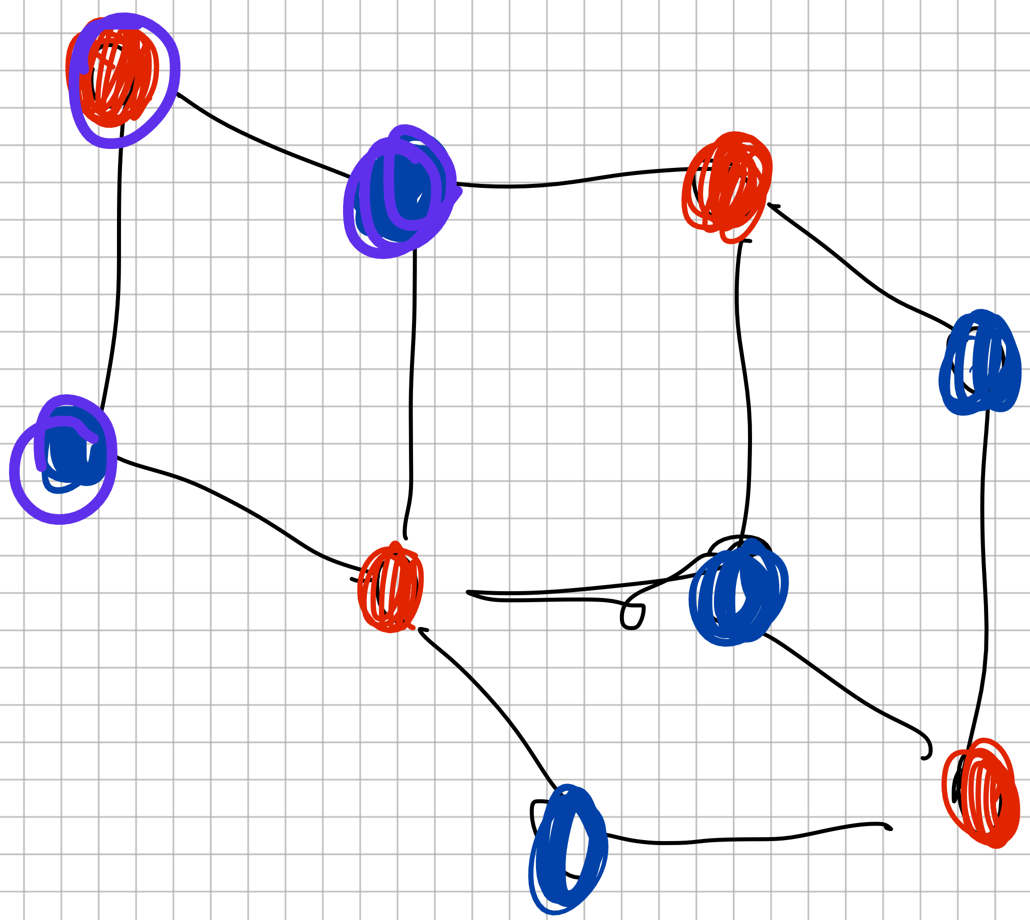
(V_1, V_2) of V :

