State-Space Realization Theory of Two-Dimensional Filters

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Abstract—**The realization problem of two-dimensional linear filters is approached from a system theoretic point of view. The input-output behavior of such a system is defined by formal power series in two variables, and a Nerode state space is constructed. This state space is, in general, infinite dimensional.**

If the power series is rational, the dynamics of the filter is described by updating equations on finite-dimensional local state space. The notions of local reachability and observability are defined in a natural way and an algorithm for obtaining a reachable and observable realization is given. In general, local reachability and observability do not imply the minimality of the realization.

I. Introduction

In the past few years there has been an increasing interest in two-dimensional filters. This type of filter is extensively used in processing two-dimensional sampled data, such as seismic data sections, digitized photographic data, and gravitational and magnetic maps.

Several methods are commonly used to represent the operations involved in image processing. A great deal of work in the processing of two-dimensional images has focused on recursive techniques since they can be used in on-line processing and require less computer memory when processing off-line.

Most papers in this area deal with analysis and synthesis of external (input-output) representations like rational transfer functions. The main results are obtained in input-output stability analysis with algebraic methods [1]–[4].

Synthesis procedures are based on least squares approximation of given impulse or frequency responses by means of rational transfer functions in two variables [5]. Algorithms using continued fraction expansion have been advanced for the synthesis of these transfer functions [6].

Very recently in the literature some papers have appeared where an "internal structure" of spatial filters is considered. This is essentially based on a set of difference equations involving a state vector [7]–[11]. All these contributions are devoted to analyzing dynamical properties of systems given by updating state equations. In some of these works the time set is endowed with the structure of a partially ordered set [7], [8], [11] and reachability and observability concepts are introduced [8], [11] under restrictive hypothesis.

The aim of the present paper is to analyze the algebraic realization problem of spatial filters defined by their input-output maps. This problem is attacked from a system theoretic point of view defining the state via Nerode equivalence classes.

If the transfer function of the filter is rational (i.e., if the series characterizing the filter is rational), a general solution of this problem is provided by updating equations on finite-dimensional local state spaces.

The notions of local reachability and observability are then defined, and we clarify the connections between these concepts and the minimality of the realizations. As we shall see, it is possible to obtain a reachable and observable realization starting from a generic one, and we shall introduce an algorithm for this purpose. This algorithm, which is very similar to the one in linear discrete-time systems, is essentially based on a generalization of the Cayley–Hamilton theorem to the two-dimensional case.

In general, local reachability and observability do not imply the minimality of the realization.

II. Input-Output Representation of a Two-Dimensional Filter

We will consider two-dimensional digital filters with scalar inputs and outputs taken from an arbitrary field K. The input-output representation of such a filter is given by

$$\mathcal{S} \triangleq (T, U, \mathcal{U}, Y, \mathcal{Y}, F) \tag{1}$$

where $T=\mathbb{Z}\times\mathbb{Z}$ (partially ordered by the product of the orderings) is the discrete plane, U and Y are one-dimensional vector spaces over the field K, $\mathcal{U}$ and $\mathcal{Y}$ are the space of truncated formal Laurent series in two variables over K (whose precise description will be given below), and $F:\mathcal{U}\rightarrow\mathcal{Y}$ is the input-output map.

A typical element of $\mathcal{U}$ or $\mathcal{Y}$ will be written

$$r=\sum_{-k}^{\infty}{}_{i,j}(r,z_1^iz_2^j)z_1^iz_2^j, \qquad \text{for some integer } k$$

where $(r,z_1^iz_2^j)$ denotes the coefficient of $z_1^iz_2^j$.

The input-output map $F:\mathcal{U}\rightarrow\mathcal{Y}$ is assumed to satisfy the following axioms.

1) *Linearity.*
2) *Two-dimensional shift invariance:*

$$F(z_1^iz_2^jr)=z_1^iz_2^jF(r), \qquad i,j\in\mathbf{Z}.$$

3) *Two-dimensional strict causality:*

$$(u_1,z_1^iz_2^j)=(u_2,z_1^iz_2^j), \qquad i<t_1, j<t_2$$

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implies

$$(Fu_1, z_1^i z_2^j) = (Fu_2, z_1^i z_2^j), \qquad i \leqslant t_1,\ j \leqslant t_2,\ \forall u_1, u_2 \in \mathcal{U}.$$

Under assumption 3) it is easy to verify that the impulse response $F(1)$ is a "strictly causal" power series, i.e.,

$$F(1) = \sum_{1}^{\infty}{}_{i,j}\big(F(1), z_1^i z_2^j\big) z_1^i z_2^j.$$

More formally we can say that

$$s \triangleq F(1) \in (z_1 z_2) K[[z_1, z_2]] \triangleq K_c[[z_1, z_2]]$$

where $K[[z_1, z_2]]$ denotes the ring of formal power series in two variables and $K_c[[z_1, z_2]]$ is the ideal of "*strictly causal*" power series.

From 1) and 2) it follows that

$$F(u) = su, \qquad \forall u \in \mathcal{U} \tag{2}$$

that is, two-dimensional filters (in their input-output representation) are in one-to-one correspondence with formal power series $K_c[[z_1, z_2]]$.

The first goal of this paper is to describe a state-space representation corresponding to an input-output description of type (1). The first such description will be given in terms of Nerode equivalence classes of inputs. We follow the usual Nerode philosophy that two inputs are equivalent if they give the same output when each one is "followed" by the same input. The precise definition of one input "following" another for two-dimensional systems must be given in terms of two-dimensional shifts and concatenations.

1) *Shift:* Two kinds of shift operators are considered in $\mathcal{U}$ and $\mathcal{Y}$:

$$\sigma_1 : \mathcal{U} \to \mathcal{U}$$
$$\sigma_1 : r \to z_1^{-1} r, \qquad r \in \mathcal{U}$$
$$\sigma_2 : \mathcal{U} \to \mathcal{U}$$
$$\sigma_2 : r \to z_2^{-1} r, \qquad r \in \mathcal{U}.$$

Analogously for $\mathcal{Y}$. The action of σ_1 and σ_2 on $\mathcal{U}$ and $\mathcal{Y}$ is naturally extended to the ring of polynomials $K[\sigma_1, \sigma_2]$. Then $\mathcal{U}$ and $\mathcal{Y}$ are endowed with a $K[\sigma_1, \sigma_2]$-module structure (or equivalently a $K[z_1^{-1}, z_2^{-1}]$-module structure). A similar set up is adopted in the algebraic realization theory of discrete-time linear systems [12].

