TWO DIMENSIONAL FILTERS AND THE PROBLEM OF REALIZATION(*)

E.Fornasini and G.Marchesini

Department of Electrical Engineering
University of Padova, Italy

ABSTRACT

Two-dimensional filters in input-output form are characterized by formal power series in two indeterminates. The realization problem consists in looking for updating equations operating on a state space.

The partial order on the time set induces two different kinds of states: an infinite dimensional global state space which derives directly from Nerode equivalence and a local state space. Whenever the input-output map is a rational power series, the local state space is finite dimensional and it is the natural framework for describing the local state evolution by appropriate updating equations. The updating equations on the local state space have the structure of a doubly-indexed dynamical system (2-D system).

The problem of constructing an efficient realization, that is a low dimensional one, is partially solved by linear computation of reachable and observable realization. Nevertheless these realization are in general not minimal and the minimality depends on the ground field.

A suitable condition of boundedness on initial states is required to introduce internal stability. Under this assumption the internal stability means that the free evolution approaches zero as the distance from initial states goes to infinity. Necessary and sufficient conditions for asymptotic stability are given and connections between internal and external stability are also considered.

1. INTRODUCTION

The aim of this communication is to report on problems and results in the study of dynamical models for two-dimensional filters.

Despite of several recent contributions [1-12], many problems are still unsolved and some of them will be mentioned in the paper. The construction of dynamical models as a generalization from the standard theory of linear systems, involves polynomials in two indeterminates, non commutative algebra techniques, partial orderings on time set, which represent a substantial difference between one-dimensional and two-dimensional systems.

2. ALGEBRAIC APPROACH TO REALIZATION

The realization problem constitutes a modern and formalized version of the classical engineering problem of constructing some devices on the basis of design specifications.

It essentially consists in determining dynamical models (i.e. state-space models) which exhibit a prescribed input-output behaviour. We therefore

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need: i) an abstract description of the particular class of input-output maps we deal with, ii) an abstract description of the dynamical models we propose to adopt, iii) an algorithm which allows us to pass from the input/output map to the dynamical model.

To give an example, let us consider the realization of discrete, linear invariant systems.

When one represents the input and the output sets by means of truncated formal Laurent series, the class of linear i/o map is the ring $K[[z]]$ of formal power series in one indeterminate. Denoting by s an element in $K[[z]]$, the output y is obtained from the input u as the Cauchy product su .

As it is well known, the linear dynamical models used in the realization of linear i/o maps, are the following

$$x(h+1) = A\,x(h) + B\,u(h)$$
$$y(h) = C\,x(h) \qquad (1)$$

In fact a linear i/o map s, is realized by model (1) (for suitable A, B, C) if and only if s is rational. In this case there are infinitely many realizations of s, and the natural problem is to single out the "most efficient" ones. The Nerode equivalence provides a canonical solution, in the sense that it is essentially unique: the state space is the set of Nerode equivalence classe, and matrices (A, B, C) are determined modulo a basis transformation.

In this section a short account will be given of state-space realization of two-dimensional filters. We therefore need an axiomatic description of their i/o behaviour as well of the class of dynamical models we consider as candidate at their realization.

The sets of input and output functions of a two-dimensional digital filter are subclasses of $K^{\mathbb{Z}\times\mathbb{Z}}$, K being a generic field. These functions are represented as formal power series in two indeterminates z_1 and z_2. To characterize the set of these functions assume in $\mathbb{Z}\times\mathbb{Z}$ the product of orderings and introduce the notions of "past" and "future" of a point (h,k)in $\mathbb{Z}\times\mathbb{Z}$.

We shall call "past" of (h,k) the set of points (i,j), such that $i \leq h$, $j \leq k$, $(i,j) \neq (h,k)$ and "future" of (h,k) the set of points (i,j) with $h \leq i$, $k \leq j$.

We say that a function $u \in K^{\mathbb{Z}\times\mathbb{Z}}$ is past-finite if the intersection of the support of u and the past of any point in $\mathbb{Z}\times\mathbb{Z}$ is a finite set. The set of functions $\mathcal{U}$ we shall assume as admissible inputs to the filter are the past-finite functions in $K^{\mathbb{Z}\times\mathbb{Z}}$.

