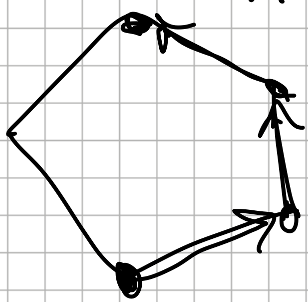
OR1 13-DEC-2021

FORD-FULKERSON alg. for MAX-FLOW

x^* OPT



feasible vertex

FLOW (FEASIBLE)

$x = 0 \quad \rightarrow \quad \varphi_0 = 0$

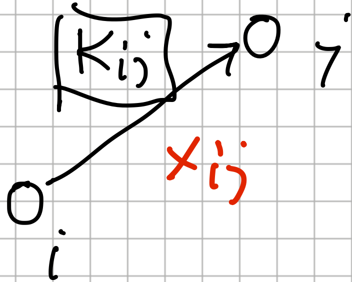
$\varphi_0 = 0 + \delta$

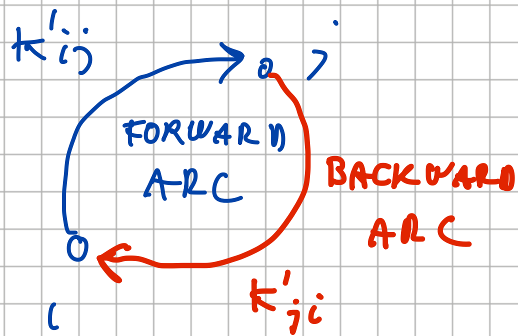
$\vdots$

x^* (PROOF OF OPT.)

CURRENT FEASIBLE FLOW x

ORIGINAL NETWORK → RESIDUAL NETWORK

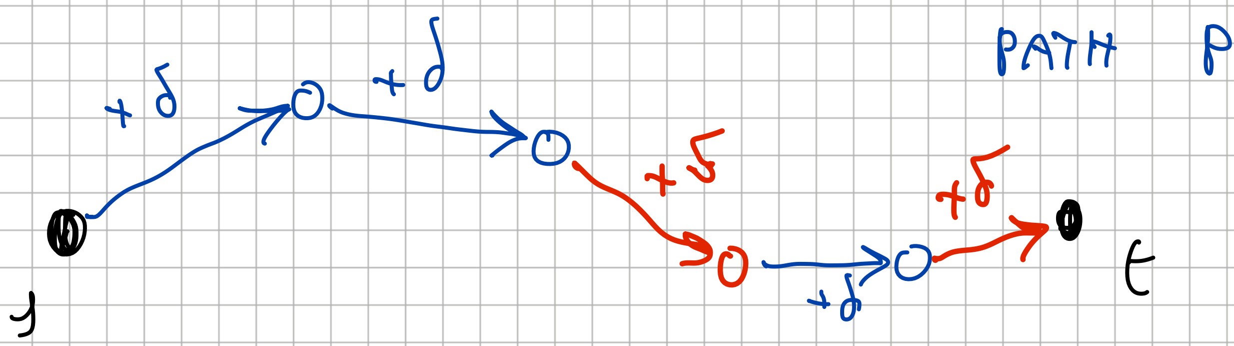




$$\left.\begin{aligned} K'_{ij} &= K_{ij} - x_{ij} \\ K'_{ji} &= x_{ij} \end{aligned}\right\}$$

