OR1 22-dec-2021

OPERATIONAL RESEARCH 1 EXAM (Prof. Fischetti) – THEME A

1) Theory

1a) Prove the equivalence between vertices and basic feasible solutions in Linear Programming

1b) Prove the correctness of the Prim-Dijkstra algorithm for the Shortest Spanning Tree problem

1c) Write an Integer Linear Programming model for the Steiner tree problem

2) Linear Programming: Solve the LP problem associated with the following tableau:

0	|	-3	-2	0	0
3	|	2	-3	1	2
4	|	3	2	-1	1

Use the two-phase method, Bland's rule. In the end, report the optimal solution identified (or "impossible" or "unlimited" depending on the case).

3) Integer Linear Programming: solve the ILP problem associated with the following tableau, using Gomory's cutting plane algorithm. At each iteration, add to the current tableau the Gomory cut from the generating row with minimum index (skipping the row associated with the objective function), and reoptimize the tableau with the dual simplex algorithm (Bland's rule). Report all the tableaux obtained, highlighting the pivot element with a circle.

-z	4	|	0	0	0	1
x3	1/3	|	0	-13/3	1	4/3
x1	4/3	|	1	2/3	0	1/3

4) Graph Theory: Write an Integer Linear Programming model for the following problem. Let G=(V,A) be an assigned directed graph in which the arcset A is partitioned into two assigned sets A1 and A2. For each (i,j) in A, a cost c(i,j) and a weight w(i,j) are assigned. You want to find a Hamiltonian circuit of minimum overall cost (travelling salesman problem) that satisfies the following additional constraints:

7/

(i) the total cost of the zero-weight arcs chosen in A1 shall not be less than the total cost of the negative-weight arcs chosen in A2.

(ii) the average weight of the arcs chosen in A2 shall not exceed an assigned value M.

(A_1, A_2) $\qquad c_{ij} \qquad w_{ij}$

$$\sum_{(i,j) \in A_1 : w_{ij}=0} c_{ij} x_{ij}$$

" $\sum_{(i,j)\in A_1^{=}} c_{ij} x_{ij}$ " " $(i,j) \in A_1 : w_{ij} = 0$ "

where $A_1^{=} := \{(i,j) \in A_1 : w_{ij} = 0\}$

Input

SOLUTIONS

LINEAR PROGRAMMING

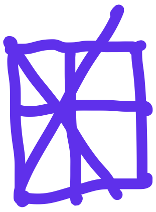
```
PHASE 1
  +---------------------------------------------------------------------------
-z|   -7   |  -5      1      0     -3      0      0
  +---------------------------------------------------------------------------
x5|    3   |   2     -3      1      2      1      0
x6|    4   |   3*     2     -1      1      0      1
  +---------------------------------------------------------------------------

  +---------------------------------------------------------------------------
-z|  -1/3  |   0    13/3   -5/3   -4/3     0     5/3
  +---------------------------------------------------------------------------
x5|   1/3  |   0   -13/3    5/3*   4/3     1    -2/3
x1|   4/3  |   1     2/3   -1/3    1/3     0     1/3
  +---------------------------------------------------------------------------

  +---------------------------------------------------------------------------
-z|    0   |   0      0      0      0      1      1
  +---------------------------------------------------------------------------
x3|   1/5  |   0   -13/5     1     4/5    3/5   -2/5
x1|   7/5  |   1    -1/5     0     3/5    1/5    1/5
  +---------------------------------------------------------------------------

PHASE II
  +---------------------------------------------
-z|  21/5  |   0   -13/5     0     9/5
  +---------------------------------------------
x3|   1/5  |   0   -13/5     1     4/5
x1|   7/5  |   1    -1/5     0     3/5
  +---------------------------------------------

  THE PROBLEM IS UNBOUNDED
```