With this in mind we define a linear, stationary, digital filter in input-output form, as a map $F.\mathcal{U}\to K^{\mathbb{Z}\times\mathbb{Z}}$ which satisfies the following axioms:

(i) linearity

(ii) stationarity:

$$F(z_1^h z_2^k u) = z_1^h z_2^k F(u), \quad \forall h,k \in \mathbb{Z},\ \forall u \in \mathcal{U}$$

(iii) causality:
the support of $F(1)$ belongs to the future of (0,0) and does not contain the point (0,0)

We can directly check that $\operatorname{Im} F \subset \mathcal{U}$ and that the impulse response $F(1)$ constitutes the transfer function of the filter, for we have (in formal power series notation):

$$F(u) = F(1)u \qquad \forall u \in \mathcal{U}$$

In this way the input-output representations of two-dimensional filters are in one-to-one correspondence with the formal power series in z_1 and z_2 with zero constant term, called "causal formal power series" and denoted by $K_c[[z_1,z_2]]$.

This result generalizes the above mentioned connection existing between input-output representations of one-dimensional systems and formal power series in one indeterminate.

The realization of one-dimensional systems is done by introducing a time vector function $x(\cdot)$, called the state of the system, which has a separation property with respect to the past, in the sense that the knowledge of this vector at any instant t is sufficient to evaluate the output at $\tau \geq t$. When one deals with two-dimensional filters it is no longer possible to attach a vector having a separation property to any point (h,k) in $\mathbb{Z}\times\mathbb{Z}$ in such a way that the knowledge of this vector and of the input in the future of (h,k) makes possible to compute the output in the future of (h,k). Clearly this fact is intrinsic to the structure of partial ordering of $\mathbb{Z}\times\mathbb{Z}$, since a separation property should interest an infinite set of points.

It is worth while to recall that the structure of model (1) can be axiomatically derived from Nerode equivalence on the input space. With the aim of singling out state space models which realize two-dimensional filters we are naturally led to extend Nerode equivalence to the input space $\mathcal{U}$.

For this consider in $\mathbb{Z}\times\mathbb{Z}$ a non empty set $\mathcal{C}$, called "separation set", which satisfies the following characteristic properties:

(i) if $h > i$, $k > j$, (h,k) and (i,j) cannot simultaneously belong to $\mathscr{C}$

(ii) if (h,k) belongs to $\mathscr{C}$, then $\mathscr{C}$ intersects the sets $\{(h-1,k), (h,k+1), (h-1,k+1)\}$ and $\{(h+1,k), (h, k-1), (h+1,k-1)\}$ and does not contain the set $\{(h+1,k), (h,k+1)\}$

(iii) if (h,k) is in the future of $\mathscr{C}$, then there is only a finite number of elements (i,j) in $\mathscr{C}$ with $(h,k) \geq (i,j)$

Thus the plane $\mathbb{Z} \times \mathbb{Z}$ is partitioned in two subsets:

$P_{\mathscr{C}} = \{(i,j): (i,j) \in \mathscr{C}$ or (i,j) belongs to the past of some point in $\mathscr{C}\}$

$F_{\mathscr{C}} = \{(i,j): (i,j) \notin \mathscr{C}$ and (i,j) belongs to the future of some point in $\mathscr{C}\}$

Assuming the input be zero in $F_{\mathscr{C}}$, then the knowledge of the input in $P_{\mathscr{C}}$ is necessary and sufficient to compute the output in $F_{\mathscr{C}}$.

Plainly there are infinitely many possibilities of shaping the set $\mathscr{C}$. In particular image processing usually refers to separation sets as in fig. 1.

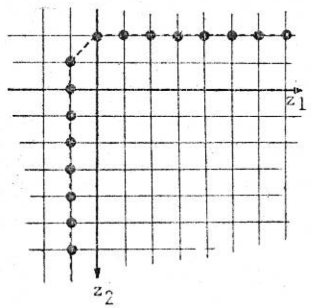

fig. 1

Let $\mathscr{U}^*$ denote the set of functions $u \in \mathscr{U}$ with support in $P_{\mathscr{C}}$ and $\mathscr{Y}^*$ the set of functions with support in $F_{\mathscr{C}}$. For every $u \in \mathscr{U}^*$ let $f(u)$ denote the restriction of $F(u)$ to $F_{\mathscr{C}}$. This defines a linear map $f: \mathscr{U}^* \to \mathscr{Y}^*$ which characterizes the filter in the same sense as F does.

After introducing a concatenation in $\mathscr{U}^*$ [2,5], it is easy to check that f-equivalence classes in $\mathscr{U}^*$ coincide with Nerode equivalence classes which turn out to be the cosets of $\mathscr{U}^*$ relative to ker f.

