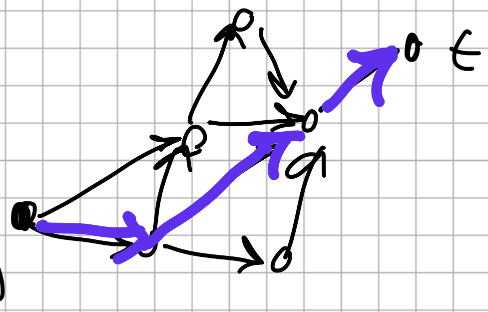
OR1 30-NOV-2021

$G = (V, A)$ directed graph

$s \in V$ source node

$t \in V$ target "



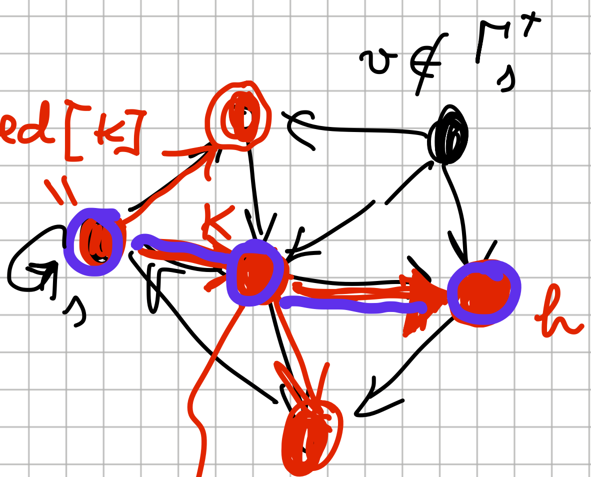
$c : A \to \mathbb{R}$

c_{ij} = cost of arc (i,j)

path $P \subseteq A$ → $cost(P) = \sum_{ij \in P} c_{ij}$

=> to be minimized

" SHORTEST PATH PROBLEM "

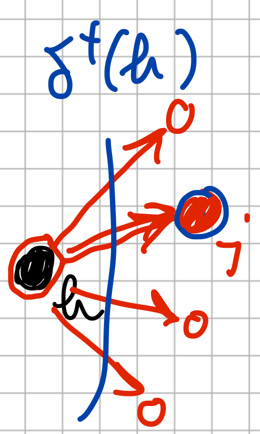


$\Gamma_s^+ = \{ v \in V : \exists \text{ path } s \to v \}$

$s \in \Gamma_s^+$ by default

pred[h] := K

PSEUDO-CODE



pred[j] := -1 , $\forall j = 1, \dots, n$

pred[s] := s ; $Q := \{s\}$;

while $Q \neq \emptyset$ do

  let $h \in Q$; $Q := Q \setminus \{h\}$;

  for each $(h,j) \in \delta^+(h)$ do

    if pred[j] = -1 then

$O(m)$

$n := |V|$

$O(m)$

$m := |A|$

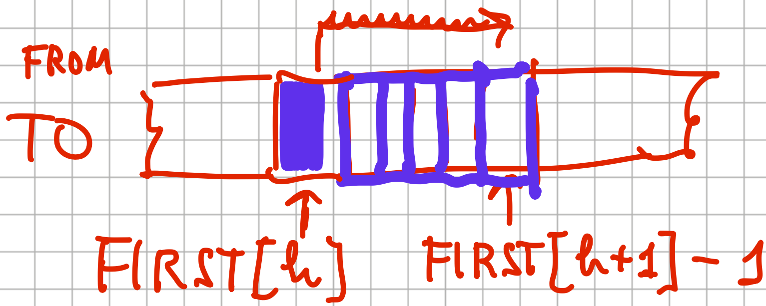
od

rof

fi

pred[j] := h ;

