OR1 7-DEC-2021

NETWORK FLOWS

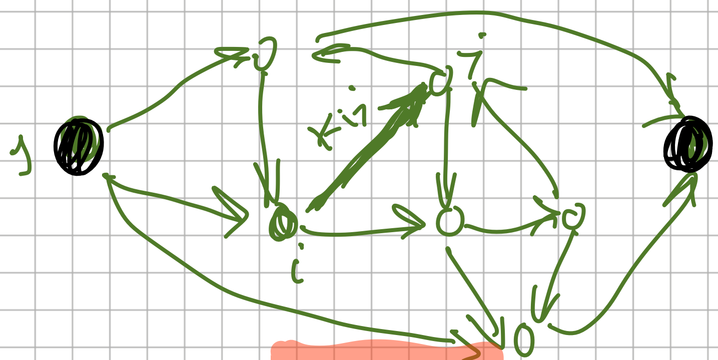
Network : $G=(V,A)$ directed graph

$$K : A \to \mathbb{R}_+$$

where $K_{ij} \geq 0$ is the CAPACITY of $(i,j) \in A$

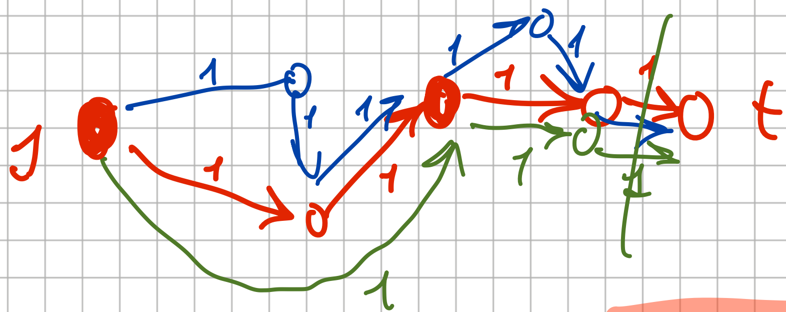
$s \in V$ SOURCE NODE

$t \in V$ TERMINAL NODE , $s \neq t$



GOOD = "GAS"

MAX-FLOW problem



"3 FLOW UNITS SENT FROM s TO t"

DEF. (FEASIBLE) FLOW from s to t on $G=(V,A)$ is

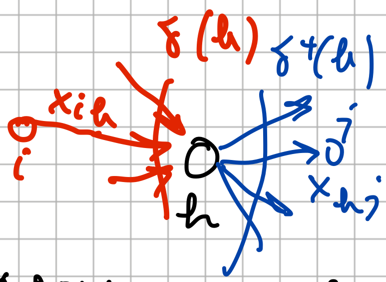
$$x : A \to \mathbb{R}$$

(so x_{ij} is the FLOW on $(i,j)\in A$):

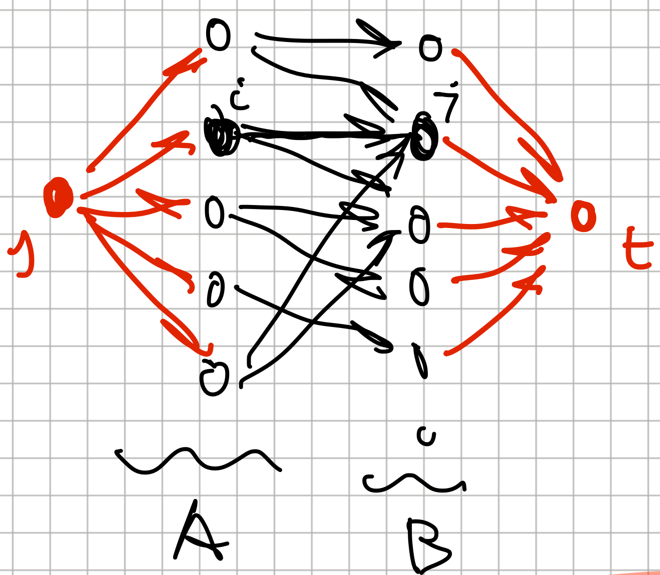
(i) $\quad 0 \le x_{ij} \le k_{ij}$, $\quad \forall (i,j)\in A$,