$\delta(V_1) = E$

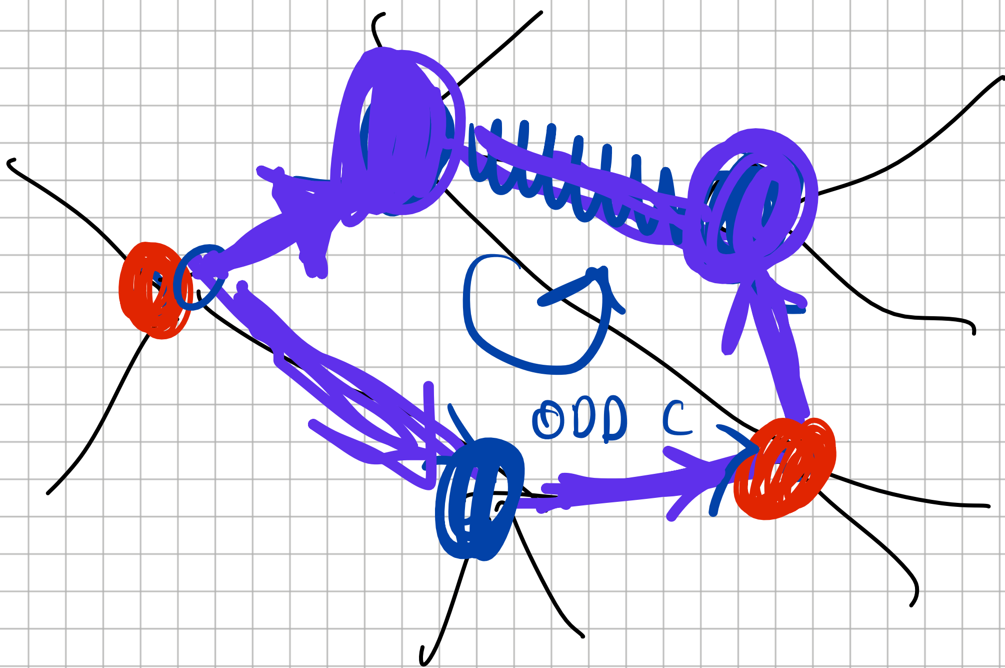


odd cycle $\Rightarrow$ G is NOT bipartite o V_1

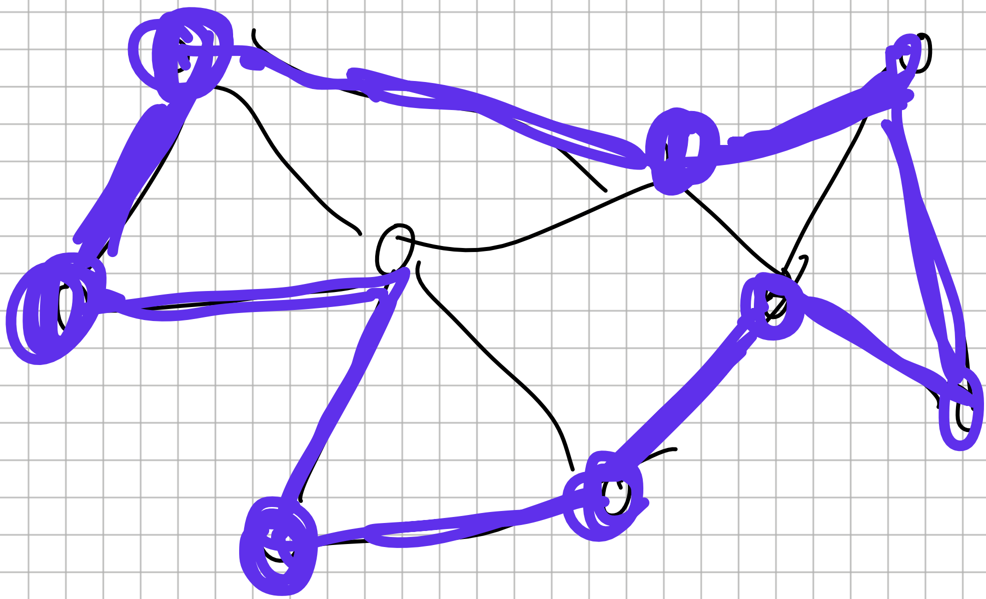
× V_2



$\Rightarrow$ G is bipartite