ARCS WITH 0 RESIDUAL CAPACITY MUST BE REMOVED

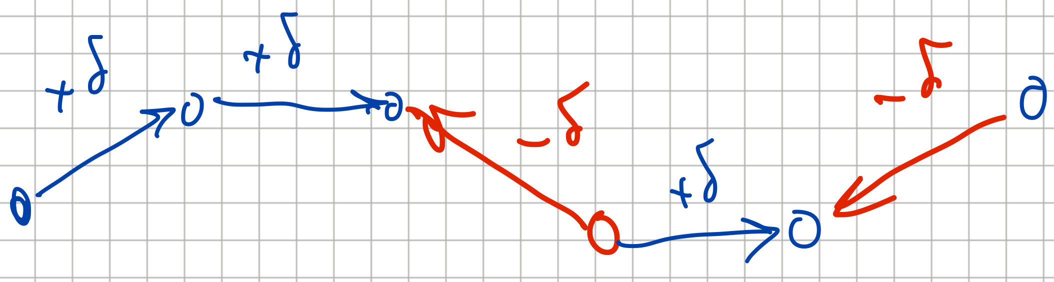
RESIDUAL NETWORK



ANY PATH from s to t in the residual network

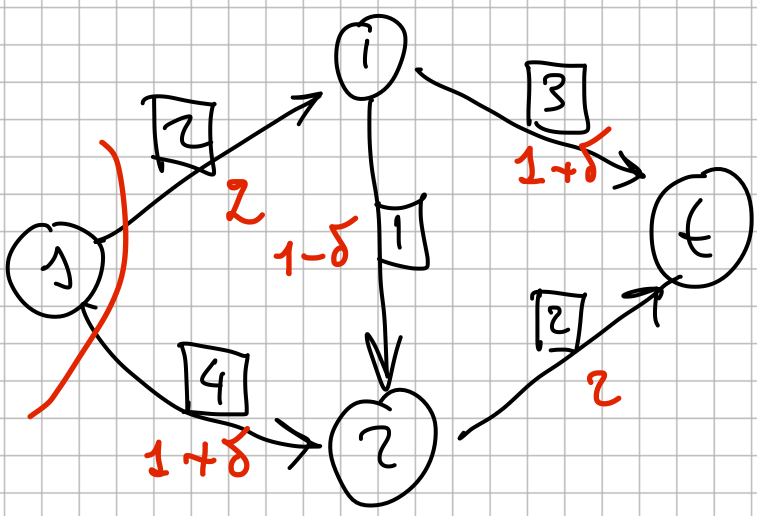
$$\delta = \min\{\, k'_{ij} : (i,j) \in P \,\}$$

ORIGINAL NETWORK



x of value φ_0 $\Rightarrow$ new flow x' of value $\varphi_0 + \delta$

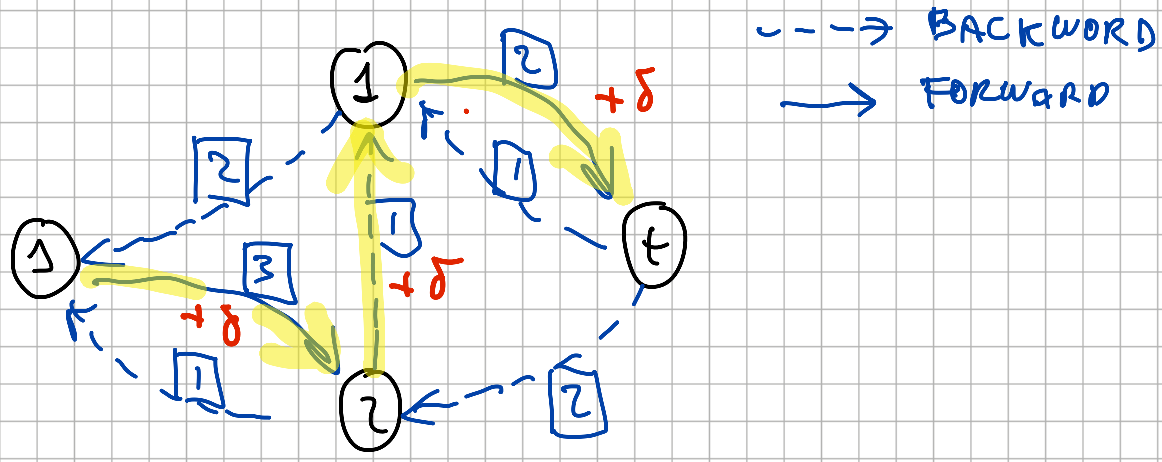
EX.