(ii) $\quad \underbrace{\sum_{(i,j)\in\delta^+(h)} x_{ij}}_{\text{OUT-FLOW}} - \underbrace{\sum_{(i,j)\in\delta^-(h)} x_{ij}}_{\text{IN-FLOW}} = 0$,

$\forall h \in V\setminus\{s,t\}$.



APPLICATIONS:



→ fictitious arcs

$k_{ij}=1$

MAXIMUM-FLOW problem:

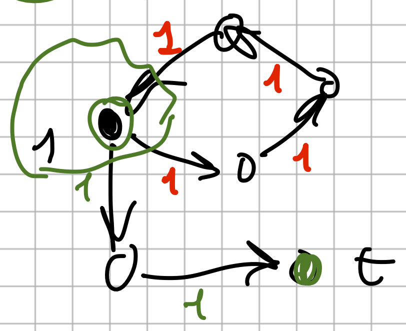
LP-problem

$$\begin{cases} \max \ \varphi_0 := \displaystyle\sum_{(i,j)\in\delta^+(s)} x_{ij} - \sum_{(i,j)\in\delta^-(s)} x_{ij} \\ \text{(i)} - \text{(ii)} \end{cases}$$

where φ_0 is the VALUE of flow x.

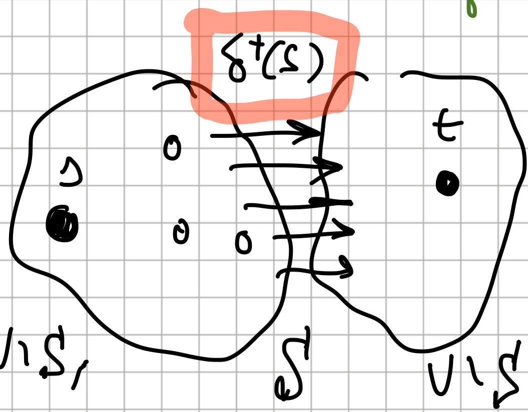
$$\min \sum_{(i,j)\in A} c_{ij} x_{ij} \quad \dots$$

MIN-COST FLOW PROBLEM



$\varphi_0 = 2 - 1 = 1$

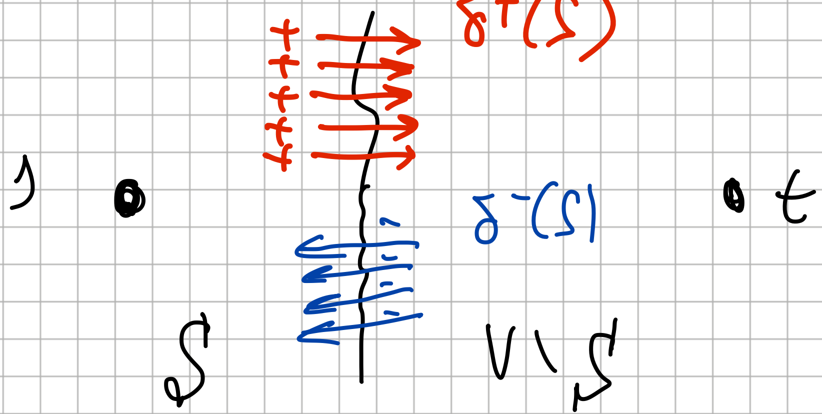
net amount from s



DEFINITIONS:

