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ALGEBRAIC REALIZATION THEORY OF TWO-DIMENSIONAL FILTERS

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The purpose of this communication is to provide a first insight into the problem of getting a recursive structure for two-dimensional filters via the algebraic realization theory. The line undertaken here has several points of contact with the algebraic realization theory of bilinear maps [1,2].

1. External representation and Nerode equivalence classes

The external representation of a two-dimensional filter is defined as:

(1) $$\mathcal{S} \triangleq (T_1 \times T_2, U, \mathcal{U}, Y, \mathcal{Y}, F)$$

where:

- $T_1 = T_2 = Z$
- $U = Y = K$ arbitrary field
- $\mathcal{U}, \mathcal{Y}$ are sets of generalized formal power series in two variables over K:

$$r = \sum_{-k = i,j}^{\infty} (r, z_1^i z_2^j)\, z_1^i z_2^j, \qquad \text{for some integer } k.$$

- $F: \mathcal{U} \to \mathcal{Y}$ (input-output map): it satisfies:
  - (i) linearity
  - (ii) two-dimensional shift invariance:

$$F(z_1^i z_2^j r) = z_1^i z_2^j F(r), \qquad \forall\, i,j \in Z$$

  - (iii) two-dimensional proper causality:

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$$(u_1, z_1^i z_2^j) = (u_2, z_1^i z_2^j), \qquad i < t_1, \quad j < t_2$$

implies:

$$(Fu_1, z_1^i z_2^j) = (Fu_2, z_1^i z_2^j), \qquad i \leq t_1, \quad j \leq t_2, \quad \forall u_1, u_2 \varepsilon \mathcal{U}.$$

Under these assumptions it is easy to verify that:

$$s \triangleq F(1) \;\varepsilon\; (z_1 z_2) K\Big[\big[z_1, z_2\big]\Big]$$

and

$$F(u) = su, \qquad \forall u \varepsilon \mathcal{U}.$$

So doing the two-dimensional filters (in their input-output representation) are in one-to-onè correspondence with the formal series $(z_1 z_2)K\Big[\big[z_1, z_2\big]\Big]$ and viceversa.

The state introduction via the Nerode equivalence classes represents the way of obtaining a recursive filter in the system theoretic sense. This requires to endow the input space with the structural properties:

(i) Truncation. Let

$$r = \sum_{i,j} (r, z_1^i, z_2^j)\; z_1^i z_2^j, \qquad r \varepsilon \mathcal{U};$$

the truncation operator $T_{(t_1,t_2)}: \mathcal{U} \to \mathcal{U}$ is defined by:

$$T_{(t_1,t_2)} r = \sum_{(i \leq t_1) \vee (j \leq t_2)} (r, z_1^i z_2^j)\; z_1^i z_2^j$$

Let $\mathcal{U}^* \triangleq \{T_{(0,0)} u : u \varepsilon \mathcal{U}\}$. Then the map

$$f: \mathcal{U}^* \to (z_1 z_2) K\Big[\big[z_1, z_2\big]\Big]$$

defined by the assignement:

$$f(u) = \sum_{i,j>0} (Fu, z_1^i, z_2^j)\; z_1^i z_2^j$$

characterizes S in the same sense as F does. This follows from two-dimensional

shift invariance.

(ii) Shift Operators. Two kinds of elementary shifts are considered:

(a) $\sigma_1 : \mathcal{U} \to \mathcal{U}$

$\sigma_2 : r \to z_1^{-1} r, \quad r \in \mathcal{U}$

(b) $\sigma_2 : \mathcal{U} \to \mathcal{U}$

$\sigma_2 : r \to z_2^{-1} r, \quad r \in \mathcal{U}$

Analogously for $\mathcal{Y}$. Then $\mathcal{U}$ and $\mathcal{Y}$ are naturally endowed with a $K[\sigma_1,\sigma_2]$ -module structure (or equivalently a $K[z_1^{-1}, z_2^{-1}]$ -module structure).

(iii) Concatenation. Let $u,v \in \mathcal{U}^*$. Then

$$u \circ v = \sigma_1^m \sigma_2^n u + v$$

$$m = \min_{z_1} \deg v, \qquad n = \min_{z_2} \deg v$$

and $u \circ v \in \mathcal{U}^*$.

Let u_1, $u_2 \in \mathcal{U}^*$, we say "u_1 is Nerode equivalent to u_2" ($u_1 \sim u_2$) iff

$$f(u_1 \circ v) = f(u_2 \circ v), \qquad \forall v \in \mathcal{U}^*.$$

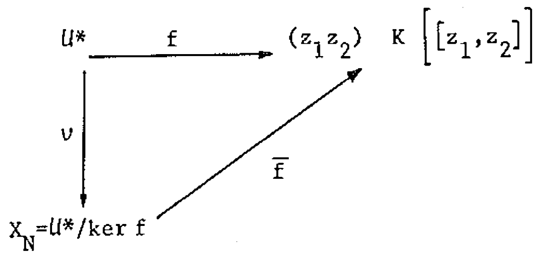
Remark. Let $u_1,u_2 \in \mathcal{U}^*$. Then

$$u_1 \sim u_2 \iff f(u_1) = f(u_2)$$

The introduced equivalence is a congruence over the vector space $\mathcal{U}^*$ and consequently $\mathcal{U}^*/\sim$ can be endowed with the same algebraic structure. In particular

$$\{u : u \sim o\} \triangleq [o]$$

is a subspace of $\mathcal{U}^*$ and $\mathcal{U}^*/\sim = \mathcal{U}^*/[o]$ is assumed as the state space X_N. The situation is reprepresented by the commutative diagram:



2. Some general properties of the input-output map

In the usual linear scalar case the following facts are equivalent:

i) $s \in K[(z)]$, $K[(z)]$ ring of rational series

ii) $\dim X_N < \infty$, X_N canonical (Nerode) state space

iii) there exist compact support non-zero inputs such that the corresponding outputs are compact support.

