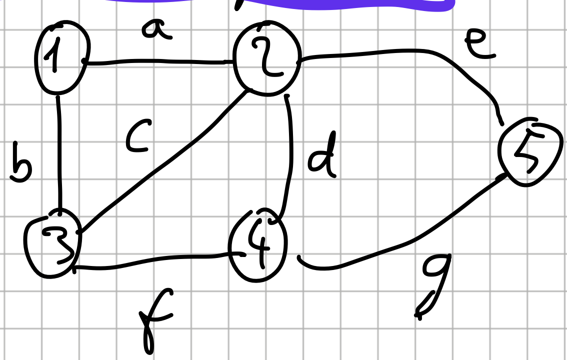
OR1 24-NOV-2021

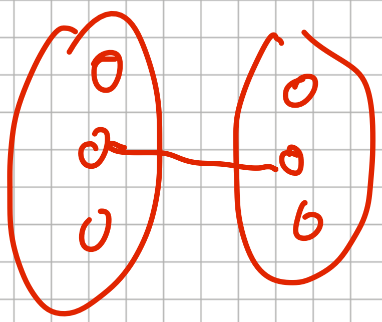
$\boxed{\text{Undirected graph}}$




$G = (V, E)$

$$
d = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\begin{array}{c} \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\
\left[\begin{array}{ccccccc}
+1 & 1 & & & & & \\
+1 & 0 & 1 & 1 & 1 & & \\
0 & 1 & 1 & & & 1 & \\
0 & 0 & & 1 & & 1 & 1 \\
0 & 0 & & & 1 & & 1
\end{array}\right] \end{array}
\begin{array}{l} \\ \left.\begin{array}{c} \\ \\ \end{array}\right\} V_1 \\ \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} V_2 \end{array}
$$

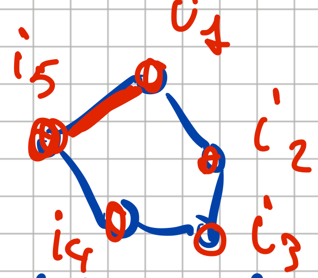
NODE–EDGE incidence matrix of $G$




$G$ is bipartite

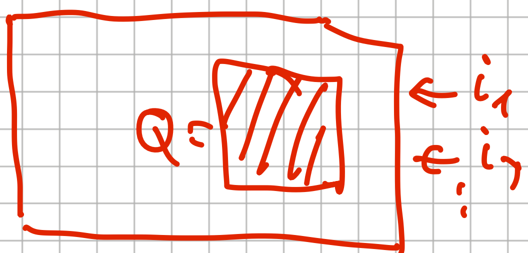
$\iff d$ is TUM

$G$ not bipartite $\Rightarrow \exists$




odd cycle

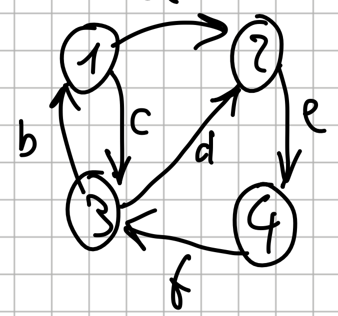
$d =$



$$Q = \begin{array}{c} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \end{array} \begin{bmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & 1 & 1 & \\ & & & 1 & 1 \\ 1 & & & & 1 \end{bmatrix} \qquad \det(Q) = \pm 2$$