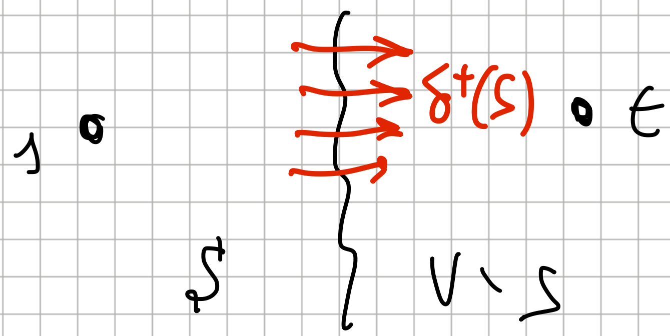
① $(S, V\setminus S)$, $s \in S$, $t \in V\setminus S$, is called a $\left\{\begin{array}{l}\text{SECTION}\\ \text{st-CUT}\\ \text{CUT}\end{array}\right.$

② FLOW through a cut $(S, V\setminus S)$:



$$\varphi(S) := \sum_{(i,j)\in\delta^+(S)} x_{ij} - \sum_{(i,j)\in\delta^-(S)} x_{ij}$$

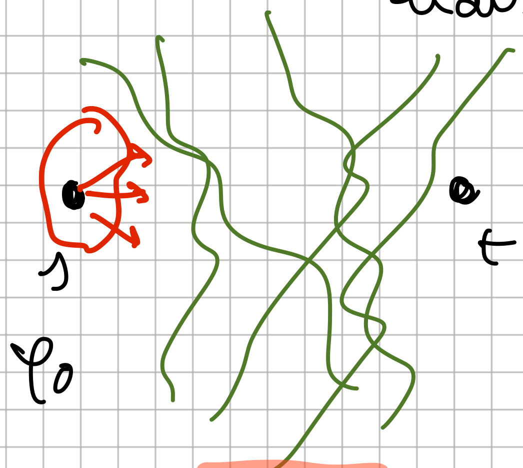
③ CAPACITY of a cut $(S, V \setminus S)$



$$\boxed{K(S) := \sum_{(i,j)\in\delta^+(S)}' K_{ij}}$$

$\boxed{\text{TH\_1}}$ Let x be a FEASIBLE flow. For every $\boxed{\text{cut}}$ $(S, V \setminus S)$ we have $\varphi(S) = \varphi_0$.

<u>Proof</u>: Take any cut $(S, V \setminus S)$:

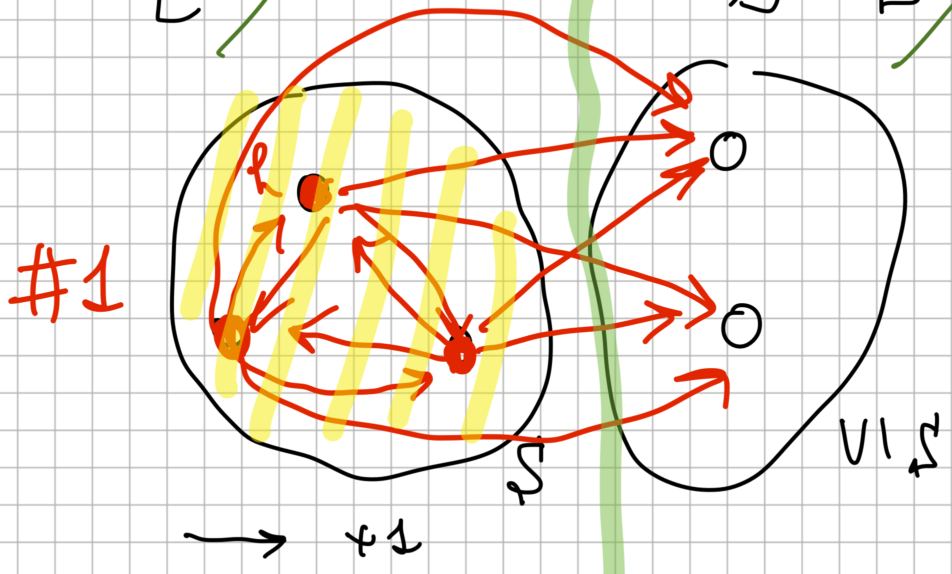


$$\boxed{\varphi_0} := \sum_{(i,j)\in\delta^+(s)}' x_{ij} - \sum_{(i,j)\in\delta^-(s)}' x_{ij} =$$