— feasible flow x

$\varphi_0 = 3$

RESIDUAL NETWORK



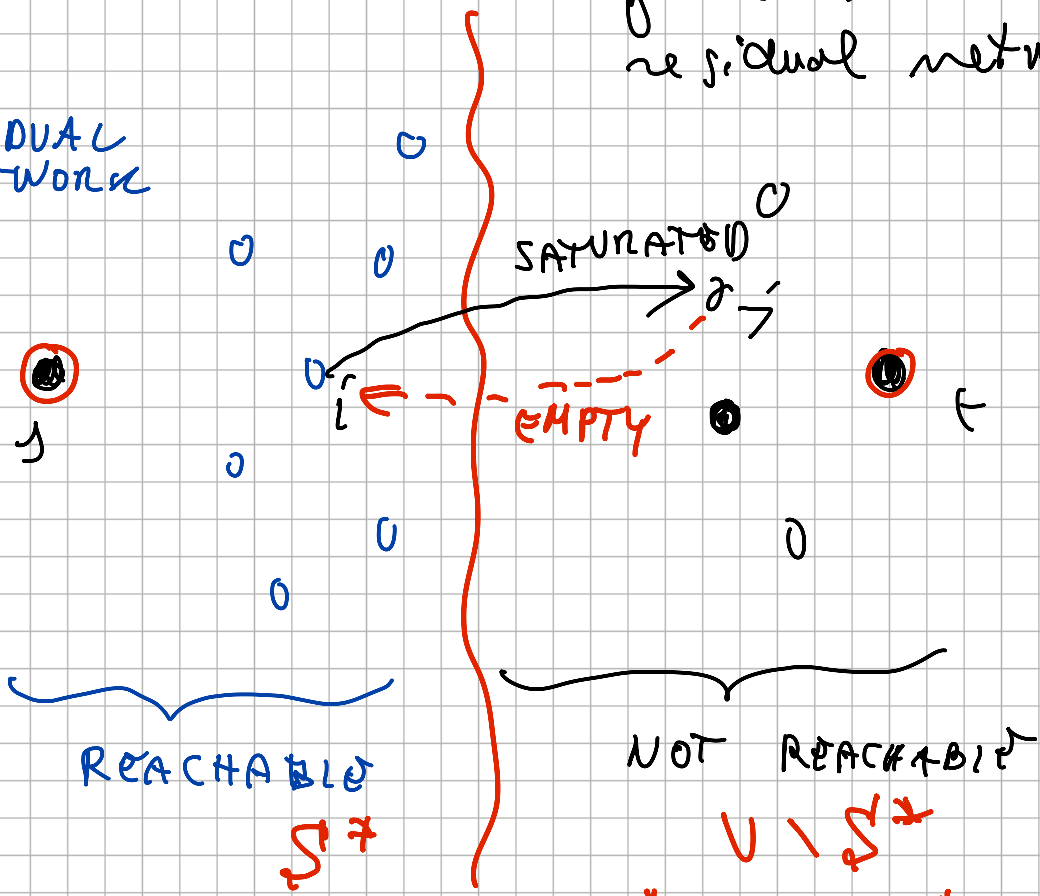
AUGMENTING PATH P

$$\delta = \min\{3, 1, 2\} = 1$$

$$\varphi_0 = 3 + \delta = 4$$

EVENTUALLY: t is NOT reachable from s in the residual network

RESIDUAL NETWORK



$s \in S^*$, $t \in V \setminus S^*$ $\Rightarrow$ $(S^*, V \setminus S^*)$ is a cut

In the residual network G',

$$\delta^+_{G'}(S^*) = \emptyset \iff$$

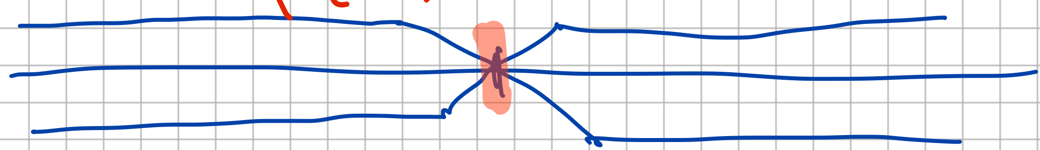
$$\boxed{\begin{array}{l} \bullet \text{ ALL ARCS } (i,j) \in \delta^+_G(S^*) \text{ ARE SATURATED, i.e., } x_{ij} = k_{ij} \\ \bullet \text{ ALL ARCS } (i,j) \in \delta^-_G(S^*) \text{ ARE EMPTY, i.e., } x_{ij} = 0 \end{array}}$$

$$\varphi_0 = \varphi(S^*) =$$

$$:= \sum'_{(i,j)\in\delta^+(S^*)} \overset{=k_{ij}}{x_{ij}} - \sum'_{(i,j)\in\delta^-(S^*)} \overset{=0}{x_{ij}}$$