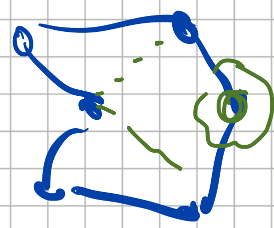


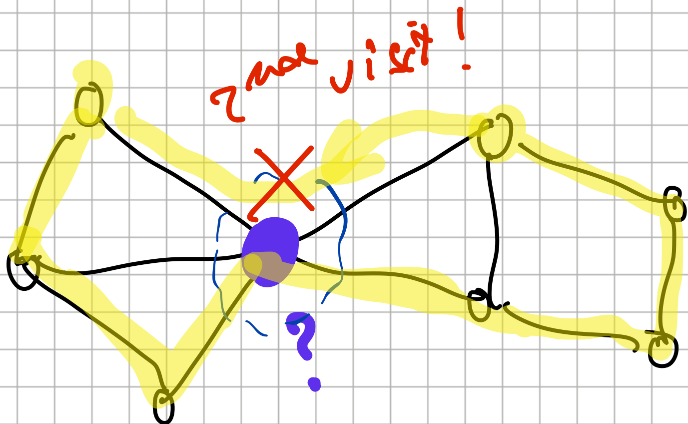
G is HAMILTONIAN



HAMILTONIAN CYCLE

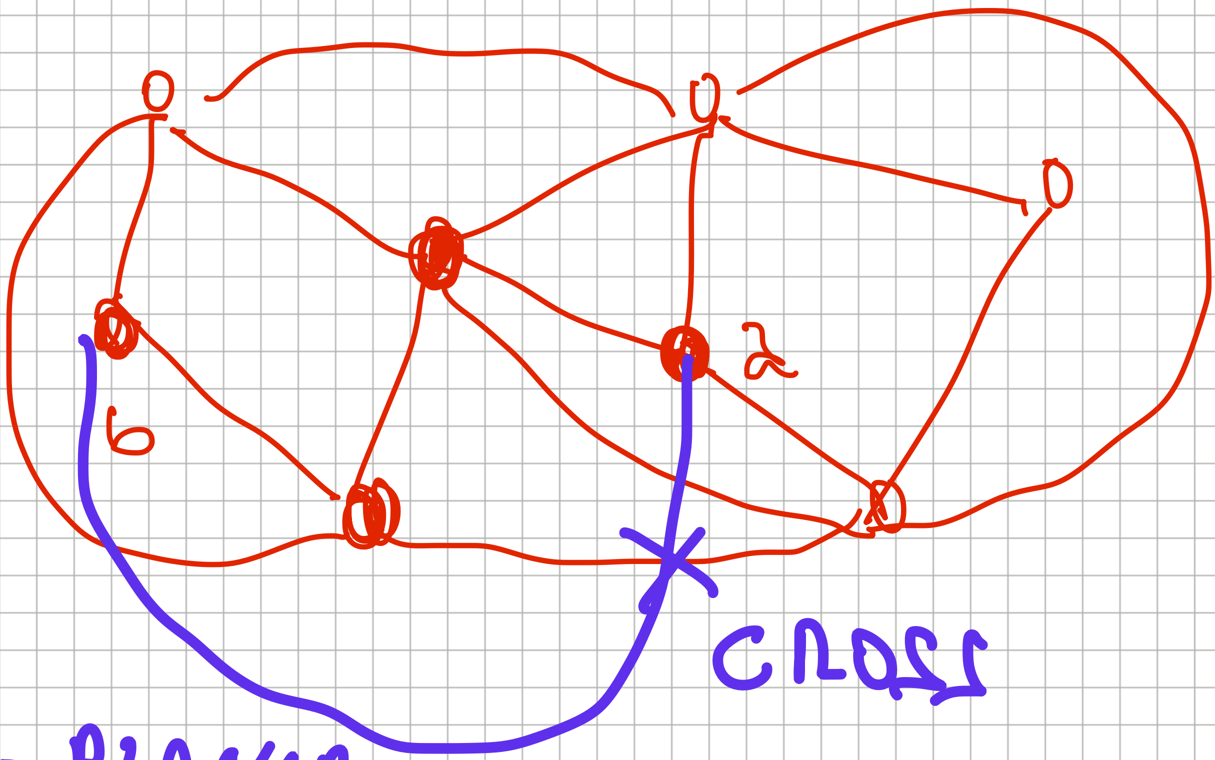
(TOUR)





IS NOT HAMILTONIAN !

