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LINEAR SYSTEMS ON FREE GROUPS

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Abstract

Basic definitions are given for an input/output theory and transform calculus for stationary, discrete-time linear systems on a time set isomorphic to a free group. Relationships with free ideal rings are pointed out, and an algorithm for computing minimal realizations is conjectured.

1. Introduction

This preliminary report investigates stationary linear systems on a time set which is isomorphic with a free group G. (In this paper, G is free on two letters, but this is just a notational convenience.) Although we have not described the connections explicitly here, most of this work represents yet another attempt to unify automaton theory and control theory. A good subtitle might be "Variations on a theme of M. Fliess [3,4]."

The first goal is to develop a Kalman-type input/output structure and a corresponding "extended" or Laurent-series i/o structure. The resulting "Z-transform" calculus seems to depend on careful choices of module actions with particular attention to right actions versus left actions. After much anguish, the authors are reasonably certain of consistency.

In addition to the basic definitions, some attractive relationships between system theory and the theory of free ideal rings are presented. In particular, the dimension of a minimal realization is closely related to the number of generators of the Nerode kernel. A conjecture giving a method to calculate these generators is given in a special case.

The methodology of this paper is derived from [7,8]. In addition to relationships with automata and formal languages, these results are closely related to two-dimensional digital filters [5].

2. Systems

Let G be the free group on two letters σ_1 and σ_2, so that a typical element of G is a word $w(\sigma,\sigma^{-1}) = x_1x_2\ldots x_k$ with each x_i either σ_1, σ_1^{-1}, σ_2, or σ_2^{-1}. The empty word e (with k=0) is the identity of G. Denote the "positive semigroups" in G by $P=\{\sigma_1, \sigma_2\}^*$ = words $w(\sigma)$ in σ_1, σ_2, and the "negative semigroups" by $N = \{\sigma_1^{-1}, \sigma_2^{-1}\}^*$ = words in σ_1^{-1}, σ_2^{-1}.

For notational clarity, denote by T a second copy of G to be used as a "time set" for the linear systems discussed below. The group G will act on the right of T by $t \mapsto t\, w(\sigma,\sigma^{-1})$ (defined since T is really the same as G). Define a partial order $\leq$ on T by $t_1 \leq t_2$ if $t_1 w = t_2$ for some w in P. The poset $(T, \leq)$ can be thought of as a homogenous tree branching in two "future directions" and two "past directions" (see Fig. 1).

A <u>stationary</u>, <u>linear</u>, <u>discrete-time</u> <u>system</u> <u>on</u> $(T,\leq)$ is given by $\Sigma = (Z,U,Y, A_1,A_2,B,C)$, where X,U and Y are vector spaces over a field k, and A_1, A_2: X → X, B: U → X, and C: X → Y are k-linear maps. The dynamical structure of Σ is defined by vector difference equations

$$x(t) = A_1 x(t\sigma_1^{-1}) + A_2\, x(t\sigma_2^{-1}) + Bu(t)$$
$$y(t) = C\, x(t) \qquad (2.1)$$

for all t in T.

The state space X will be considered as a left module over the ring $k \langle z_1\ z_2 \rangle$ of non-commuting polynomials in two variables, with $z_1x = A_1x$ and $z_2x = A_2x$ for all x in X. If w is a word over some two symbol alphabet, we abbreviate the general formula by $w(z)x = w(A)x$.

This structure is well-known and was introduced by Fliess [3] under the name "serial module".

3. Input/Output Mappings

The $k\langle z_1, z_2\rangle$ – action on the state space X (or, more precisely, the ring inclusion $k \subseteq k\langle z_1, z_2\rangle$) leads to the input and output string functors

$$\Omega U = k\langle z_1, z_2\rangle \otimes_k U$$

$$\Gamma Y = \mathrm{Hom}_k (k\langle z_1, z_2\rangle, Y),$$

as in [6,7].