2) *Concatenation:* Let $t_2 > t_1$ and let $(t_1, t_2]$ be the subset of T:

$$(t_1, t_2] \triangleq \{\tau : \tau > t_1\} - \{\tau : \tau > t_2\}.$$

Let $u \in \mathcal{U}$ and let $u_{(t,0]}$ denote the restriction of u to $(t, 0]$. Let $\mathcal{U}^*$ denote the set of all such restrictions. Let $u_1 : (t_1, 0] \to U$ and $u_2 : (t_2, 0] \to U$ be in $\mathcal{U}^*$. Then $\mathcal{U}^*$ becomes a monoid if we define $u = u_1 \circ u_2 : (t_1 + t_2, 0] \to U$ by

$$u(t) = \begin{cases} u_1(t - t_2), & t \in (t_1 + t_2, t_2] \\ u_2(t), & t \in (t_2, 0]. \end{cases}$$

Let F be as in (1) and define the map $f : \mathcal{U}^* \to K_c[[z_1, z_2]]$ by

$$f(u) = \sum_{1}^{\infty}{}_{i,j}\big(Fu, z_1^i z_2^j\big) z_1^i z_2^j.$$

Then f characterizes $\mathcal{S}$ in the same sense as F does.

If $u_1, u_2 \in \mathcal{U}^*$, we say "$u_1$ is Nerode equivalent to u_2" $(u_1 \sim u_2)$ iff

$$f(u_1 \circ v) = f(u_2 \circ v), \qquad \forall v \in \mathcal{U}^*.$$

It is standard to prove that

$$u_1 \sim u_2 \Leftrightarrow f(u_1) = f(u_2), \qquad u_1, u_2 \in \mathcal{U}^*$$

and

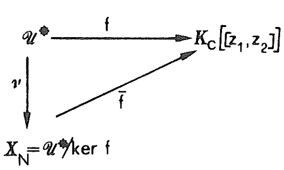
$$\ker f = \{u : u \in \mathcal{U}^*, u \sim 0\} \triangleq [0].$$

The Nerode equivalence classes are then the cosets of $\mathcal{U}^*$ in $\ker f$; hence,

$$\mathcal{U}^*/\sim\; = \mathcal{U}^*/[0] \triangleq X_N$$

is endowed in a canonical way with a linear structure.

The situation is represented by the following commutative diagram.



The space X_N is called the Nerode state space. Although it has a very natural description, if $s \neq 0$, the dimension of the canonical state space X_N is infinite. In fact, consider the above commutative diagram and restrict the input space $\mathcal{U}^*$ to $K[z_2]$. Since $K[z_2]$ is a subring of the integral domain $K[[z_1, z_2]]$, the assumption $f(u) = su = 0, s \neq 0$, implies $u = 0$ for all $u \in K[z_2]$. Since the restriction of f to $K[z_2]$, which is an infinite-dimensional K-vector space, is one-to-one, then $f(K[z_2])$ is infinite dimensional. Hence, $\dim X_N = \dim f(\mathcal{U}^*) \geqslant \dim f(K[z_2]) = \infty$.

This fact shows that the situation for two-dimensional filters is not the same as for usual discrete-time linear systems. Actually in the latter case the dimension of the canonical state space X_N is finite if and only if the input-output map is a rational power series. We recall that a formal power series $s \in K[[z_1, \cdots, z_t]]$ is rational if there exist polynomials $p, q \in K[z_1^{-1}, \cdots, z_t^{-1}]$, $\deg q > \deg p$, such that $qs = p$. The polynomial q is called a denominator of s. The ring of rational power series will be denoted by $K[(z_1, \cdots, z_t)]$.

Remark 1: In the usual linear case the rationality of the input-output map is equivalent to the existence of nonzero inputs of compact support such that the corresponding outputs are of compact support. This is also true for two-dimensional filters. In particular if s belongs to the ring $K[(z_1,z_2)]$ of rational power series in two variables, and $s=p(z_1^{-1},z_2^{-1})/q(z_1^{-1},z_2^{-1})$ and p and q have no common factors, the class of compact support inputs giving outputs with compact support is the principal ideal (q) modulo the shift semigroup generated by σ_1 and σ_2.

Remark 2: If the input space is restricted to $K[z_1^{-1},z_2^{-1}]$, then the dimension of X_N is the rank of the Hankel matrix $\mathcal{H}(s)$ associated with the series s:

$$\mathcal{H}(s)=\begin{bmatrix} (s,1) & (s,z_1) & (s,z_2) & (s,z_1^2) & (s,z_1z_2) & (s,z_2^2)\cdots \\ (s,z_1) & (s,z_1^2) & (s,z_1z_2) & (s,z_1^3) & \cdots & \cdots \\ (s,z_2) & (s,z_1z_2) & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}.$$

In [14] it is proved that the rank of $\mathcal{H}(s)$ is finite if and only if s is rational and a denominator q of s can be factorized as $q=q_1q_2$ with $q_1\in K[z_1^{-1}], q_2\in K[z_2^{-1}]$. The series satisfying this property are the elements of the ring $K[(z_1)]\otimes K[(z_2)]\triangleq K^{\text{rec}}[(z_1,z_2)]$ called the ring of "*recognizable series*."

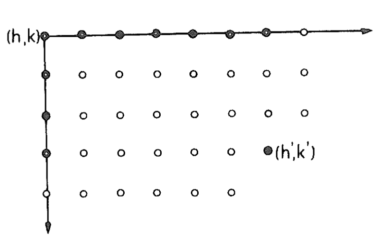
III. Realization of Two-Dimensional Filters

So far we have seen that an input-output map leads to a state-space representation by Nerode equivalence classes of inputs. This Nerode representation X_N is usually infinite dimensional, and furthermore it seems to be impossible to describe the dynamics of X_N in terms of appropriate "updating equations."