INTEGER LINEAR PROGRAMMING

```
  +---------------------------------------------------------------------------
-z|    4   |   0      0      0      1
  +---------------------------------------------------------------------------
x3|   1/3  |   0   -13/3     1     4/3
x1|   4/3  |   1     2/3     0     1/3
  +---------------------------------------------------------------------------
Generating row:           1
  +---------------------------------------------------------------------------
-z|    4   |   0      0      0      1      0
  +---------------------------------------------------------------------------
x3|   1/3  |   0   -13/3     1     4/3     0
x1|   4/3  |   1     2/3     0     1/3     0
x5|  -1/3  |   0    -2/3*    0    -1/3     1
  +---------------------------------------------------------------------------

  +---------------------------------------------------------------------------
-z|    4   |   0      0      0      1      0
  +---------------------------------------------------------------------------
x3|   5/2  |   0      0      1     7/2  -13/2
x1|    1   |   1      0      0      0      1
x2|   1/2  |   0      1      0     1/2   -3/2
  +---------------------------------------------------------------------------
 Generating row:           1
  +---------------------------------------------------------------------------
-z|    4   |   0      0      0      1      0      0
  +---------------------------------------------------------------------------
x3|   5/2  |   0      0      1     7/2  -13/2     0
x1|    1   |   1      0      0      0      1      0
x2|   1/2  |   0      1      0     1/2   -3/2     0
x6|  -1/2  |   0      0      0    -1/2   -1/2*    1
  +---------------------------------------------------------------------------
  +---------------------------------------------------------------------------
-z|    4   |   0      0      0      1      0      0
  +---------------------------------------------------------------------------
x3|    9   |   0      0      1     10      0    -13
x1|    0   |   1      0      0     -1      0      2
x2|    2   |   0      1      0      2      0     -3
x5|    1   |   0      0      0      1      1     -2
  +---------------------------------------------------------------------------
```

$$\min \sum_{(i,j)\in A} c_{ij}\, x_{ij} \qquad \text{“}\cancel{x_{ij}\, c_{ij}}\text{”}$$

$$\sum_{(i,j)\in\delta^+(h)} x_{ij} = \sum_{(i,j)\in\delta^-(h)} x_{ij} = 1, \quad \forall\, h\in V$$

(*) SEC's

“$0 \le x_{ij} \le 1$ integer”

$$x_{ij}\in\{0,1\}, \quad \forall\,(i,j)\in A$$

(i) $$\sum_{(i,j)\in A_1 :\, w_{ij}=0} c_{ij}\, x_{ij} \ \geq \sum_{(i,j)\in A_2 :\, w_{ij}<0} c_{ij}\, x_{ij}$$

(ii) $$\sum_{(i,j)\in A_2} w_{ij}\, x_{ij} \le M \sum_{(i,j)\in A_2} x_{ij}$$

$$\cancel{\text{“}\sum_{(i,j)\in A_2} (w_{ij} - M)\, x_{ij} \le 0\text{”}}$$

$$* \quad \sum_{(i,j)\in A(S)} x_{ij} \le |S|-1, \quad \forall\, S \subsetneq V : |S|\geq 2$$

1) Theory ($\geq 6/10$ to pass to exam) Time: 1.5 h

1.a Define the total unimodularity of a matrix

1.b Prove the validity of Gomory cuts

1.c Write an ILP model for the traveling salesman problem

2) (Linear Programming) Consider the following LP

$$\begin{aligned}
\min\ & 1x_1 + 3x_2 + 2x_3 \\
& 2x_1 + x_2 \geq 8 \\
& 4x_1 + 2x_2 + x_3 \geq 12 \\
& 2x_1 + 1x_2 - x_3 \leq 6 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}$$

2.1) Solve it using primal simplex method (two phase method Bland's rule) Report all the tableaux and highlight with a circle each pivot element

2.2) Write the corresponding LP dual problem.

3) (Modeling) Write an ILP model for Steiner tree problem. with the following additional constraints.

(a) At least half of the nodes of the graph must be covered (including the root)

(b) Given two distinct arcs a and b, if a selected then also b must be selected

$$\underbrace{1 + \sum_{h \in V,\ h \neq r} \ \sum_{(i,j) \in \delta^-(h)} x_{ij}}_{\text{tot. n. of covered nodes}} \geq \left\lceil \frac{|V|}{2} \right\rceil$$

BIG-M

$h \in V \setminus \{r\}$

$x_a \leq x_b$

$x_a = 1 \Rightarrow x_b = 1$

POLATO ANNA

1.a) A matrix A of size $m \times n$, ~~(m < n)~~, is TUM (totally unimodular) if and only if, for each squared submatrix Q of A, of any order, $\det(Q) = \{-1, 1, 0\}$

1.b) Gomory cuts: given ~~the~~ a ILP problem suppose x^* to be ~~the~~ an optimal solution vertex for the continuous relaxation of the problem; if x^* is fractional, then consider x^*_h ~~the~~ as any fractional component of x^*.
We call generating row the following row i of the optimal tableau

$$x_h + \sum_{\substack{j=1 \\ j \neq h}}^{m} \bar{a}_{ij} x_j = \bar{b}_i \quad (a)$$

We apply the chvàtal ~~inequality~~ procedure:

$$x_h + \sum \lfloor \bar{a}_{ij} \rfloor x_j \leq \lfloor \bar{b}_i \rfloor \quad (b)$$

→ we obtain the cut valid $\forall x_j$ but violated by x_h

→ we transform in standard form adding the variable x_{m+1} (new variable)

$$x_h + \sum_{\substack{j=1 \\ j \neq h}}^{m} \lfloor \bar{a}_{ij} \rfloor x_j + x_{m+1} = \lfloor \bar{b}_i \rfloor \quad (c)$$

The new constraint is obtained making (c) - (a) and can be added to the tableau.
With this procedure we iteratively add new constraints, valid for all x_j but x_h fractional chosen.