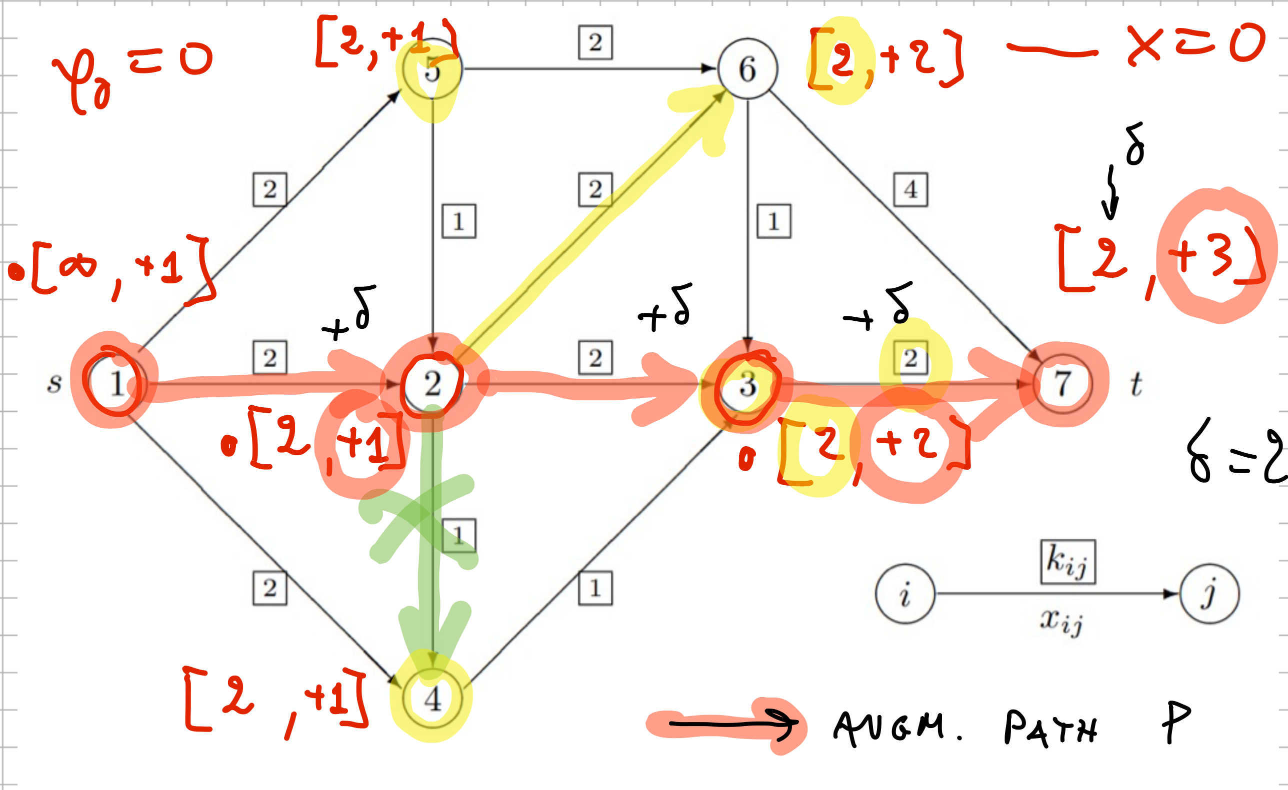
$$= \sum'_{(i,j)\in\delta^+(S^*)} k_{ij} =: k(S^*)$$