These difficulties can be overcome to some extent by introducing the notion of "local state space." We will show that under suitable conditions there exists a finite-dimensional vector space X, and matrices $A_0,A_1,A_2\in K^{n\times n}, C\in K^{1\times n}, B\in K^{n\times 1}$ such that the input-output behavior is described by the updating equations

$$\begin{aligned} x(h+1,k+1)&=A_0x(h,k)+A_1x(h+1,k)\\ &\quad +A_2x(h,k+1)+Bu(h,k) \qquad (3)\\ y(h,k)&=Cx(h,k). \end{aligned}$$

The form of the updating equations (3) will follow from an axiomatic framework which we describe below. The axioms are derived from the following intuitive picture: a finite-dimensional local state space X is attached to each point (h,k) of the plane. If $h'\geqslant h$ and $k'\geqslant k$, then a local state $x(h',k')$ depends not only on $x(h,k)$ but also on local states $x(h+1,k),\cdots,x(h',k)$ and $x(h,k+1),\cdots,x(h,k')$ as shown.



Since (h',k') is arbitrary, it is necessary to introduce a *global state space* $\mathcal{X}$ consisting of all local state spaces on the horizontal and vertical rays. Formally, we define a *doubly indexed, linear, shift-invariant dynamical system* (in state-space form) as follows:

$$\sum \triangleq (T,U,\mathcal{U},Y,\mathcal{Y},X,\mathcal{X},\phi,r)$$

where $T,U,\mathcal{U},Y,\mathcal{Y}$ are as in the definition of $\mathcal{S}$, and $X\cong K^n$ is the local state space. The *global state space* $\mathcal{X}$ is given by

$$\begin{aligned}\mathcal{X}=\{\hat{x}(h,k):\hat{x}(h,k)\\ =(\cdots,x(h+1,k),x(h,k),x(h,k+1),\cdots),\quad x(i,j)\in X\}.\end{aligned}$$

The map $\phi: T\times T\times\mathcal{X}\times\mathcal{U}\rightarrow X$ is the *state-transition function* and $r:X\rightarrow Y$ is the *readout map*.

These ingredients are assumed to satisfy the following axioms:

1) *$r:X\rightarrow Y$ is linear.*

2) *Two-dimensional determinism:* Let $u_1,u_2\in\mathcal{U}$ and $\hat{x}_1,\hat{x}_2\in\mathcal{X}$. Then

$$(u_1,z_1^iz_2^j)=(u_2,z_1^iz_2^j),\qquad h'\leqslant i<h'',\ k'\leqslant j<k''$$

and

$$\begin{aligned} &x_1(h',k')=x_2(h',k')\\ &x_1(h'+1,k')=x_2(h'+1,k'),\cdots,x_1(h'',k')=x_2(h'',k')\\ &x_1(h',k'+1)=x_2(h',k'+1),\cdots,x_1(h',k'')=x_2(h',k'') \end{aligned}$$

imply

$$\phi((h'',k''),(h',k'),\hat{x}_1,u_1)=\phi((h'',k''),(h',k'),\hat{x}_2,u_2).$$

3) *Consistency:* Let $\hat{x}=(\cdots,x(h+1,k),x(h,k),x(h,k+1)\cdots)$. Then $\phi((h,k),(h,k),\hat{x},u)=x(h,k)$ for any u in $\mathcal{U}$.

4) *Composition:* Let $h\leqslant h'\leqslant h''$ and $k\leqslant k'\leqslant k''$. Then $\phi((h'',k''),(h,k),\hat{x},u)=\phi((h'',k''),(h',k'),\hat{x}^*,u)$ where $\hat{x}^*=(\cdots,\phi((h'+1,k'),(h,k),\hat{x},u),\phi((h',k'),(h,k),\hat{x},u),\phi((h',k'+1),(h,k),\hat{x},u),\cdots)$.

5) *Shift invariance:* Let $h<h'$ and $k<k'$. Then $\phi((h'+\Delta_1,k'+\Delta_2),(h+\Delta_1,k+\Delta_2),\hat{x},\sigma_1^{\Delta_1}\sigma_2^{\Delta_2}u)=\phi((h',k'),(h,k),\hat{x},u)$.

6) *Linearity:* Let $u_1,\ u_2\in\mathcal{U}$ and $\hat{x}_1,\ \hat{x}_2\in\mathcal{X}$. Then $\phi((h',k'),(h,k),\hat{x}_1+\hat{x}_2,u_1+u_2)=\phi((h',k'),(h,k),\hat{x}_1,u_1)+\phi((h',k'),(h,k),\hat{x}_2,u_2)$.

The next lemma justifies the matrix form of the updating equations given earlier.

Lemma 3.1: Given a doubly indexed system Σ, there exist $A_0,A_1,A_2\in K^{n\times n},B\in K^{n\times 1},C\in K^{1\times n}$ such that ϕ and r are given by (3).

Proof: From 2) one gets that $x(h+1,k+1)$ depends only on $x(h,k)$, $x(h+1,k)$, $x(h,k+1)$, and $u(h,k)$. Hence, in the derivation of $x(h+1,k+1)$ from $x(h,k)$ and u it is not restrictive to assume that

$$\begin{aligned}\hat{x}(h,k)&=(\cdots,0,x(h+1,k),x(h,k),x(h,k+1),0,\cdots)\\ u(i,j)&=0,\qquad (i,j)\neq(h,k).\end{aligned}$$

Then the existence of A_0,A_1,A_2,B follows directly from 6). The existence of C is obvious by 1).

Doubly indexed systems can be used to realize input-output maps.

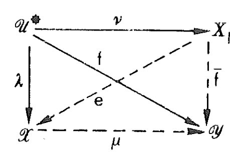
Definition: A doubly indexed dynamical system Σ is a zero state realization of a two-dimensional filter $\mathbb{S}$ if for any $i \geqslant r, j \geqslant s$

$$(Fu, z_1^i z_2^j) = r(\phi((i,j),(r,s),\hat{0},u)),$$
$$\forall (r,s) \in T, \quad \forall u \in \mathfrak{U} \text{ with } u(h,k)=0 \text{ for } h<r, k<s.$$

The *dimension of a realization* is defined as the dimension of the local state space X. We say that a realization Σ of the filter $\mathbb{S}$ is *minimal* when $\dim \Sigma \leqslant \dim \Sigma'$ for any Σ' realizing $\mathbb{S}$.

