OR1   14-DEC-2021

NP-hard problems

- KNAPSACK problem (KP)

container CAPACITY $W > 0$

items $1, 2, \dots, n$:

item j : $w_j > 0$ WEIGHT

$p_j > 0$ PROFIT

Select $S \subseteq \{1, \dots, n\}$:

$$\sum_{j \in S} w_j \le W$$

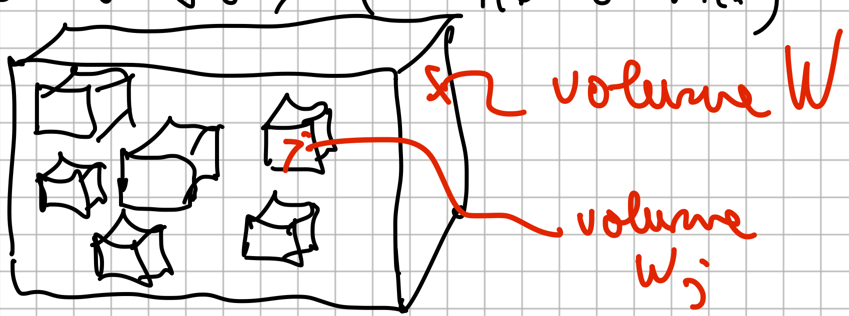
$$\sum_{j \in S} p_j \to \text{MAX.}$$

Hp: • w, W are integer

• $\sum_{j=1}^{n} w_j > W$

APPLICATIONS : ...

~~3~~-D versions (LOADING PR.)



ILP model:

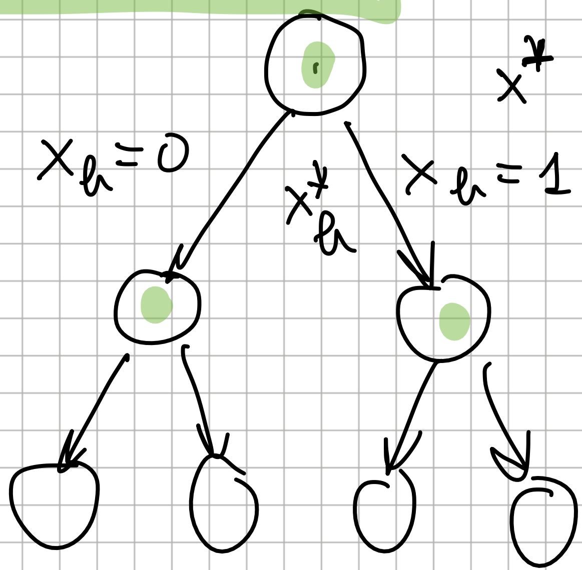
$j = 1, \dots, n$:

$$x_j = \begin{cases} 1 & \text{if item } j \text{ is loaded} \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{cases} \max \displaystyle\sum_{j=1}^{n} p_j x_j \\ \displaystyle\sum_{j=1}^{n} w_j x_j \le W \\ 0 \le x_j \le 1 \text{ integer}, \ \forall j = 1, \dots, n \end{cases}$$

" $\sum_{j \in S} p_j$ "

B & BOUND (B & ~~CUT~~)

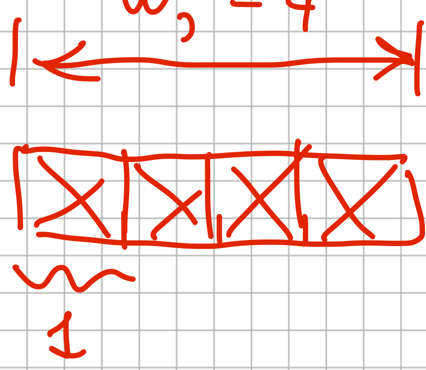


x^* LP-relax. sol.

LP-relaxation can be solved very easily ($O(n \log n)$ time) by DANTZIG's alg.