If we refer to two-dimensional filters the situation is slightly different. Actually it is direct to see that facts corresponding to i) and iii) are still equivalent. Of course point i) has to be changed into:

i) $s \in K\left[(z_1,z_2)\right]$, $K\left[(z_1,z_2)\right]$ ring of rational series in two variables.

The dimension of the canonical state space X_N in this case is always infinite. This can be proved from the above commutative diagram by restricting the input space U^* to $K\left[[z_2]\right]$ so that $\ker f = \{0\}$.

Remark 1. Let $s \in K[(z_1,z_2)]$, $s = P(z_1^{-1}, z_2^{-1})/Q(z_1^{-1}, z_2^{-1})$ and P and Q have no common factors, then the class of compact support inputs giving compact support outputs is the pincipal ideal (Q) modulo the shift sempigroup generated by σ_1 and σ_2. This situation is analogous to the usual linear case.

Remark 2. If the input space is restricted to $K\left[z_1^{-1}, z_2^{-1}\right]$, then $\dim X_N$ is finite if and only if the series s belongs to the ring $K^{rec}[(z_1,z_2)]$ of recognizable series. This can be proved noting that the rank of Hankel matrices corresponding to recognizable series is finite [3].

3. Internal representation

A double indexed, linear, shift invariant, finite dimensional dynamical system in state space form is defined as

(2) $$\Sigma = (T \times T,\ U,\ \mathcal{U},\ Y,\ \mathcal{Y},\ X,\ \mathcal{X},\ \varphi,\ r)$$

where:

- $T \times T$, U, $\mathcal{U}$, Y, $\mathcal{Y}$ are as in $\mathcal{S}$
- $X = K^n$ (local state space)
- $\mathcal{X} = \{\hat{x}(h,k): \hat{x}(h,k) = (\dots, x(h+1,k), x(h,k), x(h,k+1), \dots), x(i,j) \in X\}$
- $\varphi: T^2 \times T^2 \times \mathcal{X} \times \mathcal{U} \to X$ (state transition function)

It satisfied the axioms:

(i) (two dimensional determinism): Let $u_1, u_2 \in \mathcal{U}$ and $\hat{x}_1, \hat{x}_2 \in \mathcal{X}$. Then
$(u_1, z_1^i z_2^j) = (u_2, z_1^i z_2^j)$, $\tau' \leq i < t'$, $\tau'' \leq j < t''$ and
$x_1(\tau'; \tau'') = x_2(\tau'; \tau'')$, $x_1(\tau'+1, \tau'') = x_2(\tau'+1, \tau''), \dots, x_1(t'; \tau'') = x_2(t'; \tau'')$,
$x_1(\tau'; \tau''+1) = x_2(\tau'; \tau''+1), \dots, x_1(\tau'; t'') = x_2(\tau'; t'')$, imply
$\varphi((t'; t''), (\tau'; \tau''), \hat{x}_1, u_1) = \varphi((t'; t''), (\tau'; \tau''), \hat{x}_2, u_2)$

(ii) (consistency): Let $\hat{x} = (\dots, x(\tau'+1, \tau''), x(\tau'; \tau''), x(\tau'; \tau''+1), \dots)$. Then
$\varphi((\tau'; \tau''), (\tau'; \tau''), \hat{x}, u) = x(\tau'; \tau'')$

(iii) (composition): Let $t \leq t' \leq t''$ and $\tau \leq \tau' \leq \tau''$. Then
$\varphi((\tau''; t''), (\tau, t), \hat{x}, u) = \varphi((\tau''; t''), (\tau'; t'), \hat{x}^*, u)$
where $\hat{x}^* = (\dots, \varphi((\tau'+1, t'), (\tau, t), \hat{x}, u), \varphi((\tau'; t'), (\tau, t), \hat{x}, u),$
$\varphi((\tau'; t'+1), (\tau, t), \hat{x}, u), \dots)$

(iv) (shift invariance):
$\varphi((t+\Delta_1, \tau+\Delta_2), (t'+\Delta_1, \tau'+\Delta_2), \hat{x}, \sigma_1^{\Delta_1} \sigma_2^{\Delta_2} u) = \varphi((t,\tau), (t'; \tau'), \hat{x}, u)$

(v) (linearity)
$\varphi((t,\tau), (t'; \tau'), \hat{x}_1 + \hat{x}_2, u_1 + u_2) = \varphi((t,\tau), (t'; \tau'), \hat{x}_1, u_1) + \varphi((t,\tau), (t'; \tau'), \hat{x}_2, u_2)$

- $r: X \to Y$ (read-out function)
  It is assumed linear.