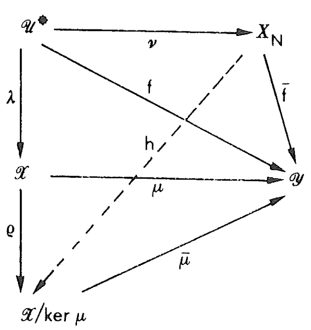
Next we discuss the connection between this construction and the Nerode theory. The canonical state space X_N of a filter $\mathbb{S}$ can be embedded in the space $\mathfrak{X}$ which characterizes a realization Σ. The following proposition shows that this embedding preserves the system theoretic properties of X_N.

Proposition 3.1: Let Σ be a zero state realization of a given $\mathbb{S}$. Then there exists a 1:1 linear map e such that the diagram



commutes along the dashed arrows (the maps λ and μ are built up in natural way from ϕ and r in Σ).

Proof: Consider the following diagram.



Since ν is onto and $\bar{\mu}$ is one-to-one, by Zeiger's lemma [12] there exists a unique linear map $h: X_N \to \mathfrak{X}/\ker\mu$ which makes the diagram commutative, and is one-to-one.

Denote by $\mathfrak{T}$ a complement of $\ker\mu$ in $\mathfrak{X}$ and by $\rho_{\mathfrak{T}}$ the restrictive of ρ to $\mathfrak{T}$. Since $\rho_{\mathfrak{T}}: \mathfrak{T} \to \mathfrak{X}/\ker\mu$ is an isomorphism, there exists a one-to-one linear map $e = \rho_{\mathfrak{T}}^{-1} \circ h$ such that $h = \rho \circ e$. The map e depends on the choice of $\mathfrak{T}$.

Obviously, the possibility of embedding the Nerode state space X_N in $\mathfrak{X}$ resulting in Proposition 3.1 does not depend on the dimension of the realization.

In Propositions 3.2 and 3.3 it will be proved that a two-dimensional filter is realizable if and only if its input-output representation is given by a rational power series s.

Proposition 3.2: Let Σ be a zero realization of a given $\mathbb{S}$. Then $s \triangleq F(1)$ is a strictly causal rational power series.

Proof: The existence of Σ implies the existence of $A_0, A_1, A_2 \in K^{n\times n}, B, C^T \in K^{n\times 1}$ such that (3) holds. By associating the local state $x(h,k)$ with the monomial $x(h,k)z_1^h z_2^k \in K^{n\times 1}[[z_1,z_2]]$ it is direct to verify that for each $u \in K[[z_1,z_2]]$

$$\sum_0^\infty {}_{h,k} x(h,k) z_1^h z_2^k = A_0 \left(\sum_0^\infty {}_{h,k} x(h,k) z_1^h z_2^k \right) z_1 z_2 + A_1 \left(\sum_0^\infty {}_{h,k} x(h,k) z_1^h z_2^k \right) z_1 + A_2 \left(\sum_0^\infty {}_{h,k} x(h,k) z_1^h z_2^k \right) z_2 + (z_1 z_2) Bu$$

and then

$$(I - A_0 z_1 z_2 - A_1 z_1 - A_2 z_2) \sum_{h,k} x(h,k) z_1^h z_2^k = (z_1 z_2) Bu.$$

The polynomial $(I - A_0 z_1 z_2 - A_1 z_1 - A_2 z_2)$ belonging to $K^{n\times n}[z_1,z_2]$ has an inverse in the ring of rational series $K^{n\times n}[(z_1,z_2)]$ and its inverse is

$$(I - A_0 z_1 z_2 - A_1 z_1 - A_2 z_2)^{-1} = \sum_0^\infty {}_i (A_0 z_1 z_2 + A_1 z_1 + A_2 z_2)^i.$$

It results that

$$\sum_{h,k} x(h,k) z_1^h z_2^k = (I - A_0 z_1 z_2 - A_1 z_1 - A_2 z_2)^{-1} (z_1 z_2) Bu$$

and the output is given by

$$y = C \sum_{h,k} x(h,k) z_1^h z_2^k = C (I - A_0 z_1 z_2 - A_1 z_1 - A_2 z_2)^{-1} (z_1 z_2) Bu.$$

The series s is expressed by

$$(z_1 z_2) C (I - A_0 z_1 z_2 - A_1 z_1 - A_2 z_2)^{-1} B$$

where $(I - A_0 z_1 z_2 - A_1 z_1 - A_2 z_2)^{-1}$ belongs to $K^{n\times n} \cdot [(z_1,z_2)] \cong K[(z_1,z_2)]^{n\times n}$.

This proves that $s = (z_1 z_2) C (I - A_0 z_1 z_2 - A_1 z_1 - A_2 z_2)^{-1} B$ belongs to $K_c[(z_1,z_2)]$.

Remark: If (A_0, A_1, A_2, B, C) is a realization of dimension n of a filter $\mathbb{S}$, and $T \in K^{n\times n}$ is nonsingular, then $(TA_0T^{-1}, TA_1T^{-1}, TA_2T^{-1}, TB, CT^{-1})$ is still a realization of $\mathbb{S}$. The matrix T is associated with a change of basis in the local state space.

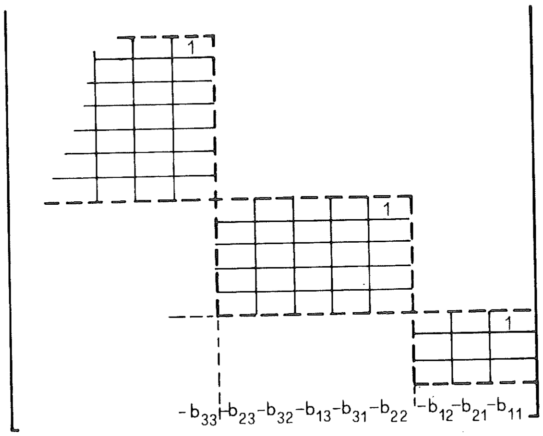
The converse of Proposition 3.2 is given by Proposition 3.3 whose proof furnishes also an effective technique for constructing a realization (A_0, A_1, A_2, B, C) of a filter with $s \in K_c[(z_1,z_2)]$.

Proposition 3.3: Let $\mathbb{S}$ as in (1) with $s \in K_c[(z_1,z_2)]$. Then $\mathbb{S}$ has a zero state realization Σ.