$\Rightarrow$ $A$ is NOT TUM

$$\boxed{\text{Directed graph } G = (V, A)}$$

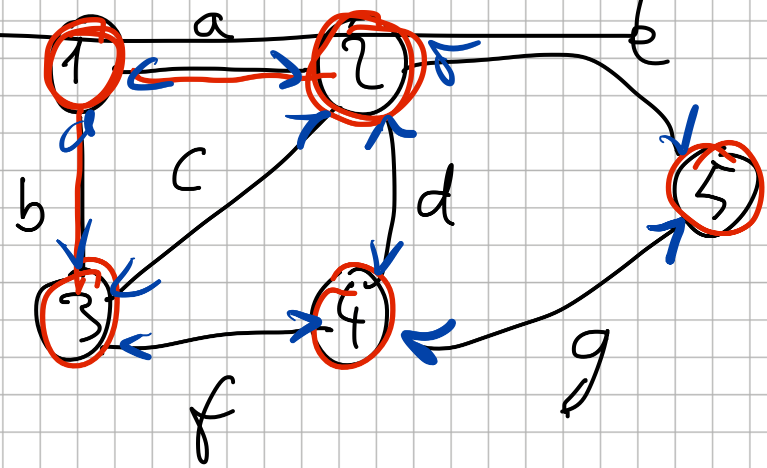



$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \left[\begin{array}{c|ccccc} +1 & -1 & +1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & +1 & 0 \\ 0 & +1 & -1 & +1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & +1 \end{array}\right] \end{array} \quad \begin{array}{l} V_1 = V \\ V_2 = \emptyset \end{array}$$

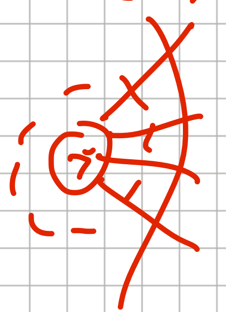
NODE-ARC incidence matrix of $G$

$A$ is always TUM

DATA STRUCTURE




$\delta(i)$




$\delta(\{i\}) = \delta(i)$

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| FROM | 1 | 1 | 2 | 2 | 2 | 2 | 3 | 3 | 3 | 4 | 4 | 4 | 5 | 5 | |
| TO | 2 | 3 | 1 | 3 | 4 | 5 | 1 | 2 | 4 | 2 | 3 | 5 | 2 | 4 | |

$n+1$

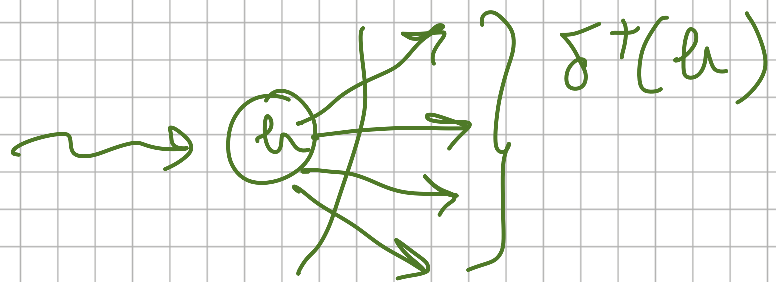
$\delta^+(1)$ $\delta^+(2)$ $\delta^+(3)$ $\delta^+(4)$ $\delta^+(5)$