1.c) Given $G = (V, A)$, let

$$x_{ij} = \begin{cases} 1 & \text{if arc } (i,j) \text{ is considered in the Hamiltonian cycle} \\ 0 & \text{otherwise} \end{cases}$$

c_{ij} = cost of arc (i,j)

$$\begin{cases} \min \sum'_{(i,j)\in A} c_{ij} x_{ij} \\ \sum'_{(i,h)\in \delta^-(h)} x_{ih} = 1 & \forall h \in V \\ \sum'_{(h,j)\in \delta^+(h)} x_{hj} = 1 & \forall h \in V \\ \sum'_{(i,j)\in \delta^+(S)} x_{ij} \geq 1 & \forall S \subset V : 1 \in S \\ x_{ij} \geq 0, \text{ integer} & \cancel{i = \{1,\dots,n\}}\ \cancel{j = \{1,\dots,n\}} \quad (i,j)\in A \end{cases}$$

2.1)

		x_1	x_2	x_3	x_4	x_5	x_6
$-z$	0	1	3	2	0	0	0
	8	2	1	0	-1	0	0
	12	4	2	1	0	-1	0
	6	2	1	-1	0	0	1

PHASE 1

ARTIF. VAR.S

?

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
$-w$	-26	-8	-4	0	1	1	-1	0	0	0
x_7	8	2	1	0	-1	0	0	1	0	0
x_8	12	4	2	1	0	-1	0	0	1	0
x_9	6	2	1	-1	0	0	1	0	0	1

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		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
$-w$	-2	0	0	2	1	-1	-1	0	2	0
x_7	2	0	0	$-\frac{1}{2}$	-1	$\frac{1}{2}$	0	1	$-\frac{1}{2}$	0
x_1	3	1	$\frac{1}{2}$	$\frac{1}{4}$	0	$-\frac{1}{4}$	0	0	$\frac{1}{4}$	0
x_9	0	0	0	$+\frac{3}{2}$	0	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	1

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
$-w$	-2	0	0	-1	1	0	1	0	1	2
x_7	2	0	0	1	-1	0	-1	1	0	-1
x_1	3	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$
x_5	0	0	0	-3	0	1	2	0	-1	2

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
$-w$	0	0	0	0	0	0	0	1	1	1
x_3	2	0	0	1	-1	0	-1	1	0	-1
x_1	4	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0
x_5	6	0	0	0	-3	1	-1	3	-1	-1

Phase 2 :

$$z = x_1 + 3x_2 + 2x_3$$

$$= 4 - \frac{1}{2}x_2 + \frac{1}{2}x_4 + 3x_2 + 2\underbrace{(2 + x_4 + x_6)}_{= x_3}$$

$$= 8 + \frac{5}{2}x_2 + \frac{5}{2}x_4 + 2x_6$$

non basic

2 (continue)

		x_1	x_2	x_3	x_4	x_5	x_6
$-z$	-8	0	$\frac{5}{2}$	0	$\frac{5}{2}$	0	2
x_3	2	0	0	1	-1	0	-1
x_1	4	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	0
x_5	6	0	0	0	-3	1	-1

$\bar{c}_j \geq 0$

$\Rightarrow$ STOP

~~$x_1 = 4$, $x_2 = 0$~~ ...

2.2)

$$\begin{cases} \max & 2u_1 + 12u_2 + 6u_3 \\ & u_1 \geq 0 \\ & u_2 \geq 0 \\ & u_3 \leq 0 \\ & 2u_1 + 4u_2 + 2u_3 \leq 1 \\ & u_1 + 2u_2 + u_3 \leq 3 \\ & u_2 - u_3 \leq 2 \end{cases}$$

3)

Given $G = (V, A)$, $T \subset V$ target set, $r \in V$ root

$$\begin{cases} \min \sum_{(i,j)\in A} c_{ij} x_{ij} \\ \sum_{(i,h)\in\delta^-(h)} x_{ih} \begin{cases} = 1 & \forall h \in T \quad (T \subset V) \\ = 0 & \text{if } h = r \quad (r = \text{root}) \\ \leq 1 & \forall h \in V \setminus (T \cup \{r\}) \end{cases} \\ \sum_{(i,j)\in\delta^+(S)} x_{ij} \geq \sum_{(i,t)\in\delta^-(t)} x_{it} \quad \forall S \subset V : r \in S; \ \forall t \in \underbrace{(V \setminus S)}_{t \in V \setminus S} \\ \sum_{(i,j)\in A} x_{ij} \geq \left\lceil \frac{1}{2} |V| \right\rceil - 1 \\ x_a \leq x_b \\ x_{ij} \geq 0, \text{ integer } (i,j) \in A \end{cases}$$