On the basis of the previous assumptions on φ and r it is direct to check that there exist $A_o, A_1, A_2 \in K^{n \times n}$, $C \in K^{1 \times n}$ $B \in K^{n \times 1}$ such that

$$x(h+1, k+1) = A_o x(h,k) + A_1 x(h+1,k) + A_2 x(h,k+1) + B u(h,k)$$

$$y(h,k) = C x(h,k)$$

These equations define (indirectly) the two maps φ and r.

A double indexed dynamical system Σ is a zero state realization of a two-dimensional filter $\mathcal{S}$ if for any $i \geq r$, $j \geq s$

$(Fu, z_1^i z_2^j) = r(\varphi((i,j),(r,s),\hat{\vartheta},u))$,

$\forall(r,s) \in T^2$; $\forall u \in \mathcal{U}$ with $u(h,k) = 0$ for $h < r$, $k < s$.

Lemma. Let S as in (1). The following facts are equivalent:

i) $s \in (z_1 z_2)\, K\left[(z_1,z_2)\right]$

ii) there exist $m, n \in N$ such that for any $u \in \mathcal{U}^*$, the map $f(u) \longmapsto f(u) \odot \sum_{(i\leq m)\vee(j\leq n)} z_1^i z_2^j$ is one-to-one(°).

proof i) $\Longrightarrow$ ii) is immediate.

ii) $\Longrightarrow$ i) Assume for sake of simplicity $m = n$.

Proper causality, linearity and shift-invariance properties imply that there exist $\{b_{ij}\}$ such that for any $u \in \mathcal{U}^*$

$$y(n+h,\ n+k) = \sum_{\substack{i,j=0\\(i,j)\neq(n,n)}}^{n} b_{ij}\, y(i+h,\ j+k),\ \forall h,k > 0$$

Suppose $u = 1$, then $y(i,j) = (s,\ z_1^i\ z_2^j)$ and

$$(z_1^{-n}\ z_2^{-n} - \sum_{ij} b_{ij}\ z_1^{-i}\ z_2^{-j})\ s = 0 \tag{4}$$

for non negative powers of z_1 and z_2.

Letting now $u = z_1^t$ and $u = z_2^t$, $t \in Z_+$ the relation (4) still holds for negative powers of z_1 and z_2. This implies that s is rational.

Proposition 3.1. Let S as in (1) with $s \in K\left[(z_1,z_2)\right]$. Then it has a zero state realization Σ.

Suppose $u \in K\left[z_1^{-1},\ z_2^{-1}\right]$ and introduce the map

$$f^{(n,n)}:\ K\left[z_1^{-1},\ z_2^{-1}\right] \to K^{n\times n}$$

defined by the assignement

$$f^{(n,n)}(u) = \left\| (\sum_{ij=1}^{n} z_1^i\ z_2^j \odot f(u),\ z_1^r\ z_2^s) \right\|_{r,s=1,\dots,n}$$

The following diagram

(°) The symbol $\odot$ denotes the Hadamard product.

$$\begin{array}{ccc} K\left[z_1^{-1},\ z_2^{-2}\right] & \xrightarrow{\ f^{(n,n)}\ } & K^{n\times n} \\ \Big\downarrow g & \nearrow & \bar{f}^{(n,n)} \\ X = K\left[z_1^{-1}\ z_2^{-2}\right]/\ker f^{(n,n)} & & \end{array}$$

commutes and $\dim X \le n^2$.

In the proof we shall use the spaces:

$$U_{ij} \triangleq \{u:\ u \in z_1^{i}\ z_2^{j}\ K\left[z_1^{-1}\ z_2^{-1}\right]\}$$

and the projection maps:

$$\pi_{ij} : U \to U_{ij},$$
$$\pi_{ij}(u) \triangleq u_{ij}$$

proof. Let $s = \sum_{o\,ij}^{n-1} a_{ij}\ z_1^{-i}\ z_2^{-j} / \sum_{o\,ij}^{n} b_{ij}\ z_1^{-i}\ z_2^{-j}$, $b_{nn} = 1$.

Introduce the linear map ψ:

$$\psi:\ K^{n\times n} \times K^{n\times n} \times K^{n\times n} \times K \to K^{n\times n}$$

$$\psi:\ (M_o,\ M_1,\ M_2,\ k) \longmapsto M$$

defined by the following relations:

$$M^{(i,j)} = -M_o^{(i+1,\ j+1)} + M_2^{(i+1,\ j)} + M_1^{(i,j+1)} + (f(k), z_1^{i}\ z_2^{j}) \quad 1 \le i,\ j \le n-1$$

$$M^{(n,j)} = M_2^{(n,j+1)} + \tilde{m}_{nj} + (f(k), z_1^{n}\ z_2^{j}), \qquad 1 \le j \le n-1$$

$$M^{(i,n)} = M_1^{(i+1,n)} + \tilde{m}_{in} + (f(k), z_1^{i}\ z_2^{n}), \qquad 1 \le i \le n-1$$

$$M^{(n,n)} = -(b_{oo} M_o^{(1,1)} + \sum_{1\,j}^{n} M_1^{(1,j)} b_{jo} + \sum_{1\,i}^{n} M_2^{(i,1)} b_{oi} + \sum_{\substack{1\,h,k \\ (h,k)\ne(n,n)}}^{n} M^{(h,k)} b_{hk}) + (f(k),\ z_1^{n}\ z_2^{n}$$