FIRST[6] = 15

FIRST[5] = 13

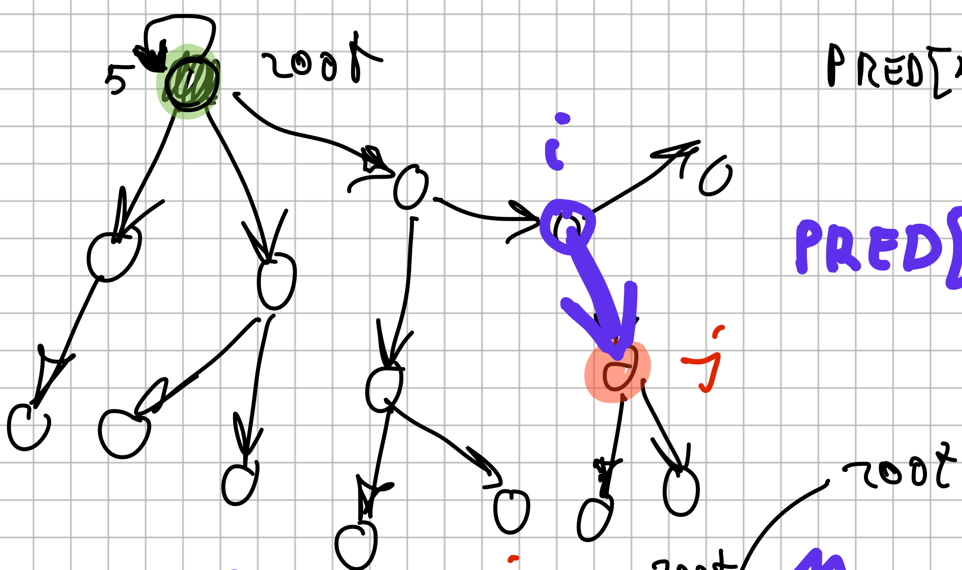
FIRST[1] = 1 FIRST[3] = 7

FIRST[2] = 3



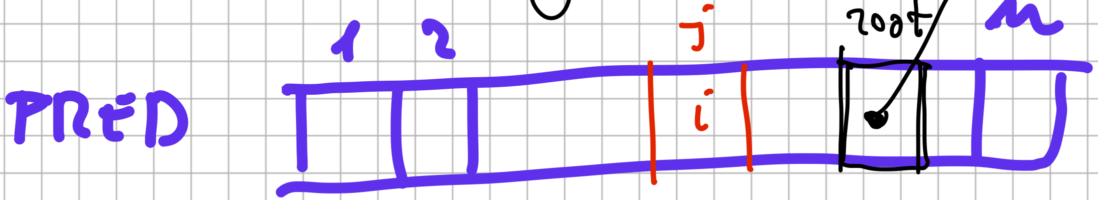

FIRST[h] ... FIRST[h+1] − 1

SPECIFIC DATA STRUCTURE FOR DIRECTED TREES




PRED[5] = 5

PRED[j] = i

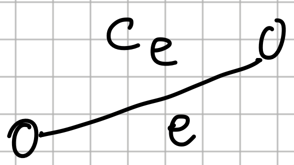


# MINIMUM SPANNING TREE PROBLEM

Undirected graph $G = (V, E)$

$$c : E \to \mathbb{R}$$

$c_e$ = cost of edge $e$




**FIND** a "spanning" tree on $G$ of MINIMUM TOTAL COST

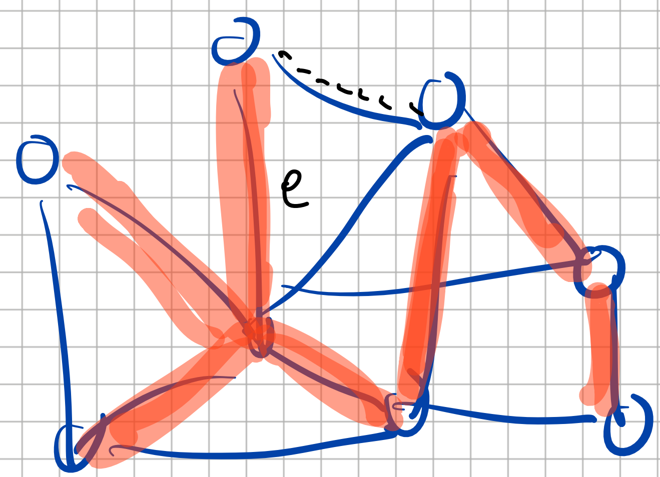
$G_T = (V, T)$ where $T \subseteq E$

$$\text{cost}(T) = \sum_{e \in T} c_e \to \text{MIN.}$$

## ILP model

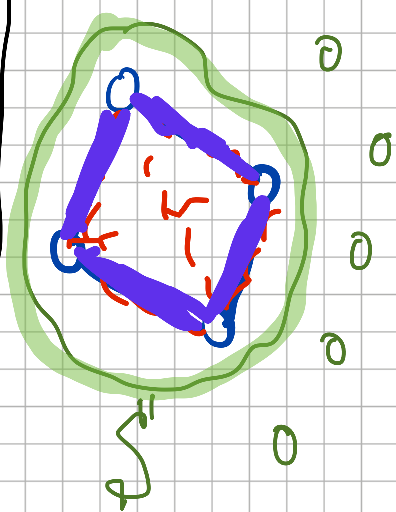
— $T$

$$x_e = \begin{cases} 1 & \text{if } e \in T \\ 0 & \text{otherwise}, \end{cases} \quad \forall e \in E$$



$$
\begin{cases}
\min \underbrace{\sum_{e\in E} c_e x_e}_{\text{“}\sum_{e\in T} c_e\text{”}} \\[2ex]
\underbrace{\sum_{e\in E} x_e}_{\text{n. of edges in } T} = |V|-1 \\[2ex]
\underbrace{\sum_{e\in E(S)} x_e}_{\substack{\text{n. of edges in}\\ T \text{ with both}\\ \text{ends in } S}} \le |S|-1 \qquad \forall\, S \subsetneq V : S\neq\emptyset \\[2ex]
x_e\in\{0,1\},\ \forall\, e\in E
\end{cases}
$$