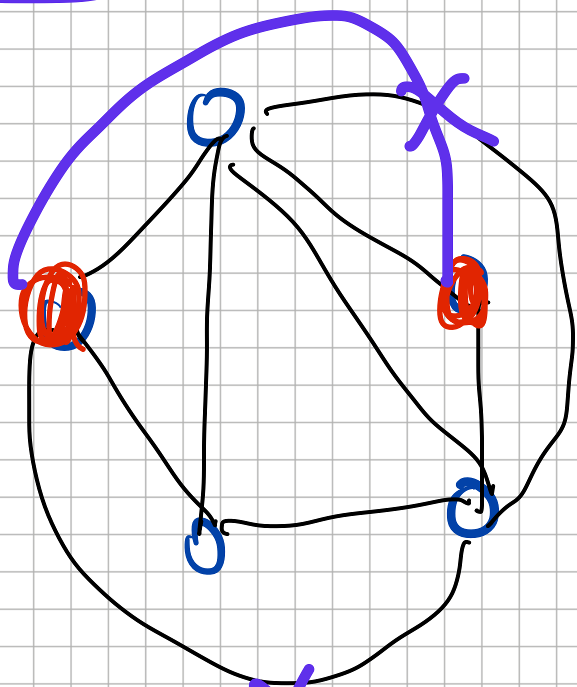
PLANAR GRAPH



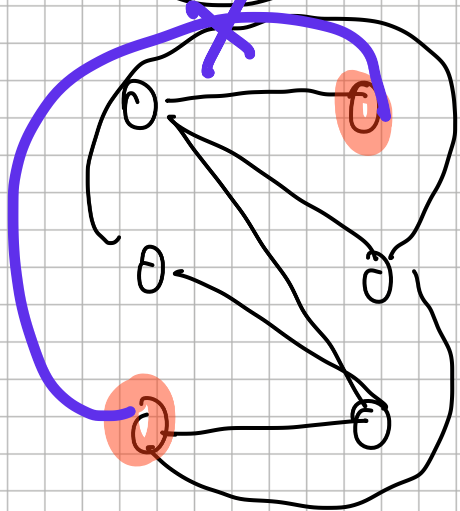
CROSS

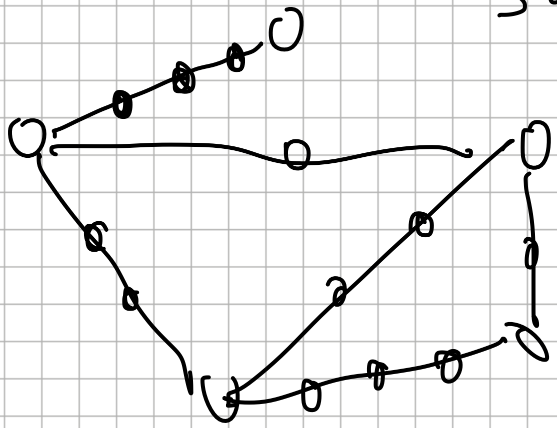
NON-PLANAR

K_5