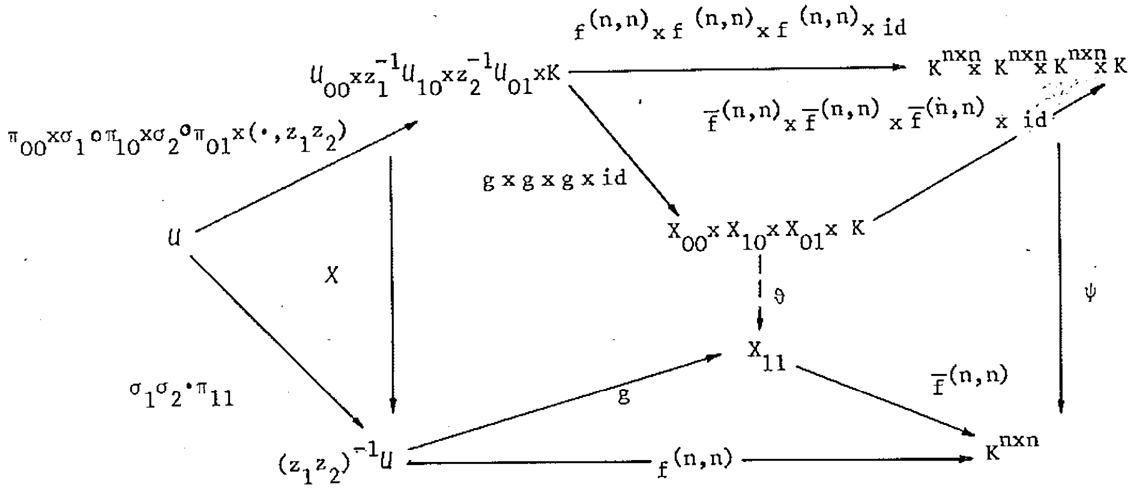
where

$$\tilde{m}_{n1} = -\sum_{o\,k}^{n-1} b_{nk} (M_1^{(k,1)} - M_o^{(k,2)}$$

$$\tilde{m}_{n2} = -\sum_{o}^{n-1}{}_{k}\, b_{nk}(M_1^{(k,2)} - M_o^{(k,3)}) - \sum_{o}^{n-1}{}_{k}(M_1^{(k,1)} - M_o^{(k,2)}) - b_{n-1,n}\,\tilde{m}_{n1}$$

....

The following diagram

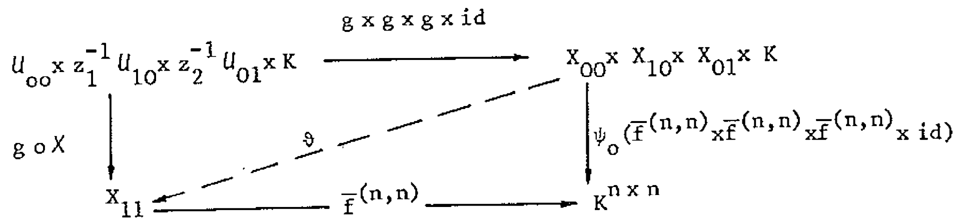


commutes along the continuous arrows. In particular

$$f^{(n,n)} \circ \sigma_1\sigma_2 \circ \pi_{11} = \psi \circ (f^{(n,n)} x f^{(n,n)} x f^{(n,n)} x id) \circ (\pi_{oo} x \sigma_1 \circ {}_{10} x \sigma_2 \pi_{01} x$$
$$x(., z_1 z_2))$$

The map X is well defined on the range of $\pi_{oo} x \sigma_1 \circ \pi_{10} x \sigma_2 \circ \pi_{01} x(\cdot, z_1 z_2)$ and assumes zero value elsewhere.

The existence of ϑ is proved by applying the Zieger Lemma to the partial diagram:



The map ϑ induces the following linear transformations:

A_o: $X_{oo} \rightarrow X_{11}$

A_1: $X_{01} \rightarrow X_{11}$

A_2: $X_1 \rightarrow X_{11}$

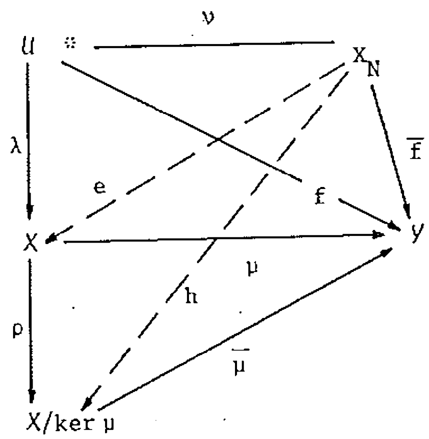
B: $K \rightarrow X_{11}$

satisfying

$$\vartheta(x_{oo}, x_{10}, x_{01}, k) = A_o x_{oo} + A_1 x_{01} + A_2 x_{10} + B k.$$

The output y(1,1) is the top left corner element of $\bar{F}^{(n,n)}(x_{11})$. This defines the read-out map.

Proposition 3.2. Let Σ a zero state realization of a given S. Then there exists a 1:1 map e such that the diagram



commutes.

(The maps λ and μ are built up in natural way from φ and r in Σ)

proof. By Zeiger Lemma there exists a linear map h: $X_N \to X/\ker\mu$ which is 1:1. Then also e exists such that $\rho \circ e = h$ and is 1:1.