$\boxed{\text{IDEA}}$

item j



$p_j = 40$

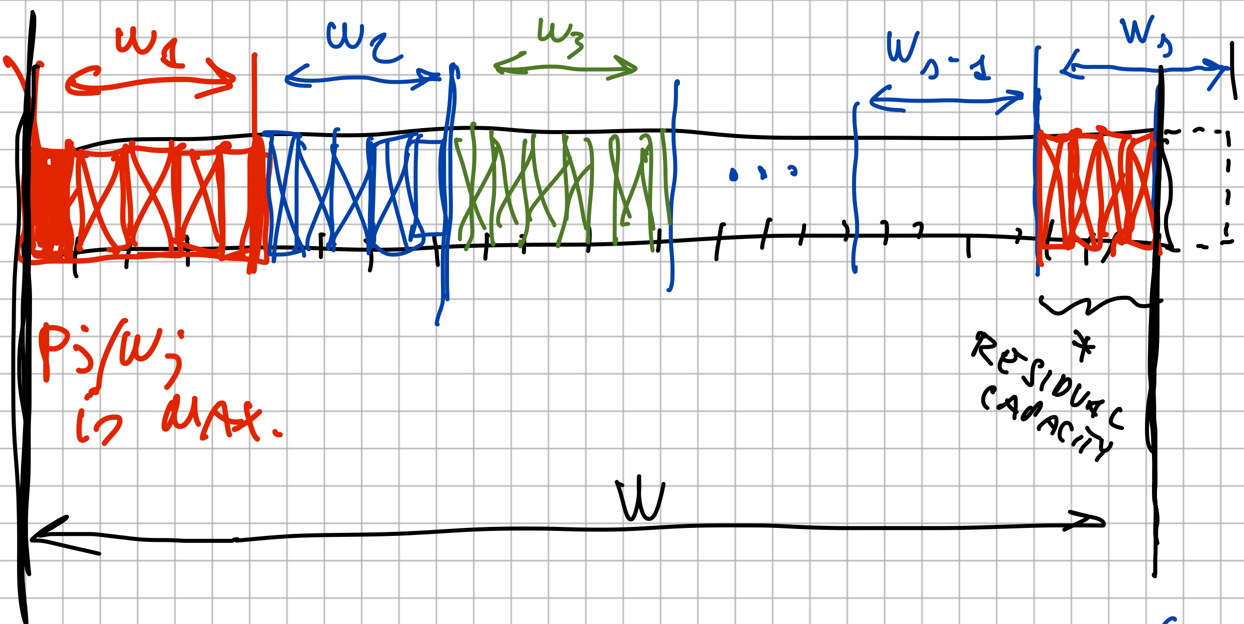
$w_j = 4$

BREAK ITEM j INTO u-ITEMS

$w = 1 \quad p = 40/4 = 10$

$\Rightarrow$ w_j u-items : weight $= 1$, profit $= p_j/w_j$

GREEDY alg: $\frac{p_1}{w_1} \geq \frac{p_2}{w_2} \geq \dots \geq \frac{p_n}{w_n}$



CRITICAL ITEM s

- SORT THE ITEMS : $\cancel{O(n \log n)}$ $O(n)$ PARTIAL SORT.
- FIND THE CRITICAL ITEM s SUCH THAT:

$$\sum_{j=1}^{s-1} w_j < W \le \sum_{j=1}^{s} w_j$$

$O(n)$ time

- OPTIMAL LP SOL. x^*:

$$x^*_1 = x^*_2 = \dots = x^*_{s-1} = 1$$

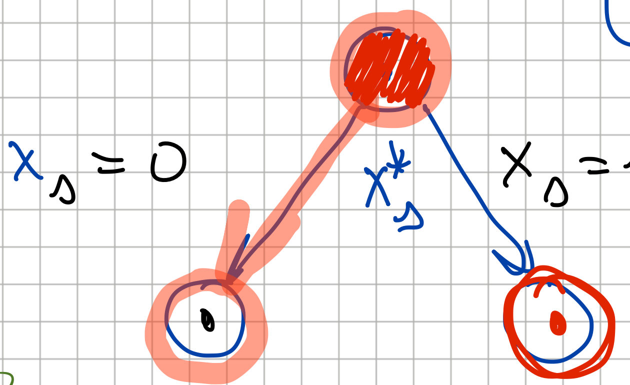
$$x^*_s = \left(W - \sum_{j=1}^{s-1} w_j\right) / w_s \quad \leftarrow \text{possibly fract. for the critical item}$$

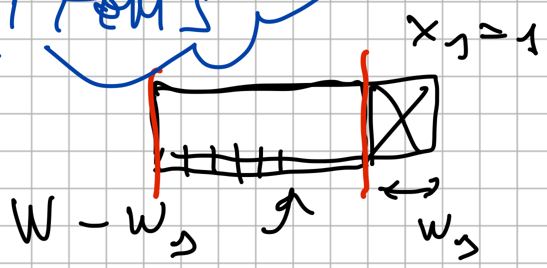
$$x^*_{s+1} = x^*_{s+2} = \dots = x^*_n = 0$$

$O(n)$ time

$\boxed{\text{B \& BOUND}}$

ROOT NODE ONLY: SORT ALL THE ITEMS





"Parametric computation of the LP solution"

move the index s to the right / left

$\Rightarrow$ ENUMERATE MILLION NODES IN VERY SHORT COMPUTING TIME !!

DYNAMIC PROGR.