$$= \sum_{h\in S}' \underbrace{\left[\sum_{(i,j)\in\delta^+(h)} x_{ij} - \sum_{(i,j)\in\delta^-(h)} x_{ij} \right]}_{=0,\ \forall h \neq s} =$$

$$= \underbrace{\sum_{h\in S}{}' \sum_{(i,j)\in\delta^+(h)}{}' x_{ij}}_{\#1} - \underbrace{\sum_{h\in S}{}' \sum_{(i,j)\in\delta^-(h)}{}' x_{ij}}_{\#2}$$

$$= \left[\cancel{\sum_{i,j\in S} x_{ij}} + \sum_{(i,j)\in\delta^+(S)} x_{ij}\right] - \left[\cancel{\sum_{i,j\in S} x_{ij}} + \sum_{(i,j)\in\delta^-(S)} x_{ij}\right] =$$



$$\#1 = \sum_{i,j\in S} x_{ij} + \sum_{(i,j)\in\delta^+(S)} x_{ij}$$

$$\boxed{=: \varphi(S)}$$

□

$\boxed{\text{TH. 2}}$ For every feasible x and for every cut $(S, V\setminus S)$:

$$\varphi(S) \leq k(S).$$

Proof:

$$\varphi(S) := \sum_{(i,j)\in\delta^+(S)} \overset{\leq k_{ij}}{x_{ij}} - \sum_{(i,j)\in\delta^-(S)} \overset{\geq 0}{x_{ij}} \leq \sum_{(i,j)\in\delta^+(S)}{}' k_{ij} - 0$$

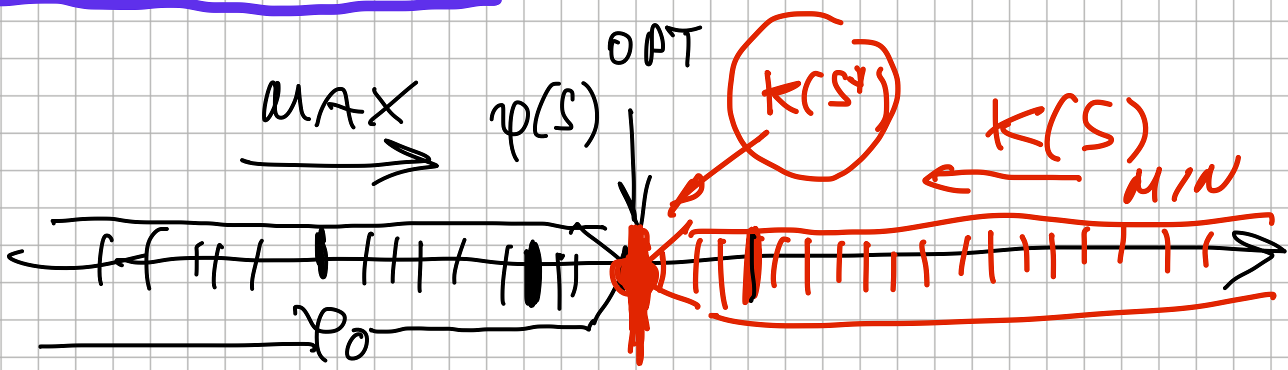
$$=: k(S)$$

□

OPTIMALITY CERTIFICATE:

$$\varphi_0 = \varphi(S^*) = \kappa(S^*)$$

DUALITY ARGUMENT:



x feasible flow $\qquad (S, V \setminus S)$

"MAX-FLOW = MIN CUT"