The definition of minimal realization is naturally related to the dimension of X in the sense that this is minimal in the class of possible realizations of the given filter S.

The construction presented in Proposition 1 does not necessarily give a minimal realization.

Obviously the possibility of embedding Nerode state space X_N in X resulting in Proposition 3.2 does not depend on the dimension of the realization.

Proposition 3. Let Σ be a zero state realization of a given S. Then $s \stackrel{\Delta}{=} F(1) \in (z_1 z_2)\, K\,[(z_1, z_2)]$.

proof. The existence of Σ implies the existence of $A_o, A_1, A_2 \in K^{n \times n}$, $B \in K^{n \times 1}$, $C \in K^{1 \times n}$ such that (3) hold. By associating the local state x(h,k) with the monomial

$x(h,k)z_1^h z_2^k \in K^{n \times 1}\left[[z_1, z_2]\right]$ it is direct to verify that for any $u \in K\left[[z_1,z_2]\right]$ we have

$$\sum_{o\,h,k}^{\infty} x(h,k)z_1^h z_2^k = A_o\left(\sum_{o\,h,k}^{\infty} x(h,k)z_1^h z_2^k\right) + A_1\left(\sum_{o\,h,k}^{\infty} x(h,k)z_1^h z_2^k\right)z_1 + $$
$$+ A_2\left(\sum_{o\,h,k}^{\infty} x(h,k)z_1^h z_2^k\right)z_2 + B(z_1z_2)u$$

and then $(I - A_o z_1 z_2 - A_1 z_1 - A_2 z_2)\sum_{h,k} x(h,k)z_1^h z_2^k = B(z_1z_2)u.$

The polynomial $(I-A_o z_1 z_2 - A_1 z_1 z_2)$ belonging to $K^{n \times n}[z_1,z_2]$ has an inverse in the ring of rational series $K^{n \times n}[(z_1, z_2)]$ and its inverse is

$$(I - A_o z_1 z_2 - A_1 z_1 - A_2 z_2)^{-1} = \sum_{o\,i}^{\infty} (A_o z_1 z_2 + A_1 z_1 + A_2 z_2)^i$$

It results that

$$\sum_{h,k} x(h,k)\, z_1^h z_2^k = (I - A_o z_1 z_2 - A_1 z_1 - A_2 z_2)^{-1}\, B(z_1 z_2)\, u$$

and the output is given by

$$y = C\sum_{h,k} x(h,k)z_1^h z_2^k = C(I - A_o z_1 z_2 - A_1 z_1 - A_2 z_2)^{-1}\, B(z_1 z_2)\, u$$

The series s is expressed by

$$C(I - A_o z_1 z_2 - A_1 z_1 - A_2 z_2)^{-1}\, B(z_1 z_2)$$

where $(I - A_o z_1 z_2 - A_1 z_1 - A_2 z_2)^{-1}$ belongs to $K^{n \times n}[(z_1,z_2)] \cong K\,[(z_1,z_2)]^{n \times n}$

This proves that the product

$$C(I - A_o z_1 z_2 - A_1 z_1 - A_2 z_2)^{-1}\, B(z_1 z_2) \text{ belongs to } (z_1 z_2)\, K\,[(z_1,z_2)].$$

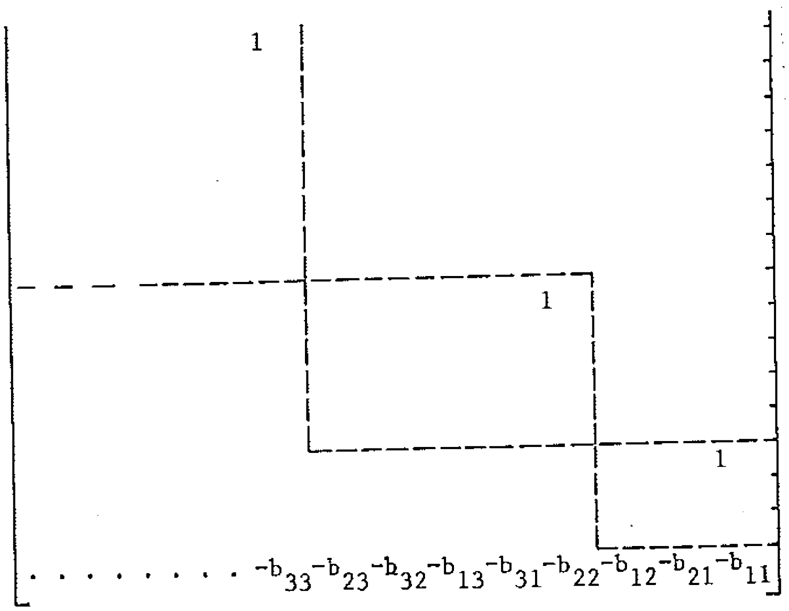
Remark. If (A_o, A_1, A_2, B, C) is a realization of a filter, $A_o, A_1, A_2 \in K^{n \times n}$, $B \in K^{n \times 1}$, $C \in K^{1 \times n}$, and $T \in K^{n \times n}$ is non singular, then $(TA_oT^{-1}, TA_1T^{-1}, TA_2T^{-1}, TB, CT^{-1})$ is still a realization of S. The matrix T is associated with a change of basis in the local state space.