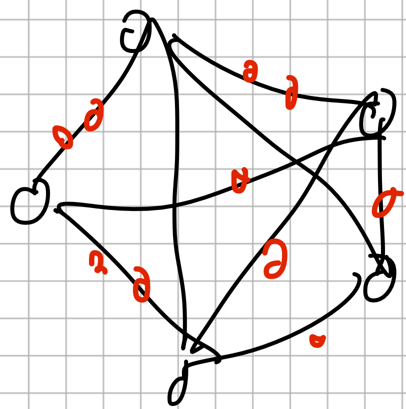
$K_{3,3}$





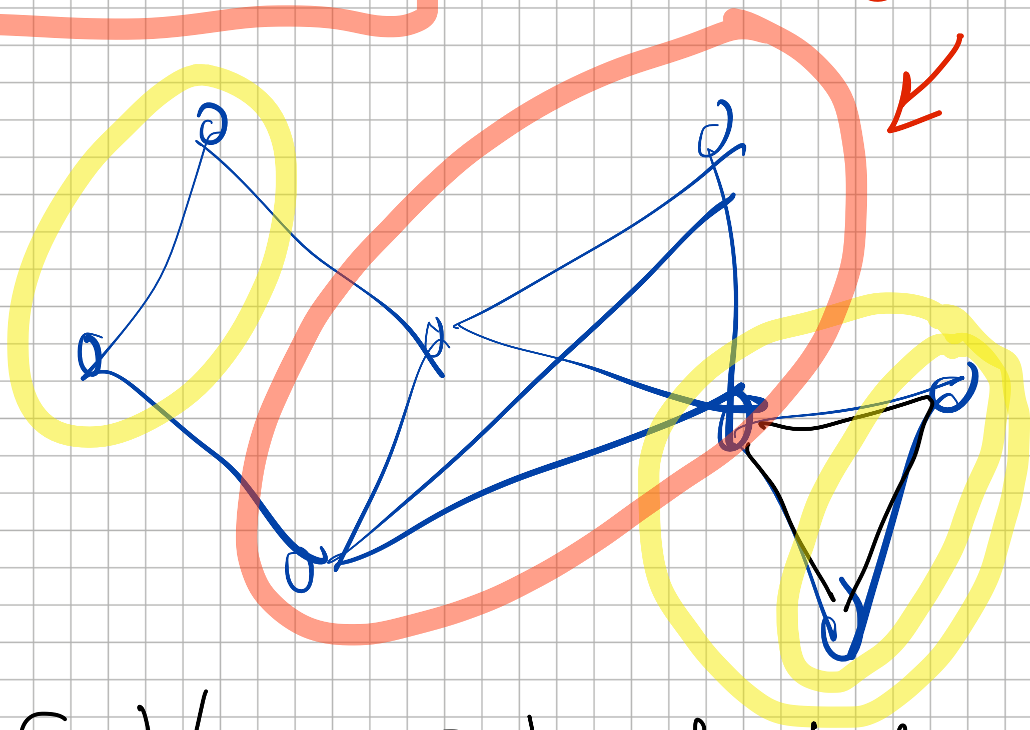
SUBDIVISION OF EDGES

K_5



CLIQUE

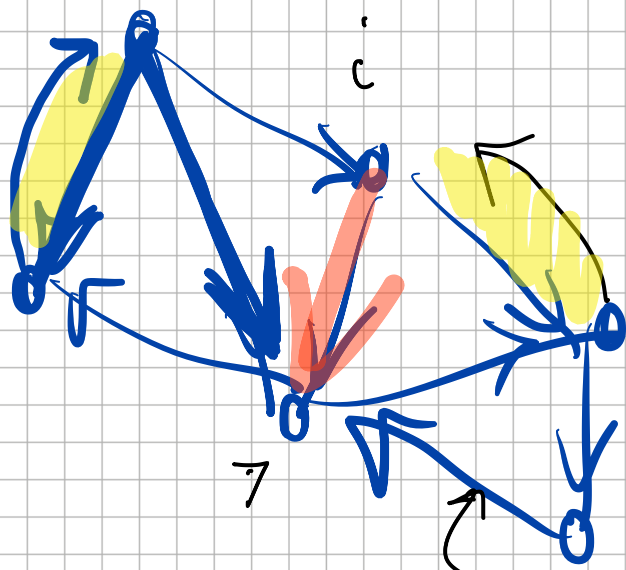
max cardinality clique



S'