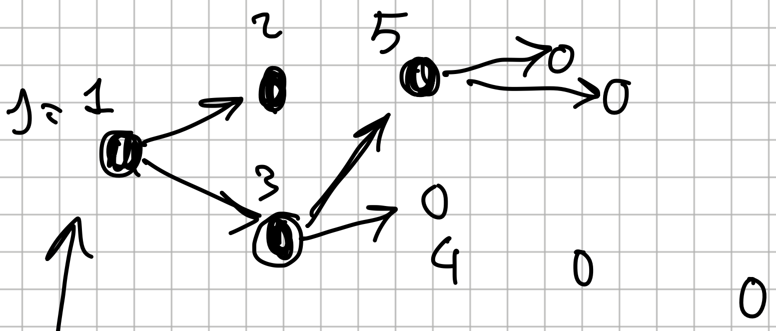
$Q := Q \cup \{j\}$



$\underbrace{\qquad\qquad}_{O(|\delta^+(h)|)}$

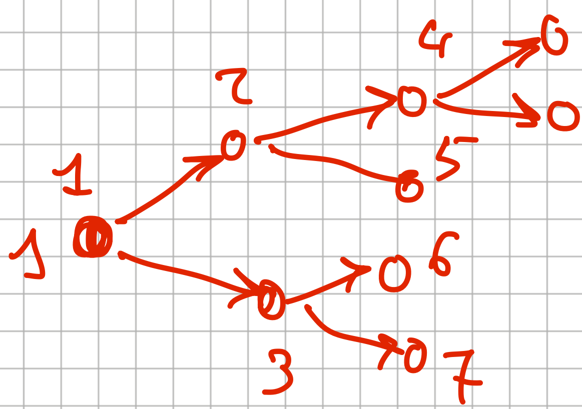
$\Rightarrow O(n+m)$ time

LINEAR in the size of the input

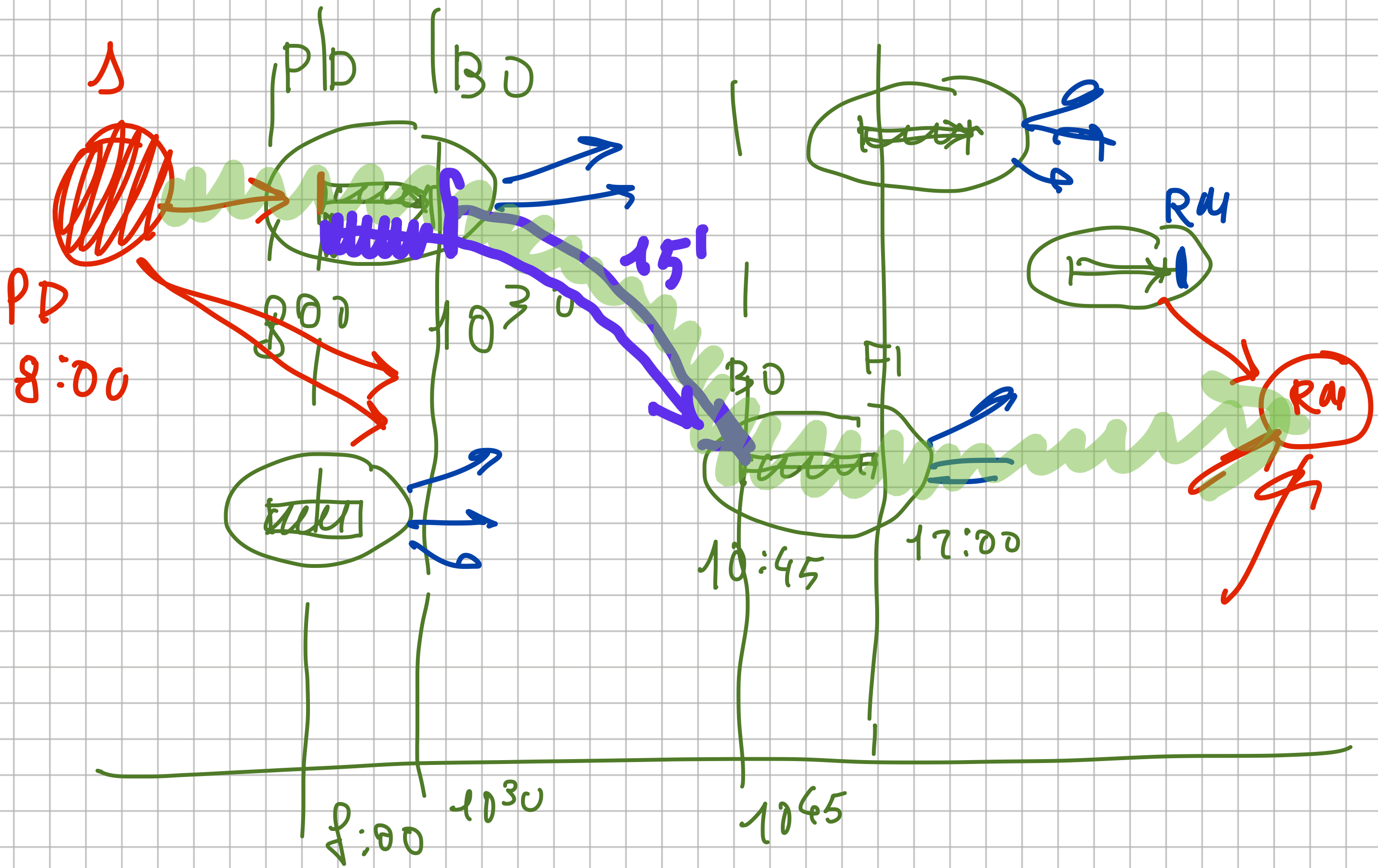


LIFO

FIFO :



SHORTEST PATH PR. (SPP)