The module ΩU is a left $k<z_1,z_2>$ - module by $z_1(w(z)\otimes u) = z_1 w(z)\otimes u$ as usual. To describe the left $k<z_1,z_2>$ - action on ΓY, write mappings h in ΓY on the right, as $w(z)\mapsto w(z)h$ in Y. Then $z_1 h$ acts by $w(z)(z_1 h) = (w(z)z_1)h$ for all words $w(z)$.

The reachability and observability maps in this context are given by

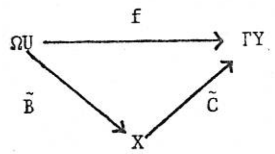
$$\tilde{B}: \Omega U \to X$$

$$(w(z)\otimes u)\tilde{B} = w(A)Bu, \text{ and}$$

$$\tilde{C}: X \to \Gamma Y$$

$$w(z)(x\tilde{C}) = C\cdot w(A)x,$$

for all words $w(z)$ in z_1 and z_2. These maps fit into the usual realization diagram of left $k<z_1,z_2>$ - modules



where the i/o map $f = \tilde{B}\tilde{C}$ is uniquely given by the Markov parameters $w(z)(1\otimes u)f = Cw(A)B$.

Existence and uniqueness of a (not necessarily finite dimensional) minimal realization of an abstract i/o map $f:\Omega U \to \Gamma Y$ is routine. Finiteness of the minimal realization follows from the Theorem of Kleine-Schutzenberger [2,p.175] just in case f is rational. (Note that a scalar i/o map $f: k<z_1,z_2> \to \Gamma k$ is essentially the same as a k-subset of $\{z_1,z_2\}^*$ as defined in [2,p.126].

4. Formal Laurent Series

The (Kalman-type) i/o-maps discussed above can be put into an "extended" or "Laurent" context. (Compare [6].) For any vector space V, define

$$LV \subseteq \{\hat{v}: T \to V\}$$

as the set of all "left-finite functions" such that for all t in T, $\{s \leq t: v(s) \neq 0\}$ is finite. To get a formal series representation, think of $\hat{v} = \Sigma\, \hat{v}(t)t$.

The set $L(V)$ accepts several useful actions. First, $L(V)$ is a left $k<z_1,z_2>$ - module by the formula $(z_i\hat{v})(t) = \hat{v}(\sigma_i t)$, and, more generally, $(w(z)\hat{v})(t) = \hat{v}(w^{-1}(\sigma^{-1})t)$, where w is any word in two symbols. For example, if $w(z) = z_1^2 z_2$, then $w^{-1}(z) = z_2^{-1} z_1^{-2}$, and $w^{-1}(\sigma^{-1}) = \sigma_2\sigma_1^2$. To motivate the action, think of z_i as acting on series as σ_i^{-1}, so $z_i(\Sigma\hat{v}(t)t) = \Sigma\hat{v}(t)\sigma_i^{-1}t = \Sigma\hat{v}(\sigma_i t)t$.

Second, $L(V)$ is a right module over the ring $k<<\sigma_1, \sigma_2>>$ of non-commutative formal power series in the group elements σ_1,σ_2. In fact, if $p=\Sigma p_s s$ is a power series, then $(\hat{v}p)(t) = \Sigma p_s\hat{v}(t\, s^{-1})$. Here s is a string in $\{\sigma_1, \sigma_2\}^*$, so s^{-1} is a string in $\{\sigma_1^{-1}, \sigma_2^{-1}\}^*$, $ts^{-1} \leq t$, and the sum is finite by the assumption of $\hat{v}$.

In order to develop an i/o-theory on the Laurent series level, it is necessary to give left $k<z_1,z_2>$ - linear maps $i:\Omega U \to LU$ and $p: LY \to \Gamma Y$ such that i is injective and p is surjective. These mappings are given as follows:

$$i: \Omega U \to LU$$

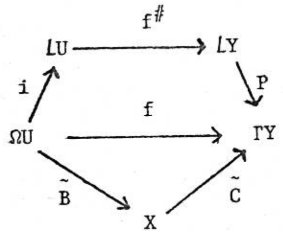
$$(w(z)\otimes u)\, i\, (t) = \begin{cases} u \text{ if } t = w(\sigma^{-1}) \\ 0 \text{ otherwise} \end{cases}$$

$$p: LY \to \Gamma Y$$

$$w(z)\ (\hat{v}p) = \hat{v}(w^{-1}(\sigma^{-1}))$$

In both these cases we have identified G with T.