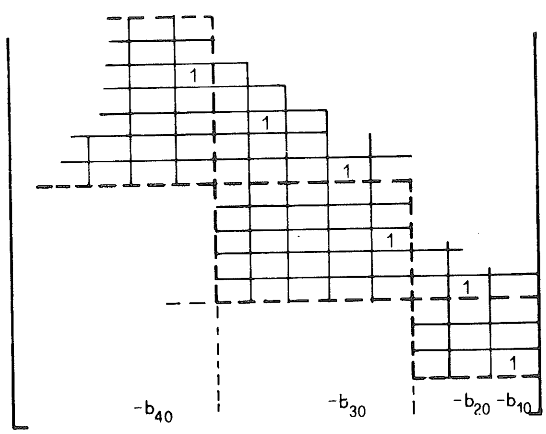
Proof: Let

$$s = \sum_0^{n-1} {}_{ij} a_{n-i,n-j} z_1^{-i} z_2^{-j} \Big/ \sum_0^{n} {}_{ij} b_{n-i,n-j} z_1^{-i} z_2^{-j}, \; b_{00} = 1.$$

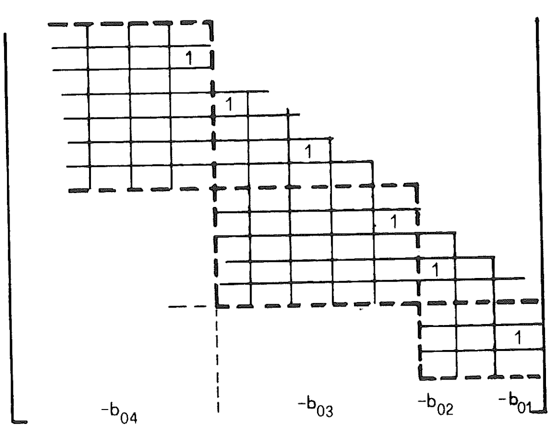
The matrices $A_0, A_1, A_2 \in K^{n^2 \times n^2}$, $B \in K^{n^2 \times 1}$, $C \in K^{1 \times n^2}$ defined by



$= A_0$



$= A_1$



$= A_2$

$$[\cdots\cdots\cdots\cdots\cdots 0\ 0\ 0\ 1] = B^T$$

$$[\cdots a_{33}\ a_{23}\ a_{32}\ a_{13}\ a_{31}\ a_{22}\ a_{12}\ a_{21}\ a_{11}] = C$$

satisfy the relation

$$s = (z_1 z_2) C (I - A_0 z_1 z_2 - A_1 z_1 - A_2 z_2)^{-1} B.$$

Consequently, the doubly indexed dynamical system $\Sigma = (A_0, A_1, A_2, B, C)$ is a (not neccessarily minimal) zero-state realization of $\mathbb{S}$.

An alternative proof of Proposition 3.3 appealing to $K[z_1^{-1}, z_2^{-1}]$-morphism properties can be found in [13].

IV. Reachability and Observability

Henceforth we shall assume that the formal power series s characterizing the input-output map of the filter $\mathbb{S}$ is rational. Hence there exists a realization given by a doubly indexed dynamical system $\Sigma = (A_0, A_1, A_2, B, C)$:

$$x(h+1,k+1) = A_0 x(h,k) + A_1 x(h+1,k) + A_2 x(h,k+1) + Bu(h,k)$$

$$y(h,k) = Cx(h,k)$$

and

$$s = (z_1 z_2) C (I - A_0 z_1 z_2 - A_1 z_1 - A_2 z_2)^{-1} B.$$

We shall now extend the notions of reachability and observability for discrete-time systems to provide notions of local reachability and observability for two-dimensional filters. If a two-dimensional filter is in a global state $\hat{x} \in \mathfrak{X}$ in (h,k), and we apply an input $u \in \mathfrak{U}$, the local state $x(i,j)$, $i \geq h$, $j \geq k$ is given by $\phi((i,j),(h,k),\hat{x},u)$. In particular if $i = j = 0$, and the system Σ starts off in a zero global state, the local state reached at $(0,0)$ is given by $\phi((0,0),(h,k),\hat{0},u)$. Using these notations and recalling that the state-transition function is shift invariant, we now introduce the following definitions.

Definition 1: A state $x \in X$ is reachable (from zero global state) if there exist $(h,k) \in T, h \leq 0, k \leq 0$ and $u \in \mathfrak{U}$ such that $x = \phi((0,0),(h,k),\hat{0},u)$.

In other words, a state $x \in X$ is reachable iff $x = ((z_1 z_2) \sum_0 {}_k (A_0 z_1 z_2 + A_1 z_1 + A_2 z_2)^k Bu, 1)$ for some $u \in \mathfrak{U}$. Hence we also have Definition 2.

Definition 2: The reachable local state space is

$$X^R = \left\{ x : x = \left((z_1 z_2) \sum_0 {}_k (A_0 z_1 z_2 + A_1 z_1 + A_2 z_2)^k Bu, 1 \right), u \in \mathfrak{U} \right\}$$

and the realization $\Sigma = (A_0, A_1, A_2, B, C)$ is L-reachable if $X = X^R$.

The reachable local state space X^R is spanned by the columns of the matrix

$$R_\infty = [M_{00}B\ \ M_{10}B\ \ M_{01}B\ \ \cdots]$$

where

$$M_{ij} = \left(\sum_0^\infty {}_k (A_0 z_1 z_2 + A_1 z_1 + A_2 z_2)^k, z_1^i z_2^j \right).$$

Since $\dim X^R = \operatorname{rank} R_\infty$, the realization $\Sigma = (A_0, A_1, A_2, B, C)$ is L-reachable if and only if R_∞ is full rank.

The notion of indistinguishable states is also extended in a very natural way.

Definition 3: A state $x \in X$ is indistinguishable from the state 0 in X if

$$\sum_0^\infty {}_i C (A_0 z_1 z_2 + A_1 z_1 + A_2 z_2)^i x = 0.$$

Notice that the left-hand term in the above relation represents the zero-input response of Σ determined by $x(0,0) = x$.

Definition 4: The indistinguishable local state space is

$$X^I \triangleq \left\{x : x \in X,\ \sum_i C(A_0z_1z_2 + A_1z_1 + A_2z_2)^i x = 0\right\}.$$

Since the space X^I is the null space of the infinite matrix

$$0_\infty \triangleq \begin{bmatrix} CM_{00} \\ CM_{10} \\ CM_{01} \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

the realization $\Sigma = (A_0, A_1, A_2, B, C)$ is L-observable if $X^I = \{0\}$, that is if 0_∞ is full rank.