SUBTOUR ELIM. CONSTR. (SEC)




where $E(S) := \{[i,j]\in E : i,j\in S\}$

$\simeq 2^n$ SECs

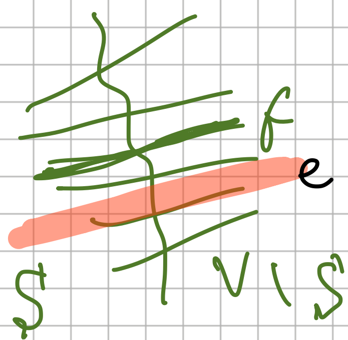
# EFFICIENT ALGORITHMS

<u>Hp</u>: (1) $G$ is connected

$G=(V,E)$ (2) $c_e \neq c_f \quad \forall e \neq f$

$\boxed{\text{TH}}$ Let $G_{T^*}=(V,T^*)$ be a minimum-cost spanning tree. For any $e \in E$ we have that

$$e \in T^* \iff \exists\, S \subsetneq V :$$
$$e = \arg\min \{ c_f : f \in \delta(S) \}.$$



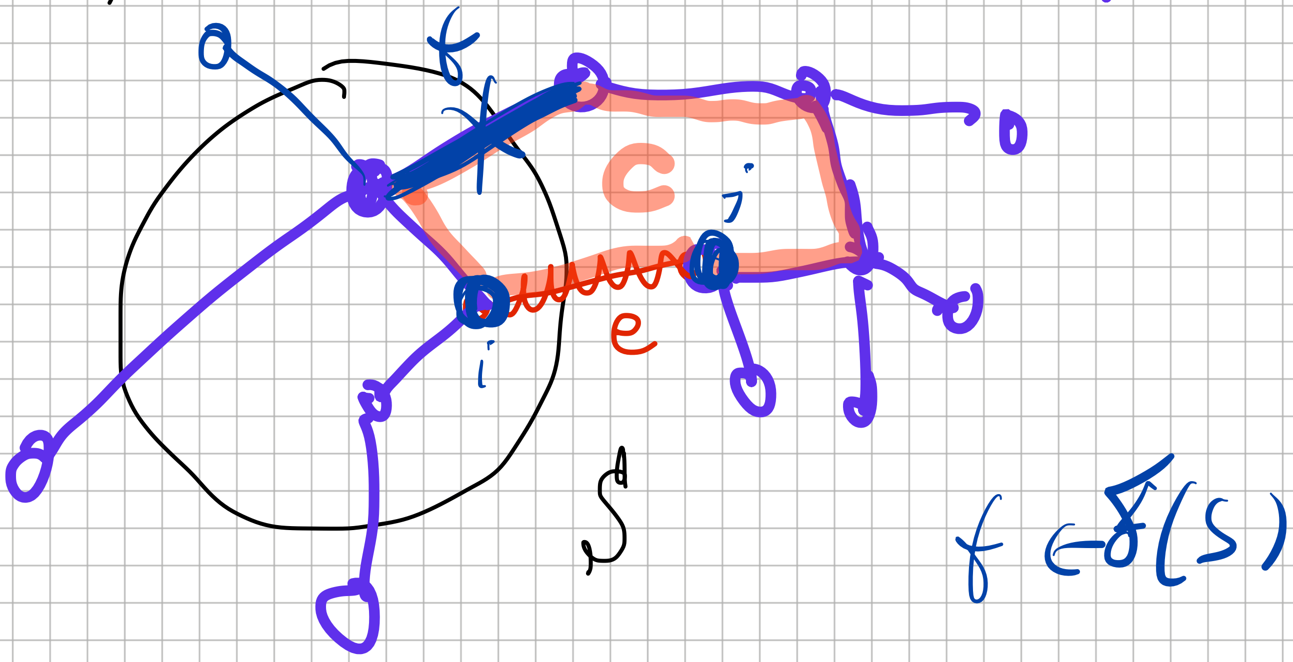

## PROOF.

① " $e = \arg\min \{ c_f : f \in \delta(S) \}$

for a certain $S \subsetneq V$, $S \neq \emptyset$,

$\Rightarrow e \in T^*$ "