4. Construction of a realization

We give here an effective technique of obtaining a realization (A_o, A_1, A_2, B, C) of a filter with $s \in K[(z_1, z_2)]$. This procedure can be interpreted as an alternative proof of Proposition 3.1. Nevertheless the intrinsic meaning of the previous proof of Proposition 3.1 is different in the sense that it is based on the concept of equivalent classes according to Nerode idea.

Let $s = \sum_{1}^{n}{}_{ij}\, a_{n-i,\, n-j}\; z_1^{-i}\; z_2^{-j} / \sum_{0}^{n}{}_{ij}\, b_{n-i,\, n-j}\; z_1^{-i}\; z_2^{-j} \qquad b_{oo} = 1$

The matrices A_o, A_1, $A_2 \in K^{n^2 x n^2}$, $B \in K^{n^2 x 1}$, $C \in K^{1 x n^2}$ defined by:



$= A_0$

$$\left[\begin{array}{cccccccc} & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \\ \cdots\cdots & -b_{40} & & -b_{30} & & & -b_{20} & -b_{10} \end{array}\right] = A_1$$

$$\left[\begin{array}{cccccccc} & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \\ \cdots\cdots & -b_{04} & & -b_{03} & & & -b_{02} & -b_{01} \end{array}\right] = A_2$$

$$\begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix} = B$$

$$C = \left[\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ a_{22}\ a_{12}\ a_{21}\ a_{02}\ a_{20}\ a_{11}\ a_{01}\ a_{10}\ a_{00}\right]$$

satisfy the relation

$$s = C(I - A_o z_1 z_2 - A_1 z_1 - A_2 z_2)^{-1}\ B(z_1 z_2)$$

5. Minimality of the realization and Hankel matrices

The Hankel matrix associated with a series $s \varepsilon (z_1 z_2)\ k\left[[z_1, z_2]\right]$ is given by

$$H(s) = \begin{bmatrix} (s, z_1 z_2) & (s, z_1^2\ z_2) & (s, z_1 z_2^2) & (s, z_1^3 z_2) & (s, z_1^2 z_2^2), & (s, z_1 z_2^3) \\ (s, z_1^2 z_2) & (s, z_1^3 z_2) & (s, z_1^2 z_2^2) & (s, z_1^4 z_2), \ldots & & \\ (s, z_1 z_2^2) & (s, z_1^2 z_2^2) & \ldots\ldots\ldots & & & \\ \ldots\ldots\ldots\ldots & & & & & \end{bmatrix}$$

Adopting an indexing which agrees with H the input $u \varepsilon K[z_1^{-1}, z_2^{-1}]$ is represented by the vector $\bar{u} = \left[(u,1)\ (u, z_1^{-1})\ (u, z_2^{-1})\ (u, z_1^{-2})\ (u, z_1^{-1} z_2^{-1}) \ldots\right]^T$ so that the corresponding output is represented by the vector $H(s)\ \bar{u}$. Of course if $u \varepsilon U^*$, the output in (i,j) is given by the first component of $H(s)\ (\sigma_1^i\ \sigma_2^i\ u_{ij})$.

The rank of $H(s)$ with $s \varepsilon K\ [(z_1, z_2)]$ is in general infinite and it becomes finite if and only if the series s belongs to the ring of recognizable series K^{rec} $[(z_1, z_2)]$. Moreover the matrix $H(s)$ with s rational has the following properties:

i) There exists a set of scalars $\{b_{r,s}\}$, $r, s = 0, \ldots,\ n$ such that for any $h, k \geq n$ the rows of $H(s)$ indexed by $z_1^{h-r}\ z_2^{k-s}$ are linearly dependent with coefficients $\{b_{r,s}\}$.

ii) consider the submatrices $H_k(s)$ obtained considering in $H(s)$ the columns indexed by z_1^i, $i \geq 0$ and the rows indexed by $z_1^i\ z_2^k$, $i \geq 0$ and the submatrices $H^k(s)$ obtained considering the columns indexed by z_2^i, $i \geq 0$ and the rows indexed by $z_2^i\ z_1^k$, $i \geq 0$. For any positive k, the matrices $H_k(s)$ and $H^k(s)$ are finite rank.

If we assume that the series s is recognizable we can exploit some useful properties of this class of series for getting a minimal realization in the set of representations. In fact if $s \varepsilon K^{rec}(z_1, z_2)$ there exist an integer $m \geq 1$, a representation μ on $K^{m \times m}$ of the commutative monoid generated by z_1 and z_2, two matrices $C \varepsilon K^{1 \times m}$ and $B \varepsilon K^{m \times 1}$ such that

$$s = \sum_{ij} (C\mu(z_1^i z_2^j)\ B,\ z_1^i z_2^j)\ z_1^i z_2^j$$

The minimum value of m is given by rank $H(s)$.

Proposition 5.1. Let $s \in (z_1 z_2)\ K^{rec}[(z_1,z_2)]$ and (C,μ,B) a representation of s.Then $(-\mu(z_1 z_2),\ \mu(z_1),\ \mu(z_2),\ B,\ C)$ is a realization of the filter S corresponding to s and $\mu(z_1 z_2) = \mu(z_1)\ \mu(z_2) = \mu(z_2)\mu(z_1)$.Viceversa let $(A_o,\ A_1,\ A_2,\ B,\ C)$ be a realization of S satisfying $-\ A_o = A_1 A_2 = A_2 A_1$. Then the filter S is characterized by a series $s \in (z_1 z_2)\ K^{rec}[(z_1,z_2)]$

proof. Let $s \in (z_1 z_2)\ K^{rec}\ [(z_1,z_2)]$ so that

$$s = (z_1 z_2) \sum_{o\ ij}^{\infty} C\ A_1^i\ A_2^j\ B\ z_1^i z_2^j$$

with $A_1 \triangleq \mu(z_1)$, $A_2 \triangleq \mu(z_2)$ and $A_1 A_2 = A_1 A_2$.