The space $\mathscr{U}^*/\ker f$ displays the "memory function" of the i/o map and it can be assumed as the state space X_N of a dynamical system which realizes the two-dimensional filter.

This construction is clearly canonical but suffers from the drawback that the dimension of X_N is infinite even if the i/o map is given by a rational function. It seems therefore impossible to describe the dynamics of X_N in terms of appropriate updating equations.

This fact represents the first substantial difference with respect to discrete, linear systems where the Nerode state space is finite dimensional if and only if the i/o map is rational.

These difficulties can be overcome to some extent by introducing the notion of "local state space". A dynamical model, based on the introduction of a local state vector, is required to have a structure such that: (i) every point (i,j) in $\mathbb{Z} \times \mathbb{Z}$ is associated with a finite dimensional vector, called local state at (i,j), which linearly depends on the past of (i,j), (ii) the set of local states on a separation set $\mathscr{C}$ is necessary and sufficient to compute with linear operations the free evolution on $F_{\mathscr{C}}$.

These conditions are not sufficient for deriving univocally the structure of the updating equations. The class of models we shall define, is constituted by the so called "2-D systems" and has the property that it contains the other local state models.

DEFINITION. A double-indexed linear, stationary, finite-dimensional, dynamical system (2-D system) $\Sigma = (A_1, A_2, B_1, B_2, C)$ is defined by the first order partial difference equation:

$$x(h+1,k+1) = A_1x(h+1,k) + A_2x(h,k+1) + B_1u(h+1,k) + B_2u(h,k+1) \tag{2}$$

$$y(h,k) = Cx(h,k)$$

where $u(h,k)$, the input value at (h,k), and $y(h,k)$, the output value at (h,k), are in K and $h,k \in \mathbb{Z}$, $A_i \in K^{n\times n}$, $B_i \in K^{n\times 1}$, $C \in K^{1\times n}$, $i = 1,2$ and $x \in X = K^n$ (local state space).

It is straightforward to verify that the i/o relationship (transfer function) of a 2-D system, when it starts from zero local states on a separation set, is the same as a two-dimensional filter and it is given by the following rational series:

$$s_\Sigma = C(I - A_1z_1 - A_2z_2)^{-1} (B_1z_1 + B_2z_2)$$

Then it is clear that 2-D systems constitute a class of dynamical systems which can be used to realize two-dimensional filters. Actually every 2-D system can be viewed as a realization of its transfer function s_Σ.

Two problems naturally arise: the first is to specify the subclass of the class of two-dimensional filters which can be realized by 2-D systems; the second consists in setting up some techniques for obtaining the most "efficient" realization in the sense of the dimension of the local state.

The solution to the first problem recalls very closely the solution of the realization problem for discrete linear systems and is given by the following proposition [2,8]:

PROPOSITION. Let $s \in K[[z_1,z_2]]$. Then there exists a 2-D system which realizes s (i.e. whose trasfer function is s) if and only if s belongs to the set of rational series with zero constant term.

3. MINIMALITY

So far the second problem, that is to find realizations whose local state space shows the lowest dimension, has not been solved in a satisfactory way though several partial related results have been reached yet [5,9].

In this section we shall briefly report some partial solutions to this problem and some questions which still wait for a complete answer.

If we start from a rational transfer function, we can use existing techniques for computing the matrices (A_1,A_2,B_1,B_2,C) of a realization [5,6]. In general these procedures do not provide minimal realizations even if the numerator and denominator of the trasfer function are coprime polynomials. Nevertheless there exhist algorithms we can use to reduce the dimension whenever the realization we start with is "locally-unreachable" and/or "locally -unobservable".

Referring to [5] for definitions, we recall here that a test for local-reachability and local-observability consists in checking the full rank of the following matrices:

$$\mathcal{R} = [B_1 \vdots B_2 \vdots A_1B_1 \vdots A_1B_2 + A_2B_1 \vdots \ldots\ldots$$
$$\ldots\ldots \vdots (A_1{}^{i-1} \sqcup\!\sqcup {}^{j} A_2)B_1 + (A_1{}^{i} \sqcup\!\sqcup {}^{j-1} A_2)\, B_2 \vdots \ldots],\quad i+j < n$$

"reachability matrix"

and

$$\mathcal{O}^T = [C^T \vdots (CA_1)^T \vdots (CA_2)^T \vdots \ldots \vdots C(A_1{}^{i} \sqcup\!\sqcup {}^{j} A_2) \vdots \ldots],\quad i+j < n$$

"observability matrix"