$S \subseteq V$: subgraph induced by S is complete

DIRECTED GRAPHS



ARC

$G = (V, A)$

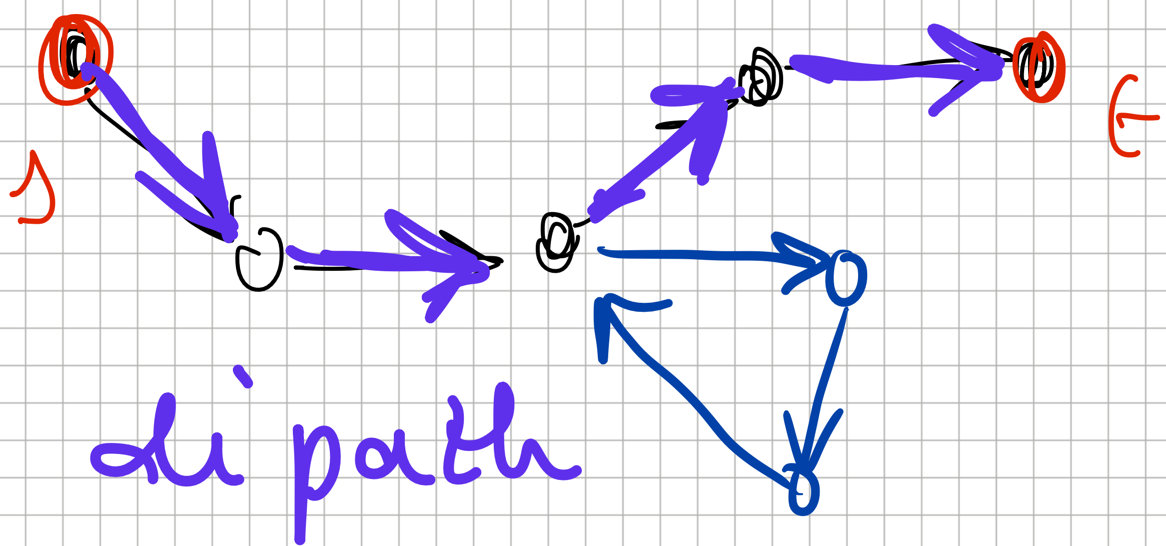
$V = \{1, \dots, n\}$

A set of arcs

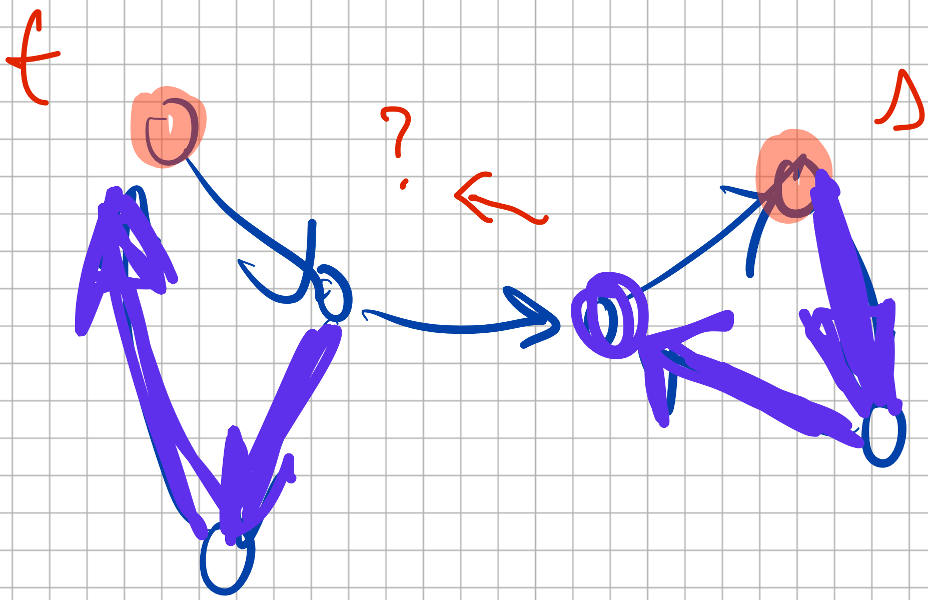
$(i, j) \in A$

ordered pair

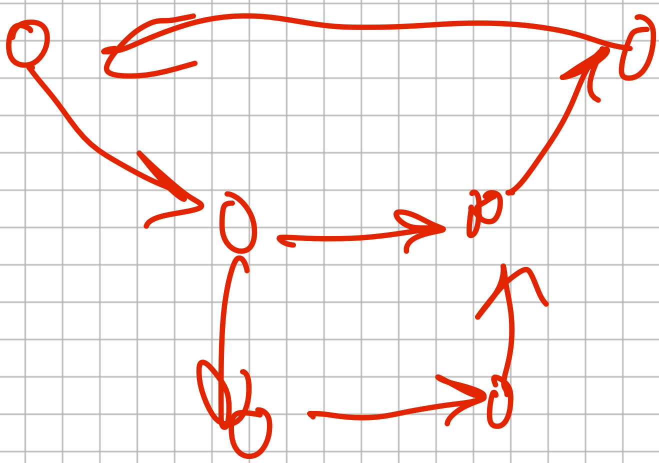
“$(i, j) \neq (j, i)$”