All these mappings fit together into a big diagram



where $f^{\#}$ is defined as follows: let $M_w = Cw(A)B$: $U \to Y$ be the Markov parameters, and let u be in U. Then

$$(\hat{u}f^{\#})(t) = \sum_w M_w\, \hat{u}(t\, w^{-1}(\sigma)).$$

Furthermore, $f^{\#}$ is a left $k<z_1,z_2>$ - module map which is uniquely determined by the M_w, and vice-versa.

5. Transform Calculus

In terms of the module actions defined above, the standard updating equations (2.1) can be rewritten as

$$\hat{x} = A_1\, \hat{x}\, \sigma_1 + A_2\, \hat{x}\, \sigma_2 + B\hat{u}$$
$$\hat{y} = C\, \hat{x}\,. \tag{5.1}$$

Here A_1, A_2: $LX \to LX$, B: $LU \to LX$, and C: $LX \to LY$ are given their natural meaning (e.g. $(C\hat{x})(t) = Cx(t)$) and $\hat{X}$ is considered as a right $k<<\sigma_1,\sigma_2>>$ - module. Since mixed actions of the type $\hat{X} \mapsto A_1\hat{X}\sigma_1$ are hard to use, the enveloping algebra $M_n(k)^{op} \otimes_k k<<\sigma_1,\sigma_2>>$ can be introduced, with right actrons defined by, e.g.,

$$\hat{X}(A_1^{op} \otimes \sigma_1) = A_1 \hat{X} \sigma_1.$$

Thus works since $(A_1A_2)^{op} = A_2^{op}A_1^{op}$ in the opposite ring $M_n(k)^{op}$. Then we have

$$\hat{X}(1 - A_1^{op}\sigma_1 - A_2^{op}\sigma_2) = B\hat{u}$$
$$\hat{y} = C\hat{x} \quad (5.2)$$

so that

$$\hat{y} = \hat{u} f^{\#} = C B\hat{u} (1 - A_1^{op}\sigma_1 - A_2^{op}\sigma_2)^{-1},$$

or, by expanding the right hand side,

$$\hat{y}(t) = \sum_w C w(A) B\hat{u} (t w^{-1}(\sigma)), \quad (5.3)$$

which corresponds to the familiar formula on the line.

The same technique can be used to study high-order scalar recurrences on a free group. Suppose an i/o-map is defined by

$$y(t) = \sum_w d_w y(t w^{-1}(\sigma)) + bu(t), \quad (5.4)$$

where w runs over words in two letters, and only finitely many of the scalars d_w are not zero. This gives

$$\hat{y} = \sum_w d_w \hat{y} w(\sigma) + b \hat{u}$$
$$\hat{u} f = \hat{y} = \hat{u}b (1 - \sum d_w w(\sigma))^{-1}. \quad (5.5)$$

Thus a type of recurrence familar to control theorists defines a rational function (equivalently, a regular language) of a rather special type. This case will be studied more closely in the next section.

6. Realizing Scalar Recurrences

Suppose given a scalar recurrence of the form (5.4) above. The transfer function (5.5) defines an i/o map f: $\Omega k \to \Gamma k$ as usual, and in this case $\Omega k = k\langle z_1, z_2\rangle$. The left ideal I = ker(f) may be called the Nerode ideal, and the minimal state space for f is just the cyclic left $k\langle z_1, z_2\rangle$ - module $k\langle z_1, z_2\rangle/I$. One important goal is the explicit computation of I, say by giving a list of generators, and the explicit construction of the state module from the ideal. This procedure is more or less equivalent to the construction of a (non-deterministic) k-valued finite state automation from a given regular language.