By contradiction:




$T^* \cup \{e\} \setminus \{f\} \to \bar{T}$ "still a tree"

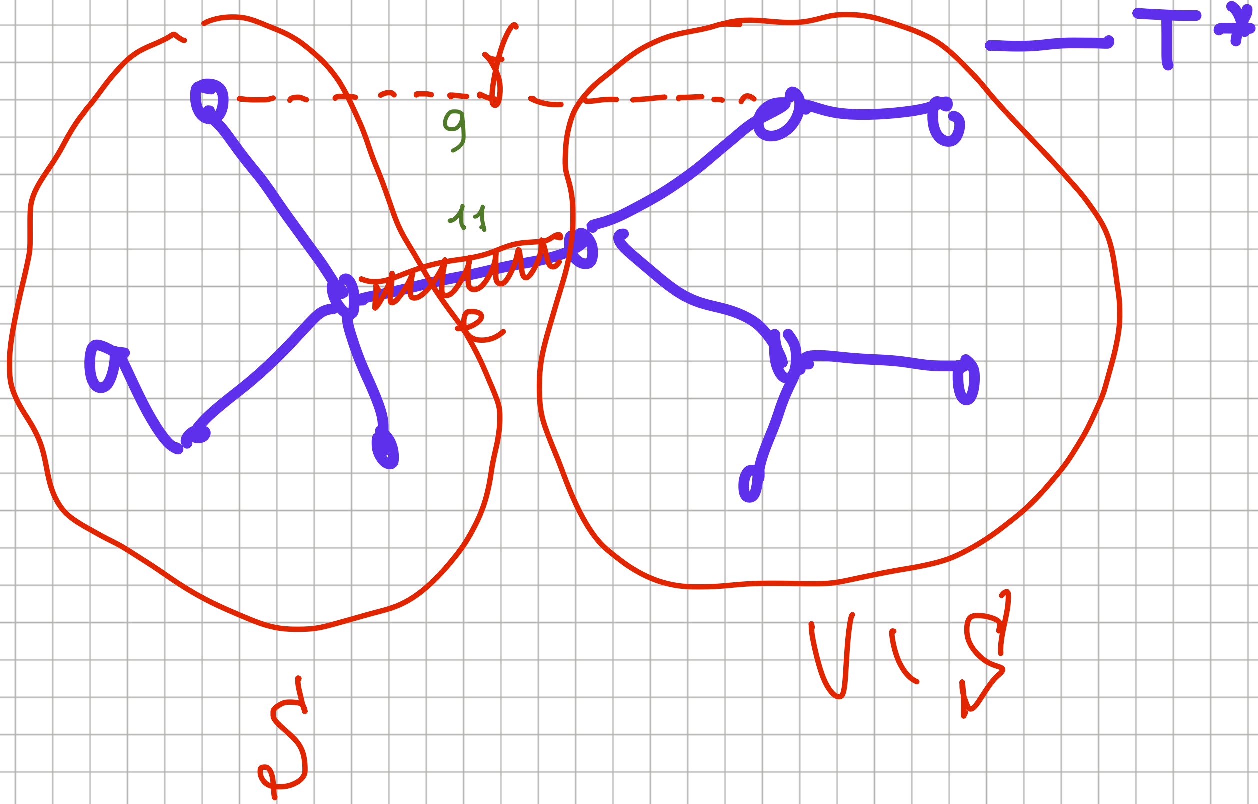
$$\boxed{\operatorname{cost}(\bar{T})} = \operatorname{cost}(T^*) + \underbrace{(c_e - c_f)}_{<0}$$

$$\boxed{< \operatorname{cost}(T^*)}$$

→ CONTRADICTION

② " $e \in T^* \Rightarrow \exists\, S \subsetneq V$ :

$e = \arg\min \{ c_f : f \in \delta(S) \}$"




if, by contradiction, $c_f < c_e$,
then $T^* \setminus \{e\} \cup \{f\} = \bar{T}$
with smaller cost $\Rightarrow$ impossible.

$\boxtimes$

A POSSIBLE SOLUTION SCHEME (based on the theorem above):