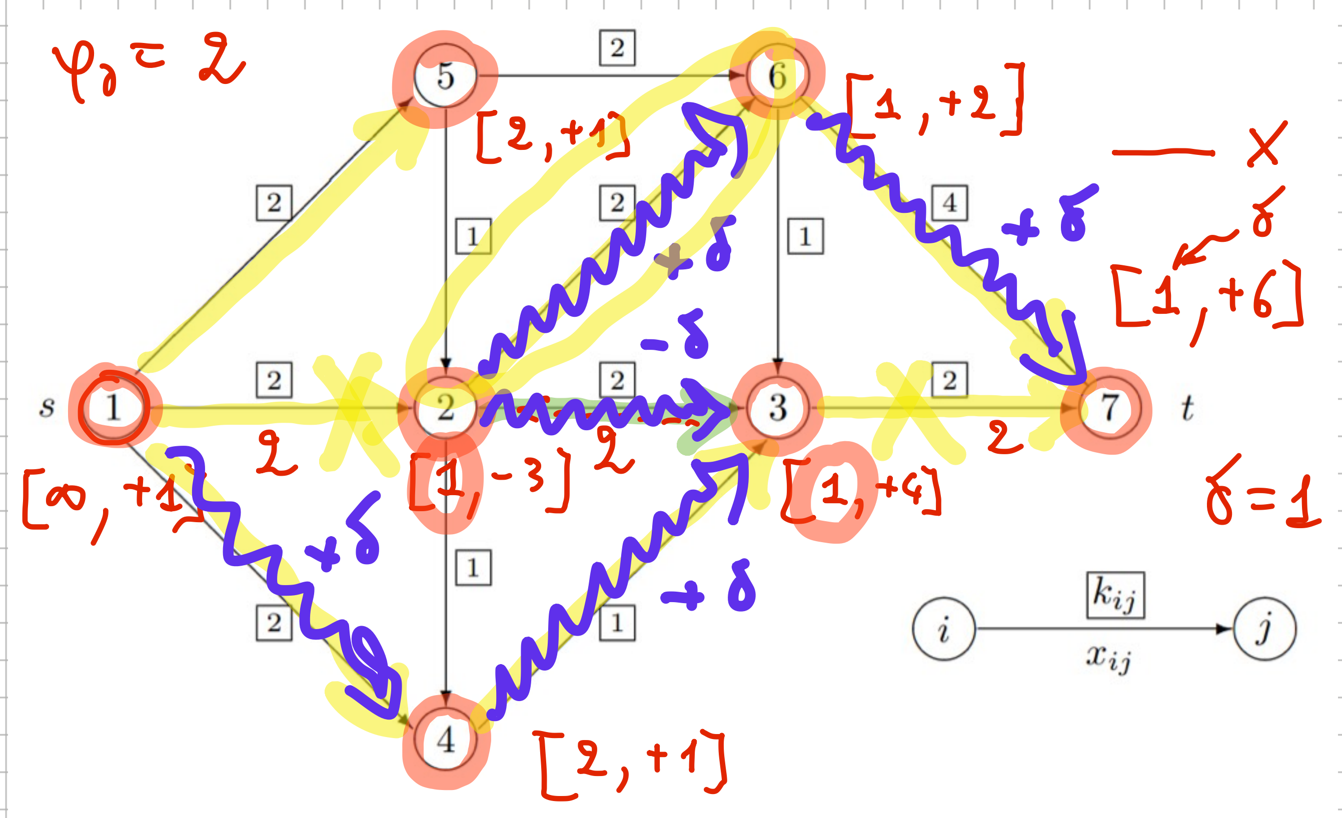
$$\varphi(S^*) = k(S^*)$$



EXAMPLE :