The following ring theoretic result is very suggestive (1, p.85, bottom): let $R = k\langle z_1, z_2, \ldots, z_d\rangle$, and let I be an ideal of R. It is known that I is a free R-module (R is a "free ideal ring"). Suppose that I has finite rank r, so that I has r free generators, and suppose also that R/I is finite dimensional of dimension n. Then r-1 = n(d-1). In the case considered here d=2, so r-1 = n.

We can use this result in specific calculations as follows: suppose we find elements $p_1, \ldots, p_t$ in I, and define the left ideal $I_0 = Rp_1 + \ldots + Rp_t$. Then I_0 is free of rank $r \leq t$ (with equality of the P_i are independent). Then either R/I_0 is infinite dimensional, or else $\dim_k (R/I_0) = r-1$.

A rather detailed conjecture is available in the scalar case. Consider the recurrence (5.5) and set $D = 1 - \Sigma d_w w(\sigma)$. For u(z) in Ωk, compute u(z)D, a polynomial in $z_1, z_2, \sigma_1, \sigma_2$, and set $z_i \sigma_i = 1$, i = 1,2. Finally, define the truncation Tr(u(z)D) by omitting all mixed terms. (For example, $z_1 \sigma_2$ and $z_2^2 \sigma_1 \sigma_2$ would be omitted.)

Conjecture: Let f: $\Omega k \to \Gamma k$ be given by $u(z) f = u(z)D^{-1}$ (in the sense of (5.5).) Then ker f is generated by the set {Tr(u(z)D)}, where u(z) runs through all polynomials such that Tr(u(z)D) lies in $k z_1, z_2$.

As an illustration, consider the recurrence $y(t) = y(t\sigma_1^{-1}) + y(t\sigma_1^{-1}\sigma_2^{-1}) + u(t)$, or $y = u(1 - \sigma_1 - \sigma_2\sigma_1)^{-1}$. Let $D = 1 - \sigma_1 - \sigma_2\sigma_1$. Then

$$z_1D = z_1 - 1 - z_1\sigma_2\sigma_1 \to z_1 - 1$$
$$z_2D = z_2 - z_2\sigma_1 - \sigma_1 \to z_2 - \sigma_1$$
$$z_2^2D = z_2^2 - z_2^2\sigma_1 - z_2\sigma_1 \to z_2^2$$
$$z_1z_2D = z_1z_2 - z_1z_2\sigma_1 - 1 \to z_1z_2 - 1.$$

Here the arrows represent truncation. Of these, the second is not usable, and the conjecture would suggest that the Nerode kernel I is generated by $z_1 - 1$, z_2^2, $z_1z_2 - 1$. Direct calculation shows that $k\langle z_1,z_2\rangle/I$ is 2 dimensional, so n = r - 1, and these are free generators of the kernel. Further calculation verifies that the resulting system is minimal, and it is easy to draw a two-state non-deterministic state diagram for this problem. More details, and hopefully a proof of the conjecture, will be submitted elsewhere.

7. Acknowledgements

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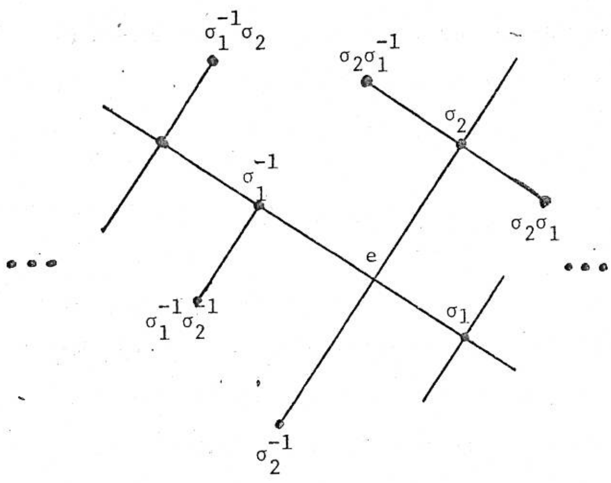

Figure 1