Define

$$z[K,j] = \begin{cases} \text{max profit for a KP} \\ \text{with capacity } K \text{ and} \\ \text{item set } 1,\dots,j \end{cases}$$

for

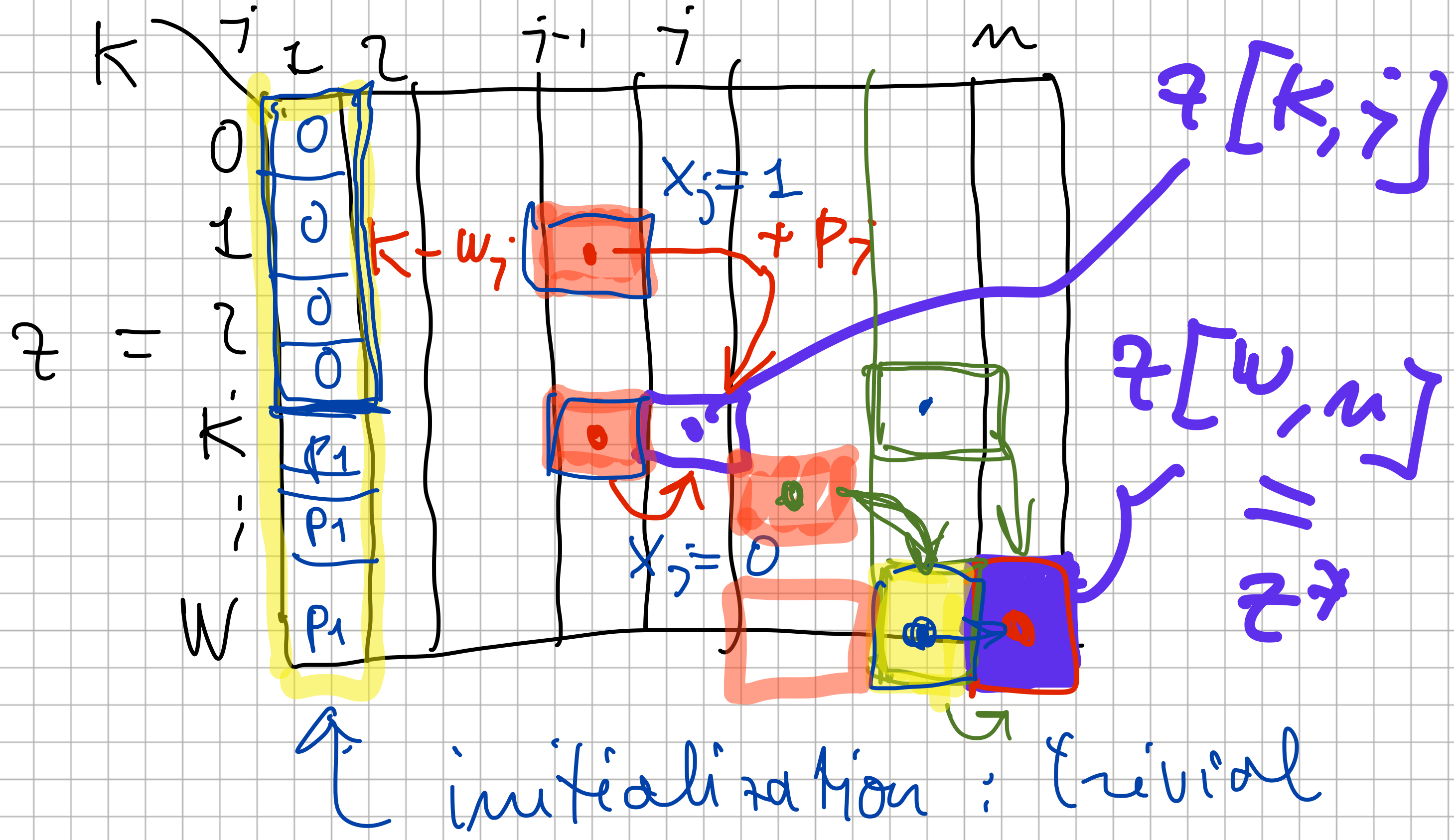
$$K = 0,1,\dots,W$$
$$j = 1,\dots,n$$

$$\rightarrow z^* = z[W,n] =$$

= opt. value of the original KP.

RECURSIVE FORMULA

$$z[K,j] = \max\Big\{\underbrace{z[K,j-1]}_{x_j=0},\ \underbrace{p_j + z[K-w_j,\, j-1]}_{x_j=1}\Big\}$$



OPTIMAL x^* → BACKTRACKING

$x_n = 0$, $x_{n-1} = 1$, $x_{n-2} = \dots$

$O(n)$ time

TOTAL COMPLEXITY: $\boxed{O(\underbrace{W}_{\text{not polynomial in the size of input}} \cdot n)}$

time (and space)

$W \approx 1{,}000 \Rightarrow$ OK

$W = 1{,}000{,}000 \Rightarrow$ TOO SLOW!!