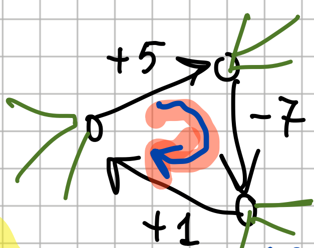


$\boxed{\text{POSITIVE COSTS}}$ $c_{ij} \geq 0$

POS/NEGATIVE COSTS "$c_{ij} < 0$ allowed"

$\boxed{\text{TIME COMPLEXITY :}}$

- $\boxed{\text{EASY}}$ if $c_{ij} \geq 0$
- EASY if $\nexists$ negative cycles



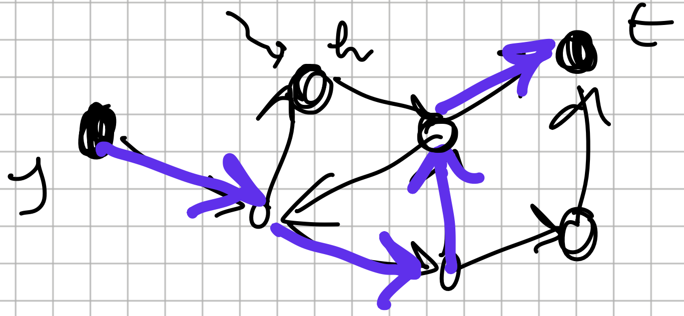
NEGATIVE CYCLE !

- HARD if $\exists$ negative cycles !!

ILP model for the SPP

"SIMPLE PATHS ONLY"

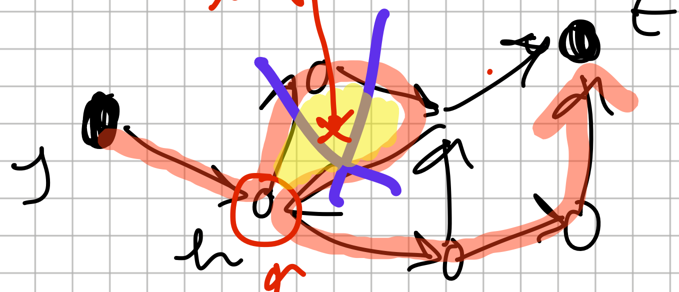
unvisited node



— SIMPLE PATH

(nodes are visited, at most, once)

can be negative!



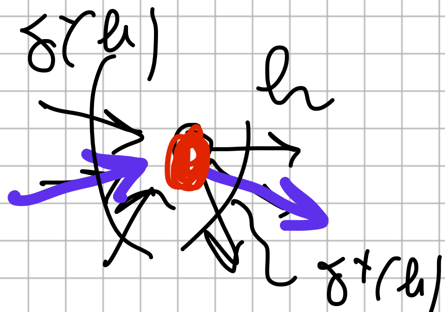
— "WALK"

NON-SIMPLE PATH

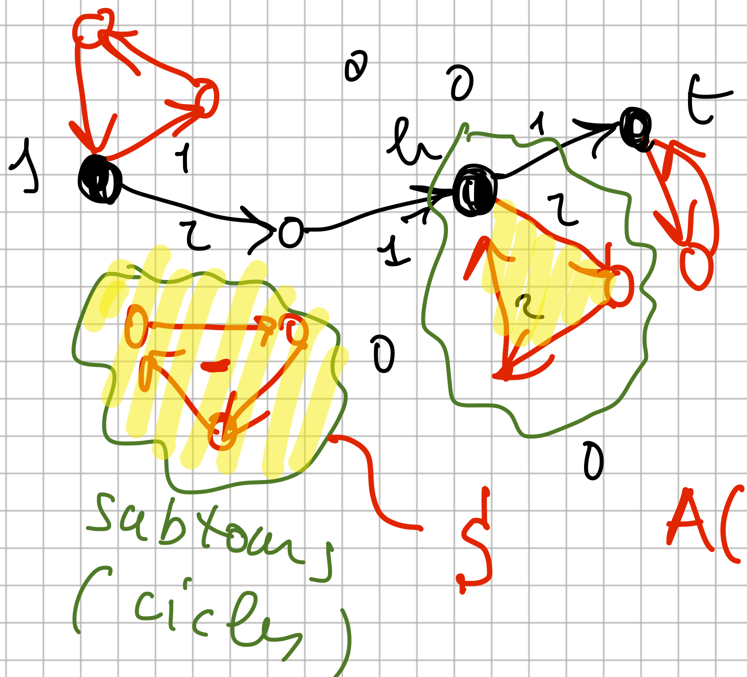
node visited twice

$\forall (i,j) \in A$:

$$x_{ij} = \begin{cases} 1 & \text{if } (i,j) \text{ is chosen in the optimal path} \\ 0 & \text{otherwise} \end{cases}$$



$$\begin{cases} \min \sum_{(i,j) \in A} c_{ij} x_{ij} \\ \underbrace{\sum_{(i,j) \in \delta^+(h)} x_{ij}}_{\text{n. of selected arcs leaving node } h} - \underbrace{\sum_{(i,j) \in \delta^-(h)} x_{ij}}_{\text{n. of selected arcs entering node } h} = \begin{cases} +1, & h = s \\ -1, & h = t \\ 0, & \forall h \in V \setminus \{s,t\} \end{cases} \\ 0 \le x_{ij} \le 1 \text{ integer}, \quad \forall (i,j) \in A \end{cases}$$