The matrices R_∞ and 0_∞ contain an infinite number of elements. Nevertheless the evaluation of their ranks, which is essential to reachability and observability analysis, can be confined to check the rank of two submatrices $R \in K^{n \times n^2}$ and $0 \in K^{n^2 \times n}$ given by

$$R = [M_{00}B\ \ M_{10}B\ \ \cdots\ \ M_{n-1,n-1}B]$$

and

$$0 = \begin{bmatrix} CM_{00} \\ CM_{10} \\ \cdot \\ \cdot \\ \cdot \\ CM_{n-1,n-1} \end{bmatrix}.$$

This statement will be proved in Lemma 4.1 and Proposition 4.1. Operating in a different context Roesser presented in [11] an analogous result.

Lemma 4.1: Let A_0, A_1, A_2 belong to $K^{n \times n}$ and let

$$M_{ij} = \left(\sum_0^\infty {}_h (A_0z_1z_2 + A_1z_1 + A_2z_2)^h, z_1^iz_2^j\right) \qquad i,j \in \mathcal{Z}$$

be the coefficients of $z_1^iz_2^j$ in the series $(I - A_0z_1z_2 - A_1z_1 - A_2z_2)^{-1}$. Then there exist $b_{ij} \in K$, $i,j = 0,1,2,\cdots,n$, $b_{nn} \neq 0$ such that

$$\sum_0^n {}_{ij} M_{i-h,j-k} b_{ij} = 0$$

for all

$$(h,k) \notin \{(1,1),(1,2),\cdots,(n,n)\}.$$

The scalars b_{ij} can be assumed as the coefficients in the polynomial

$$\det(Iz_1^{-1}z_2^{-1} - A_0 - A_1z_2^{-1} - A_2z_1^{-1}).$$

Proof: Since

$$\sum_0^\infty {}_h (A_0z_1z_2 + A_1z_1 + A_2z_2)^h$$
$$= \frac{(z_1z_2)^{-1}\operatorname{adj}(Iz_1^{-1}z_2^{-1} - A_0 - A_1z_2^{-1} - A_2z_1^{-1})}{\det(Iz_1^{-1}z_2^{-1} - A_0 - A_1z_2^{-1} - A_2z_1^{-1})}$$
$$= \frac{1}{\sum_0^n {}_{ij} b_{ij} z_1^{-i}z_2^{-j}} \sum_1^n {}_{r,s} N_{rs} z_1^{-r}z_2^{-s}, \qquad b_{nn} = 1, \quad N_{rs} \in K^{n \times n}$$

it follows that

$$\sum_1^n {}_{rs} N_{rs} z_1^{-r}z_2^{-s} = \sum {}_{hk} M_{hk} z_1^hz_2^k \sum_0^n {}_{ij} b_{ij} z_1^{-i}z_2^{-j}$$
$$= \sum {}_{hk}\left(\sum_0^n {}_{ij} M_{i-h,j-k} b_{ij}\right) z_1^{-h}z_2^{-k}.$$

Equating the coefficients of the same powers in both sides one gets the proof.

Proposition 4.1: Let M_{ij} as in Lemma 4.1. Then

$$\operatorname{span}(M_{ij}, i,j \in \mathbf{Z}) = \operatorname{span}(M_{ij}, i,j = 0,1,\cdots,n-1) \triangleq \mathfrak{M}.$$

Proof: It is sufficient to prove that if r,s are nonnegative integers and either $r \geqslant n$ or $s \geqslant n$, then

$$M_{ij} \in \mathfrak{M}, \qquad i \leqslant r, j \leqslant s,\ (i,j) \neq (r,s)$$

implies

$$M_{rs} \in \mathfrak{M}.$$

In fact, by Lemma 4.1,

$$\sum_0^n {}_{ij} M_{i-n+r,j-n+s} b_{ij} = 0$$

so that

$$M_{rs} = -\frac{1}{b_{nn}} \sum_{\substack{0\\(i,j)\neq(n,n)}}^n {}_{i,j} M_{i-n+r,j-n+s} b_{ij}.$$

Remark 1: The above result can be defined when $r \geqslant n$ and $s < n$. In fact,

$$M_{rs} \in \operatorname{span}(M_{ij},\ i = 0,\cdots,n-1, j = 0,\cdots,s).$$

Remark 2: The Cayley–Hamilton theorem is a particular case of Proposition 4.1 when $A_0 = A_1 = 0$.

Applying Proposition 4.1 we can write $\operatorname{rank} R_\infty = \operatorname{rank} R$, $\operatorname{rank} 0_\infty = \operatorname{rank} 0$, which proves the following.

Proposition 4.2: A realization Σ is L-reachable (L-observable) if and only if $R\,(0)$ is full rank.

The matrices R and 0 are called reachability matrix and observability matrix associated with the realization $\Sigma = (A_0, A_1, A_2, B, C)$.

V. Computation of a Reachable and Observable Realization

So far we have seen that reachability and observability of a realization are strictly connected with the ranks of the reachability matrix R and observability matrix 0. We are now concerned with the following problem: suppose that we obtained by some algorithm (see Section III) a realization $\Sigma=(A_0,A_1,A_2,B,C)$ of dimension n and we would like to construct a L-reachable and an L-observable realization, starting from Σ. To solve this problem we introduce two algorithms which act independently to give an L-reachable and an L-observable realization, respectively. Of course the alternate application of these provides realizations which are eventually both L-reachable and L-observable.

Let us assume that $\Sigma=(A_0,A_1,A_2,B,C)$ is a realization of dimension n of a given filter $\mathcal{S}$, and let R be the reachability matrix of Σ. Assume $\operatorname{rank} R=r<n$. The algorithm for constructing a L-reachable realization of dimension r is based on the following two steps.

Step 1: Construct a nonsingular matrix $T\in K^{n\times n}$ having the last $n-r$ rows orthogonal to the space spanned by the columns of R. Consider then the realization $(\hat{A}_0, \hat{A}_1,\hat{A}_2,\hat{B},\hat{C})$ characterized by

$$\hat{A}_i=TA_iT^{-1},\qquad i=0,1,2$$
$$\hat{B}=TB$$
$$\hat{C}=CT^{-1}.$$

The matrix T induces a change of basis in X. The first r elements of this new basis are a basis for X^R.