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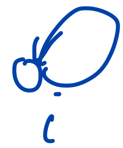
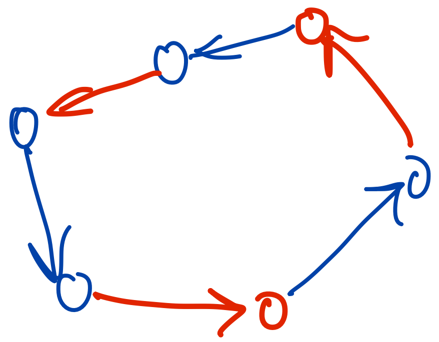
$G = (V, A)$ TSP solution

$$(A_1, A_2) = A$$

"no loops"



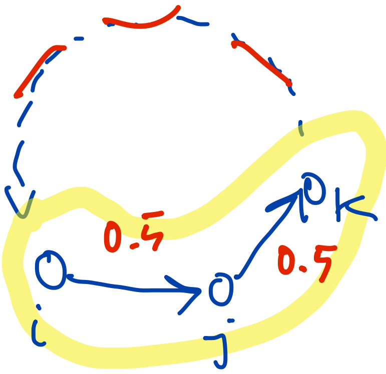


$|V|$ even

RULE: Alternate between arcs in A1 (blue) and A2 (red)

"1st idea"

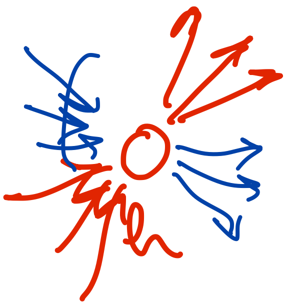
MINIMAL VIOLATION OF THE RULE



1) $x_{ij} + x_{jk} \le 1, \quad \forall\, i,j,k \in V:$
$(i,j) \in A_1,\ (j,k) \in A_1,$
$|\{i,j,k\}| = 3$

2) $x_{ij} + x_{jk} \le 1, \quad \forall\, i,j,k \in V:$
$(i,j) \in A_2,\ (j,k) \in A_2$

OR, BETTER:



$$\sum_{(i,j) \in \delta^-(h) \cap A_1} x_{ij} = \sum_{(i,j) \in \delta^+(h) \cap A_2} x_{ij}, \quad \forall\, h \in V$$

TELEMATIC EXAMINATION MODE OF THE COURSE
OPERATIONS RESEARCH 1
(PROF. FISCHETTI)

- The exam will be held electronically using the ZOOM platform; students must participate in the meeting by using their University e-mail account name.surname@studenti.unipd.it

- The link of the ZOOM meeting will be sent through UNIWEB to all the student enrolled in the exam.

- At the agreed start time, all students enrolled in the exam must be present at the ZOOM meeting and will be identified by the teacher via camera and valid identification document.

- A PDF file with the exam assignment will be made available to all participants at the beginning of the exam; the file cannot be shared (not even at a later time) and **must be cancelled after 15 minutes from the beginning of the exam**--hence the students must copy it to their own white sheets.

- **After the first 15 minutes, the students cannot touch the keyboard nor look at the screen of their own PC/notebook/tablet/smartphone (unless explicitly requested by the teacher).**

- Students will have 90 minutes to carry out the task (in total autonomy and without the help of any teaching material), only using their own white sheets, a pen, and a basic calculator (no smartphone or alike).

- During the exam, students will be constantly monitored through the camera, which must be positioned **laterally** (not in front of the student) and give a **lateral view of the student's hands and head**; at any time, both the camera and the **microphone** must be on.

- At the end of the agreed time, each student till scan his/her "final sheets" in a **single PDF file** and send it by e-mail to the address

  esame.fischetti@gmail.com

  using his/her own institutional e-mail account name.surname@studenti.unipd.it. This operation will be monitored by the teacher via camera: the student will be allowed to disconnect only when the teacher has confirmed that he has received the e-mail with the attachment.

- For scanning the exam sheets in a single PDF file, it is recommended to install an app like CamScanner on a smartphone and to test it before the exam**: no photos in jpeg or similar format will be evaluated.**

- The meeting will be registered to identify, even later, any suspicious or unacceptable behavior.

- At the discretion of the teacher, some students may be asked for an oral integration covering all the course's topics.

Padua, January 7th, 2021 Prof. Matteo Fischetti