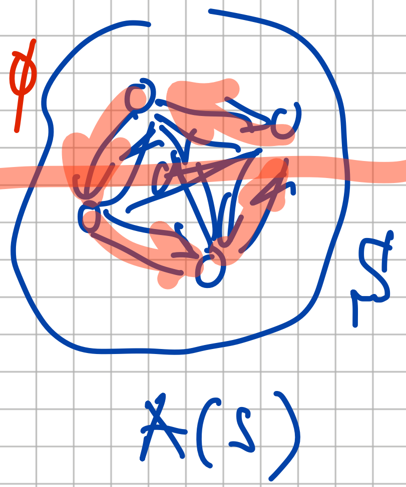
— $x_{ij} = 1$

NOT A SIMPLE PATH!

$A(S) := \{(i,j) \in A : i,j \in S\}$

SUBTOUR ELIMINATION CONSTR.S

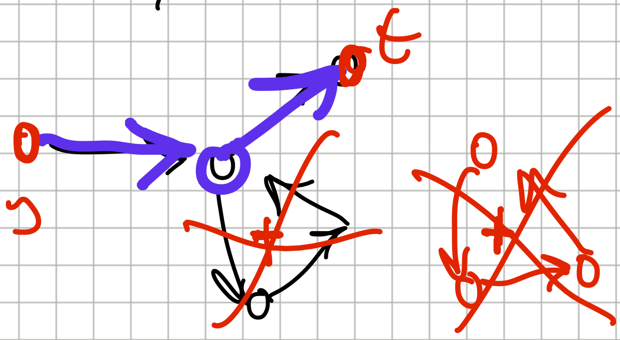
$$\boxed{\sum_{(i,j) \in A(S)} x_{ij} \leq |S| - 1, \quad \forall S \subseteq V, \; S \neq \emptyset}$$



$\approx 2^n$ constr.s !!

SPECIAL CASE:

"$\nexists$ negative-cost cycle"



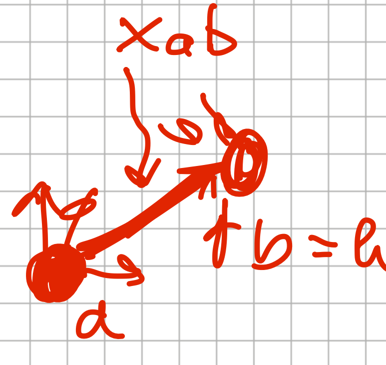
$\Rightarrow$ "IN AN OPT. SOL., YOU DON'T NEED TO FORBID SUBTOURS!"

⇒ SIMPLER MODEL WITHOUT SUBTOUR ELIM. CONST.S

$$
\begin{cases}
\min \displaystyle\sum_{(i,j)\in A} c_{ij}\, x_{ij} \\[2ex]
\underbrace{\displaystyle\sum_{(i,j)\in\delta^+(h)} x_{ij}}_{\text{n. of selected arcs leaving node } h} - \underbrace{\displaystyle\sum_{(i,j)\in\delta^-(h)} x_{ij}}_{\text{n. of selected arcs entering node } h} = \begin{cases} +1, & h=s \\ -1, & h=t \\ 0, & \forall h\in V\setminus\{s,t\} \end{cases} \\[2ex]
0 \le x_{ij} \le 1 \text{ integer}, \quad \forall (i,j)\in A
\end{cases}
$$

REDUNDANT

~~S. ELIM CONSTR.S~~



LINEAR SYSTEM:

$$A x = b \quad \text{where}$$

$$
A = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ \\ \\ n \end{array}
\begin{array}{c} x_{ab} \\ \left[\begin{array}{ccc} & 0 & \\ & 0 & \\ & +1 & \\ & 0 & \\ & -1 & \\ & 0 & \\ & 0 & \end{array}\right] \end{array}
\begin{array}{c} \\ \\ \\ a \\ \\ b \\ \\ \\ \end{array}
\qquad b = \begin{bmatrix} +1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

A = NODE-ARC INCIDENCE

$\Rightarrow$ MATRIX OF G !

$\Rightarrow$ TUM

because b is integer

"x ~~integer~~" REDUNDANT

$\Rightarrow$ "ILP $\Rightarrow$ LP" !

$\Rightarrow$ SPP can be solved efficiently !!