Step 2: Write $\hat{A}_0,\hat{A}_1,\hat{A}_2$ in partitioned form

$$\hat{A}_k=\begin{bmatrix}\hat{A}_{11}^{(k)} & \hat{A}_{12}^{(k)}\\ \hat{A}_{21}^{(k)} & \hat{A}_{22}^{(k)}\end{bmatrix},\qquad \hat{A}_{11}^{(k)}\in K^{r\times r},\ k=0,1,2$$

and partition $\hat{B}$ and $\hat{C}$ conformably,

$$\hat{B}=\begin{bmatrix}\hat{B}_1\\ 0\end{bmatrix},\qquad \hat{C}=[\hat{C}_1\ \ \hat{C}_2],\qquad \hat{B}_1,\hat{C}_1^T\in K^{r\times 1}.$$

Then $(\hat{A}_0,\hat{A}_1,\hat{A}_2,\hat{B},\hat{C})$ and $(\hat{A}_{11}^{(0)},\hat{A}_{11}^{(1)},\hat{A}_{11}^{(2)},\hat{B}_1,\hat{C}_1)$ realize the same filter, and $(\hat{A}_{11}^{(0)},\hat{A}_{11}^{(1)},\hat{A}_{11}^{(2)},\hat{B}_1,\hat{C}_1)$ is L-reachable. In fact, let $x(h,k)$ be the state reached by the effect of an input $u\in\mathcal{U}$, and assume as a basis in X the basis corresponding to $(\hat{A}_0,\hat{A}_1,\hat{A}_2,\hat{B},\hat{C})$. With respect to such a basis the last $n-r$ components of $x(h,k)$ are zero and the system

$$\begin{bmatrix}\hat{A}_{11}^{(0)} & \hat{A}_{12}^{(0)}\\ \hat{A}_{21}^{(0)} & \hat{A}_{22}^{(0)}\end{bmatrix}\begin{bmatrix}x_1(h,k)\\ 0\end{bmatrix}+\begin{bmatrix}\hat{A}_{11}^{(1)} & \hat{A}_{12}^{(1)}\\ \hat{A}_{21}^{(1)} & \hat{A}_{22}^{(1)}\end{bmatrix}\begin{bmatrix}x_1(h+1,k)\\ 0\end{bmatrix}$$
$$+\begin{bmatrix}\hat{A}_{11}^{(2)} & \hat{A}_{12}^{(2)}\\ \hat{A}_{21}^{(2)} & \hat{A}_{22}^{(2)}\end{bmatrix}\begin{bmatrix}x_1(h,k+1)\\ 0\end{bmatrix}$$
$$+\begin{bmatrix}\hat{B}_1\\ 0\end{bmatrix}u(h,k)=\begin{bmatrix}x_1(h+1,k+1)\\ 0\end{bmatrix}$$
$$y(h,k)=[\hat{C}_1\ \ \hat{C}_2]\begin{bmatrix}x_1(h,k)\\ 0\end{bmatrix}$$

realizes the same input-output map as

$$\hat{A}_{11}^{(0)}x_1(h,k)+\hat{A}_{11}^{(1)}x_1(h+1,k)+\hat{A}_{11}^{(2)}x_1(h,k+1)$$
$$+\hat{B}_1u(h,k)=x_1(h+1,k+1)$$
$$y(h,k)=\hat{C}_1x_1(h,k).$$

Assume now $\operatorname{rank} 0=r'<n$. The algorithm for obtaining a L-observable realization is also based on two steps and is substantially the same as the above reachability algorithm, although the proof of the second step is based on somewhat different reasonings.

Step I: Construct a nonsingular matrix $Q^{-1}\in K^{n\times n}$ having the last $n-r'$ columns orthogonal to the space spanned by the rows of 0. The realization $(\tilde{A}_0,\tilde{A}_1,\tilde{A}_2,\tilde{B},\tilde{C})$ defined by

$$\tilde{A}_i=QA_iQ^{-1},\qquad i=0,1,2$$
$$\tilde{B}=QB$$
$$\tilde{C}=CQ^{-1}$$

satisfies

$$C(I-A_0z_1z_2-A_1z_1-A_2z_2)^{-1}B$$
$$=\tilde{C}(I-\tilde{A}_0z_1z_2-\tilde{A}_1z_1-\tilde{A}_2z_2)^{-1}\tilde{B}=\sum_{ij}\tilde{C}\tilde{M}_{ij}\tilde{B}z_1^iz_2^j.$$

In the associated observability matrix

$$\tilde{0}=\begin{bmatrix}\tilde{C}\tilde{M}_{00}\\ \tilde{C}\tilde{M}_{10}\\ \cdot\\ \cdot\\ \cdot\\ \tilde{C}\tilde{M}_{n-1,n-1}\end{bmatrix},$$

the elements in the last $n-r'$ columns are zeros.

Step II: Write $\tilde{A}_0,\tilde{A}_1,\tilde{A}_2,\tilde{B}_1,\tilde{C}$ in partitioned form as in Step 2 above, and notice that $\tilde{C}=[\tilde{C}_1\ \ 0], \tilde{C}_1\in K^{1\times r'}$. To show that $(\tilde{A}_{11}^{(0)},\tilde{A}_{11}^{(1)},\tilde{A}_{11}^{(2)},\tilde{B}_1,\tilde{C}_1)$ is a realization we have to prove that

$$\sum_{ij}\tilde{C}\tilde{M}_{ij}\tilde{B}z_1^iz_2^j=\tilde{C}(I-\tilde{A}_0z_1z_2-\tilde{A}_1z_1-\tilde{A}_2z_2)^{-2}\tilde{B}$$
$$=\tilde{C}_1(I-\tilde{A}_{11}^{(0)}z_1z_2-\tilde{A}_{11}^{(1)}z_1-\tilde{A}_{11}^{(2)}z_2)^{-1}\tilde{B}_1$$
$$\triangleq\sum_{ij}\tilde{C}_1M_{ij}^*\tilde{B}_1z_1^iz_2^j,$$

namely that

$$\tilde{C}\tilde{M}_{ij}\tilde{B}=\tilde{C}_1M_{ij}^*\tilde{B}_1,\qquad i,j=0,1,\cdots.$$