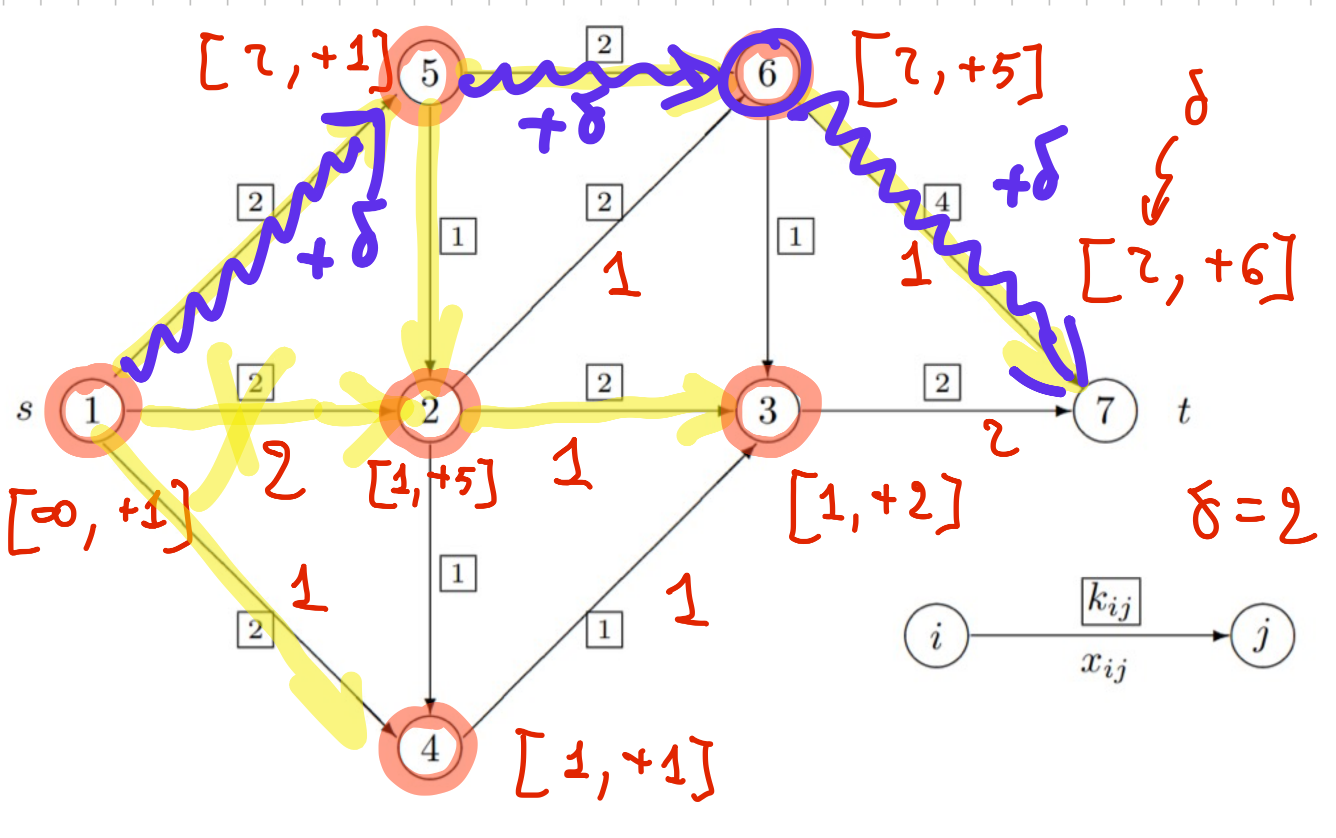
$[\delta_h, \pm \text{pred}_h]$

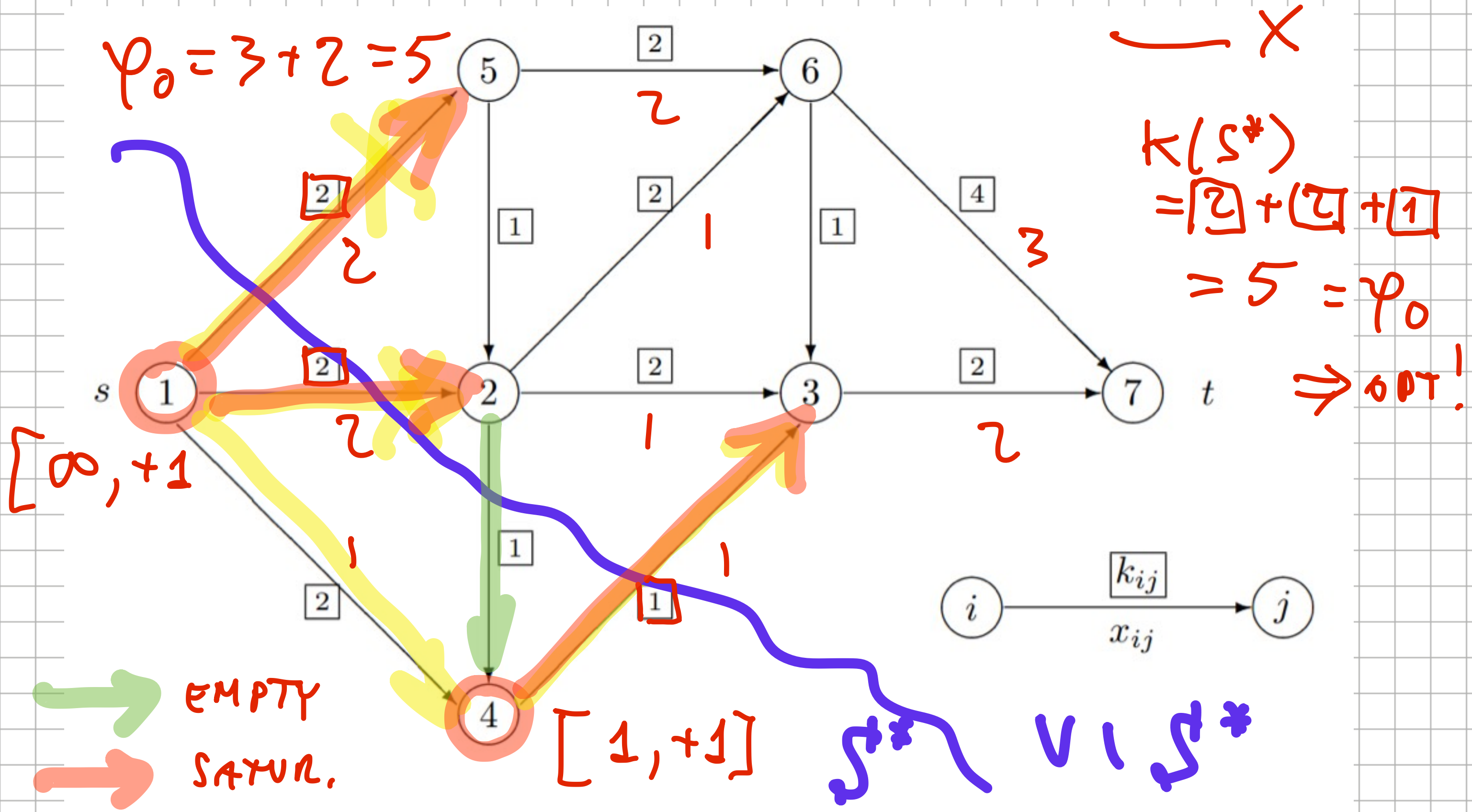




→ AUGM. PATH P

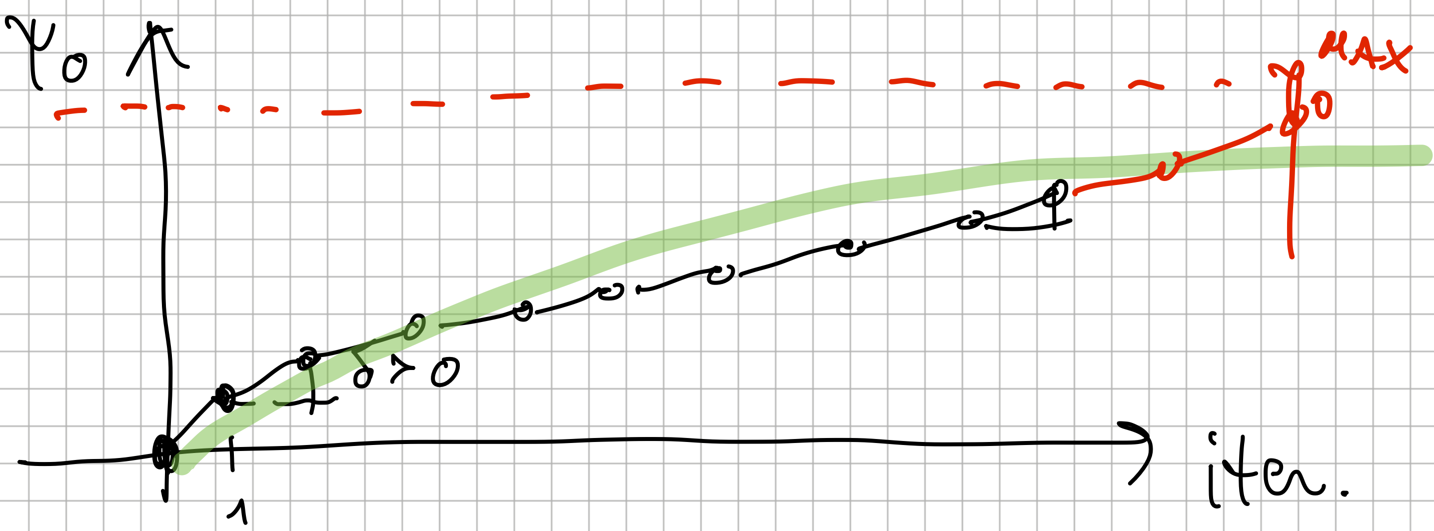
$\varphi_0 = 2 + 1 = 3$ — X





$\varphi(S^*) = \dots = 5$

FINITE CONVERGENCE

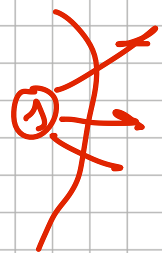


$$K_{ij} \in \mathbb{Z} \Rightarrow \delta \in \mathbb{Z} \Rightarrow \delta \geq 1$$

$\Rightarrow$ FINAL CONVERGENCE in at most, φ_0^{MAX} iterations,

where

$$\varphi_0^{MAX} \leq K(\{s\}) = \sum_{(i,j) \in \delta^+(s)} K_{ij}$$



is FINITE.

WARNING: not a POLYNOMIAL-TIME alg. ...