Then

$$s = (z_1 z_2) C (\sum_{o\ ij}^{\infty} A_1^i A_2^j z_1^i z_2^j)\ B = (z_1 z_2) C \sum_{o\ i}^{\infty} (A_1 z_1 + A_2 z_2 - A_1 A_2 z_1 z_2)^i B =$$

$$= (z_1 z_2)\ C(I - A_1 z_1 - A_2 z_2 + A_1 A_2 z_1 z_2)^{-1} B.$$

This proves that $(-A_1 A_2,\ A_1, A_2,\ B, C)$ is a realization of S. The converse is immediate.

Remark 1. The rank of $H(s)$, $s \in K^{rec}\ [(z_1,z_2)]$ provides the dimension of a minimal realization in the class of realization $(A_o,\ A_1,\ A_2,\ B, C)$ satisfying $A_o = -A_1 A_2 = -\ A_2 A_1$.

In the sequel the realizations belonging to this class will be called representations.

Remark 2. If $(-A_1 A_2,\ A_1,\ A_2,\ B,\ C)$ and $(-\hat{A}_1 \hat{A}_2,\ \hat{A}_1,\ \hat{A}_2,\ \hat{B},\ \hat{C})$ are two minimal representations of dimension $\bar{m}$=rank $H(s)$, then there exists a non singular matrix $T \in K^{\bar{m} \times \bar{m}}$ such that:

$$TA_1T^{-1} = \hat{A}_1 \qquad TB = \hat{B}$$

$$TA_2T^{-1} = \hat{A}_2 \qquad CT^{-1} = \hat{C}.$$

When s is rational but not recognizable, the minimization procedure cannot be directly correlated with the Hankel matrix $H(s)$ of the series s. Nevertheless it is possible to get a partial minimization with the aid of a non-commutative structure. To this end we recall that any non commutative rational series is recognizable. As a consequence for a non commutative rational series $r \in K <(x_1,x_2,x_3)>$ there exist an integer m, a representation $\mu: X^* \to K^{m \times m}$, two matrices $B \in K^{m \times 1}$ and $C \in K^{1 \times m}$ such that

$$r = \sum_{w \in X^*} C\mu(w)\, B\, w$$

where X^* is the free monoid generated by x_1, x_2, x_3.

In this case $\mu(x_1) = A_1$, $\mu(x_2) = A_2$, $\mu(x_3) = A_o$ are not necessarily commutative. The series r can also be expressed as A minimal representation $(\hat{A}_o, \hat{A}_1, \hat{A}_2, \hat{B}, \hat{C})$ of r, whose dimension is of course $\bar{m}$=rank $H(r)$, constitutes the minimal realization of s in the class of realizations which are also representations for r.

For any realization (A_o, A_1, A_2, B, C) of s in this class with dimension m, there exists a matrix $P \in K^{\bar{m} \times m}$ of full rank, such that

$$\hat{A}_o P = P A_o$$

$$\hat{A}_1 P = P A_1$$

$$\hat{A}_2 P = P A_2$$

$$\hat{C} P = C$$

$$\hat{B} = P B$$

In this class any minimal realization can be obtained from any other minimal by similarity.

6. - Reachability and observability

In this paragraph s will be always assumed to be rational so that it will be possible to consider finite dimensional local state space.

Let $X = K^n$ be the local state space of a realization (A_o,A_1,A_2,B,C) for the given s.

Defintion 1 (local reachability): the local state $x \in X$ is reachable in (i,j) if there exists $u \in U$ such that

$$x = ((z_1 z_2) \textstyle\sum_k (A_o z_1 z_2 + A_1 z_1 + A_2 z_2)^k B\, u,\ z_1^i z_2^j)$$

<u>Definition 2</u> The reachable local state space in (i,j) is

$$X^R(i,j) \triangleq \{x : x = ((z_1 z_2) \textstyle\sum_h (A_o z_1 z_2 + A_1 z_1 + A_2 z_2)^h B\, u,\ z_1^i z_2^j),\ \exists u \in \mathcal{U}\}$$

By shift invariance property $X^R(i,j) = X^R(1,1) = X^R$.
The state space X is completely 1-reachable if $X^R = X$.

$$r = \sum_i C(A_1 x_1 + A_2 x_2 + A_o x_3)^i B = C(I - A_1 x_1 - A_2 x_2 - A_o x_3)^{-1} B$$

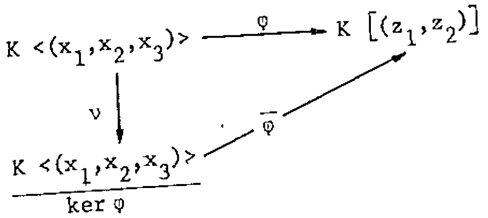
The dimension of the minimal representation (A_o, A_1, A_2, C, B) is given by rank $H(r)$.