Observe that for $i,j>0$

$$\tilde{C}\tilde{M}_{ij}=\tilde{C}\tilde{M}_{i-1,j}\tilde{A}_1+\tilde{C}\tilde{M}_{i,j-1}\tilde{A}_2+\tilde{C}\tilde{M}_{i-1,j-1}\tilde{A}_0$$
$$\tilde{C}_1M_{ij}^*=\tilde{C}_1M_{i-1,j}^*\tilde{A}_{11}^{(1)}+\tilde{C}_1M_{i,j-1}^*\tilde{A}_{11}^{(2)}+\tilde{C}_1M_{i-1,j-1}^*\tilde{A}_{11}^{(0)}$$

and assume by induction that the first r' elements of $\tilde{C}\tilde{M}_{i,j-1}$, $\tilde{C}\tilde{M}_{i,j-1}$, $\tilde{C}\tilde{M}_{i-1,j-1}$ coincide with $\tilde{C}_1M_{i-1,j}^*$, $\tilde{C}_1M_{i,j-1}^*$, $\tilde{C}_1M_{i-1,j-1}^*$, respectively. Then

$$\tilde{C}\tilde{M}_{i-1,j}\tilde{A}_1=[\tilde{C}_1M_{i-1,j}^*\ \ 0]\begin{bmatrix}\tilde{A}_{11}^{(1)} & \tilde{A}_{12}^{(1)}\\ \tilde{A}_{21}^{(1)} & \tilde{A}_{22}^{(1)}\end{bmatrix}=[\tilde{C}_1M_{i-1,j}^*\tilde{A}_{11}^{(1)}\ \ *]$$

and similarly,

$$\tilde{C}\tilde{M}_{i,j-1}\tilde{A}_2 = \left[\tilde{C}_1 M^*_{i,j-1}\tilde{A}^{(2)}_{11} \;\; *\right]$$

$$\tilde{C}\tilde{M}_{i-1,j-1}\tilde{A}_0 = \left[\tilde{C}_1 M^*_{i-1,j-1}\tilde{A}^{(0)}_{11} \;\; *\right]$$

from which it is clear that the first r' elements in $\tilde{C}\tilde{M}_{ij}$ are the same as in $\tilde{C}_1 M^*_{ij}$. Of course the last $n-r'$ elements are zero because of the structure of $\tilde{0}$.

Evidently, the result that we have proved can also be stated in the following form.

Proposition 5.1: Let $\Sigma = (A_0, A_1, A_2, B, C)$ be any realization of $\mathcal{S}$. Then a L-reachable and L-observable realization can be constructed in a finite number of steps from Σ following the procedure introduced above.

Corollary: Every minimal realization is completely L-reachable and L-observable.

VI. Remarks on Minimal Realizations

As it was pointed at the end of the previous section, the minimality of a realization implies that this is L-reachable and L-observable.

In this section we will show by means of a counterexample that the converse of this fact does not hold. In the example we deal with filters characterized by recognizable series. The properties of these filters have been already presented and we refer to [3] for the details. We summarize in Proposition 6.1 below some relevant aspects connected with the realization problem.

Proposition 6.1: Let $s \in K_c^{\text{rec}}[(z_1, z_2)]$. Then the following statements are equivalent.

1) Rank $\mathcal{H}((z_1z_2)^{-1}s) = n < \infty$.

2) There exist L-reachable and L-observable realizations $\Sigma = (A_0, A_1, A_2, B, C)$ of dimension n satisfying

$$A_0 = -A_1A_2 = -A_2A_1.$$

Example: Assume $s \in \mathfrak{R}^{\text{rec}}[(z_1, z_2)]$ is given by

$$s = (z_1z_2)\frac{1+z_1+z_2}{1+z_1+z_2+z_1z_2}.$$

Therefore, the formal power series expansion of $(z_1z_2)^{-1}s$ is expressed by

$$(1+z_1+z_2)\sum_0^\infty {}_k(-1)^k(z_1+z_2+z_1z_2)^k$$
$$= 1 - z_1z_2 + z_1z_2^2 + z_1^2z_2 \cdots.$$

By Proposition 6.1 there exist L-reachable and L-observable realizations whose dimension is rank $\mathcal{H}((z_1z_2)^{-1}s)$. It is easy to check that rank $\mathcal{H}((z_1z_2)^{-1}s) \geq 3$. On the other hand, the doubly indexed dynamical system $\Sigma = (A_0, A_1, A_2, B, C)$ with

$$A_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = [1 \;\; 1]$$

is a realization with dimension two. This proves the existence of L-reachable and L-observable realizations which are not minimal.

From the above remarks it appears that the Hankel matrix $\mathcal{H}(s)$ is not relevant for evaluating the dimension of minimal realizations of s. In fact when s is rational, but not recognizable, rank $\mathcal{H}(s)$ is infinite and for s recognizable, rank $\mathcal{H}(s)$ furnishes solely the minimal dimension of realizations satisfying

$$A_0 = -A_1A_2 = -A_2A_1.$$

VII. Conclusions

In this paper the algebraic realization problem of two-dimensional linear filters has been approached from a system theoretic point of view. The input-output behavior of such systems is defined by formal power series in two variables and the state is introduced by means of Nerode equivalence classes. The Nerode state space is in general infinite-dimensional; nevertheless if the formal power series which characterizes the input-output map is rational, a finite-dimensional local state space is defined and the dynamics of the filter is then described by updating equations on the local spaces. An explicit algorithm for constructing the matrices of a realization is given.

The notions of local reachability and local observability have been introduced and an algorithm is presented which allows us to obtain a reachable and observable realization starting from a generic one.

In general, the reachability and observability properties do not guarantee that we are dealing with a minimal realization, as we have proved by means of an example.

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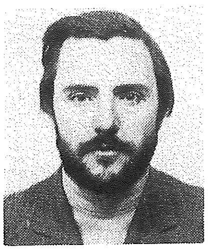

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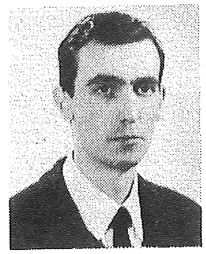

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