CONNECTIVITY



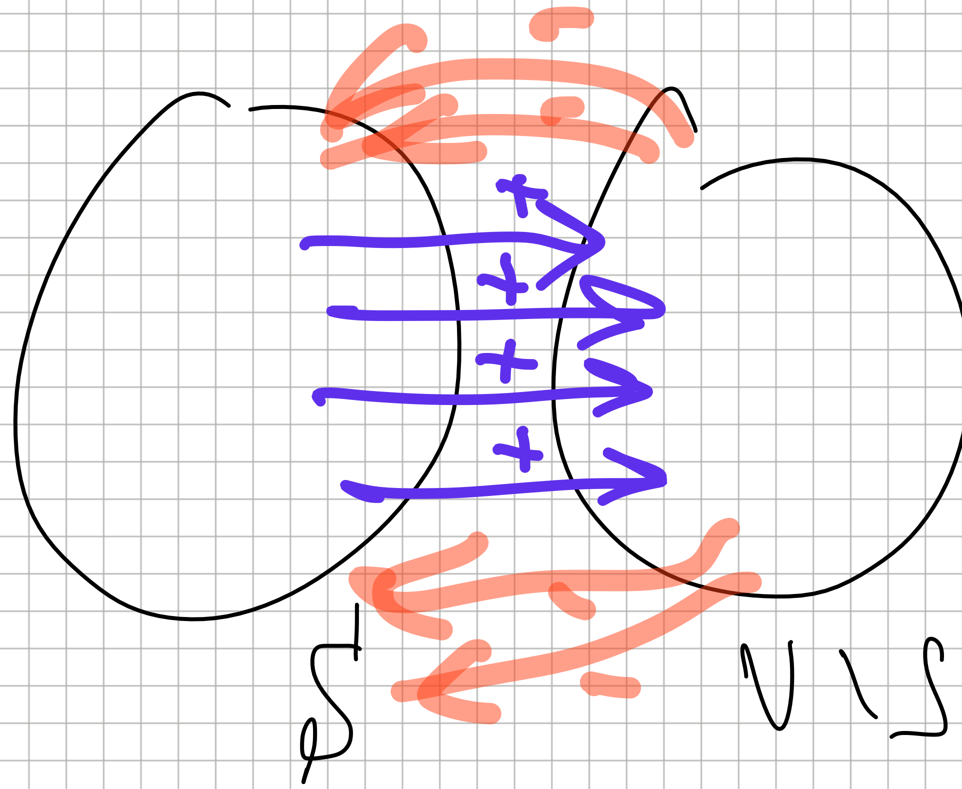
NOT STRONGLY CONN.



STRONGLY CONN.

dipath $s \to t \quad \forall\, s,t \in V$

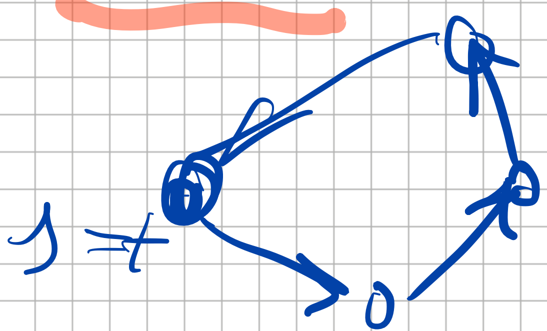
DI-CUT (or simply CUT)



$$\boxed{\delta^+(S)} = \{(i,j) \in A : i \in S, j \in V \setminus S\}$$

$$\boxed{\delta^-(S) =} \{(i,j) \in A : i \in V \setminus S, j \in S\}$$

CIRCUIT = directed cycle



HAMILTONIAN CIRCUIT