The morphism φ: $K\langle(x_1,x_2,x_3)\rangle \to K[[z_1,z_2]]$ defined by the assignements $\varphi(k)=k$, $\forall k \in K$, $\varphi(x_1)=z_1, \varphi(x_2)=z_2, \varphi(x_3)=z_1 z_2$ is a morphism of algebras whose image is $K[(z_1,z_2)]$. This comes from the fact that

$$\varphi:\ C(I - A_1 x_1 - A_2 x_2 - A_o x_3)^{-1} B \to C(I - A_1 z_1 - A_2 z_2 - A_o z_1 z_2)^{-1} B$$

so that the image of a recognizable series is a rational series. Since all rational series are obtained by varying A_1, A_2, A_o, B, C, φ is onto and to each representation of a non-commutative series $r \in K\langle(x_1,x_2,x_3)\rangle$ corresponds a realization of $\varphi(r) \in K[(z_1,z_2)]$.

The map φ is factorized as in the following commutative diagram



where $\bar\varphi$ is an isomorphism.

Given $s \in K[(z_1,z_2)]$ consider the coset $\bar\varphi^{-1}(s)$. A minimal realization of s is then a minimal representation in the class of representations associated to $\bar\varphi^{-1}(s)$. For each $r \in \bar\varphi^{-1}(s)$ the dimension of a minimal representation is rank $H(r)$ so that the dimension of a minimal realization of s is $\min_{r \in \bar\varphi^{-1}(s)}$ rank $H(r)$.

In general starting from a realization (A_0, A_1, A_2, B, C) of a series $s \in K[(z_1,z_2)]$ we can reduce the dimension of the realization by minimizing the representation of the non-commutative series $r = C(I - A_1 x_1 - A_2 x_2 - A_0 x_3)^{-1} B$.

In a way analogous to the usual linear case a 1-reachability matrix R, associated with the realization (A_0, A_1, A_2, B, -) can be introduced

$$R = \left[B\ M_{00} \ \vdots\ B\ M_{10} \ \vdots\ B\ M_{01} \ \vdots\ \dots \right]$$

where $M_{ij} = (\sum_{o}^{\infty}{}_h (A_0 z_1 z_2 + A_1 z_1 + A_2 z_2)^h,\ z_1^i z_2^j)$

The image of R is X^R so that the state space is completely 1-reachable if and only if rank (R) = n.

Proposition 6,1. The minimal realizations are completely 1-reachable.

proof. Let ($\hat{A}_0$, $\hat{A}_1$, $\hat{A}_2$, $\hat{B}$, $\hat{C}$) be a minimal realization for s with dimension $\bar{n}$. Suppose $n = \dim X^R = \text{rank}\ (\hat{R}) < \bar{n}$ and decompose X as $X^R \oplus (X^R)^{\perp}$. Then by definition of reachability a zero-state realization of dimension n can be constructed in X^R. This contradicts the assumption of minimality

Remark. The realization associated to ϑ in Proposition 3.1 and the realization presented in paragraph 4 are both completely 1-reachable even if they are not necessarily minimal.

Definition 3. A local state $x \in X$ is indistinguishable from $0 \in X$ if:

$$\sum_i C\ (A_0 z_1 z_2 + A_1 z_1 + A_2 z_2)^i x = 0$$

The left hand term in the above relation represents the zero-input response of $\sum$ starting from $\hat{x}(0,0) = 0$ and assuming $x(1,1) = x$.

Definition 4 (indistinguishable 1-state space):

$$X^I \triangleq \{x : x \in X,\ \sum_i C(A_0 z_1 z_2 + A_1 z_1 + A_2 z_2)^i x = 0\}$$

The space X^I is the null space of the observability matrix:

$$O \triangleq \begin{bmatrix} C\ M_{00} \\ C\ M_{10} \\ C\ M_{01} \\ \vdots \end{bmatrix}$$

The state space X is completely 1-observable if $X^I = \{0\}$ i.e. rank $(0) = n$.

The analysis of reachability and observability can be done in a more standard way when the series s is recognizable.(°) In this case if $(-A_1A_2, A_1, A_2, B, C)$ is a representation of dimension n for s, the reachability and observability matrices assume the form

$$R = \left[B \vdots A_1B \vdots A_2B \vdots A_1^2B \vdots A_1A_2B \vdots A_2^2B \vdots \ldots.. \right]$$

$$0 = \begin{bmatrix} C \\ CA_1 \\ CA_2 \\ CA_1^2 \\ CA_1A_2 \\ CA_2 \\ \vdots \end{bmatrix}$$

so that

$$H(s) = 0\ R.$$

By Cayley-Hamilton theorem the ranks of R and 0 can be evaluated considering only the first n^2 block submatrices. Hence for the computation of rank $H(s)$ we shall restrict to evaluate the rank of $H_{n^2,n^2}(s)$.

On this basis we have directly the following result:

Proposition 6.2. Let $s \in K^{rec}[(z_1,z_2)]$ and $\bar{n}$ be the dimension of a minimal representation. Then the local state space X is completely 1-reachable and 1-observable.

This follows from $H(s) = 0\ R$ and rank $H(s) = \bar{n}$.

(°) The systems in two variables presented in [4] can be considered as filters characterized by a recognizable series s. The "realizations" introduced there are actually representations of s, so that a result similar to Prop 6.2 is proved.

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