Matrices $A_1{}^{r} \sqcup\!\sqcup {}^{s} A_2$ are inductively defined as

$$A_1^{r} \sqcup\!\sqcup^{0} A_2 = A_1^{r}\ ,\quad A_1{}^{0} \sqcup\!\sqcup^{s} A_2 = A_2^{s}$$

$$A_1{}^{r} \sqcup\!\sqcup^{s} A_2 = A_1\,(A_1{}^{r-1} \sqcup\!\sqcup^{s} A_2) + A_2\,(A_1{}^{r} \sqcup\!\sqcup^{s-1} A_2)$$

If one of the above matrices is not full rank, there is a finite, linear procedure [5,10] which leads to a locally-reachable and locally-observable realization which has lower dimension than the one we started with. This procedure has a structure which follows the usual one in linear system theory. The main difference is that the locally-reachable and locally observable realizations we obtain are not necessary minimal. For instance the following 2-D systems:

$$A_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ B_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix},\ C = [0\ 1]$$

and

$$\bar{A}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},\ \bar{A}_2 = \begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},\ \bar{B}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},\ \bar{B}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ \bar{C} = [0\ -1\ 0]$$

are both locally-reachable and locally-observable realizations of the transfer function:

$$s = \frac{-z_2}{1 - z_1^2 - z_2^2}$$

Obviously the second one does not constitute a minimal realization.

As a matter of fact the analysis of the structure of minimal realizations, as well as the design of algorithms which explicitly give such realizations, constitute the bottleneck of 2-D systems theory.

To get some insight into the algebraic nature of the problem we observe that the dimension of minimal realizations depends on the field we use for constructing the realizations. This fact seems to signify that a solution should resort to mathematical tools beyond linear algebra.

To illustrate that, consider the transfer function

$$s = \frac{2\, z_1 z_2}{1+z_1^2+z_2^2}$$

which admits the following complex realization of dimension 2:

$$A_1 = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix},\ B_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ B_2 = \begin{bmatrix} i \\ -i \end{bmatrix},\ C = [1\ 1]$$

The identity $1+z_1^2+z_2^2 = \det\,(1 - A_1z_1 - A_2z_2)$ cannot be satisfied if A_1,A_2 belong to $\mathbb{R}^{2\times 2}$. For the contrary, assume

$$1 + z_1^2 + z_2^2 = \det \begin{bmatrix} 1+p & r \\ t & 1+q \end{bmatrix}$$

with p,q,r,t homogeneous polynomials of degree one in $\mathbb{R}[z_1,z_2]$. Since the left hand side does not include monomials of degree one, we get p=-q. Letting $p=\alpha z_1+\beta z_2$, we have

$$(\alpha^2+1)z_1^2 + 2\,\alpha\beta\, z_1z_2 + (\beta^2+1)z_2^2 = -rt$$

which would imply that a positive definite quadratic form admits a proper factorization on $\mathbb{R}[z_1,z_2]$.

It is significant to point out that Kung, Lévy, Morf and Kailath [7] tried to get minimal realizations resorting to a less general model than (2), and yet they fell with an algebraic non linear problem exactly like that we mentioned above.

4. STABILITY

The notion of internal stability of a 2-D system is related to the behaviour of the free evolution of local states resulting from a generic local state assignment on a separation set $\mathscr{C}$.

Let assume in $\mathbb{Z}\times\mathbb{Z}$ the distance

$$d((i,j),(h,k)) \triangleq |(i-h)| + |(j-k)|$$

and denote by

$$d((i,j),\mathscr{C}) = \min_{(h,k)\in\mathscr{C}} d((i,j),(h,k))$$

the distance between (i,j) and the set $\mathscr{C}$. Introduce the following notations:

(i) $\mathscr{X}_{\mathscr{C}} = \{x(h,k),\ (h,k)\in\mathscr{C}\}$, "global state" on $\mathscr{C}$

(ii) $\|\mathscr{X}_{\mathscr{C}}\| = \sup_{x\in\mathscr{X}_{\mathscr{C}}} \|x\| < \infty$

were $\|x\|$ denotes the euclidean norm of $x\in X$.

DEFINITION. Let $\mathscr{C}$ be a separation set in $\mathbb{Z}\times\mathbb{Z}$ and assume $u\equiv 0$. The 2-D system (2) is asymptotically stable with respect to $\mathscr{C}$ if given $\epsilon>0$, for every $\mathscr{X}_{\mathscr{C}}$ with $\|\mathscr{X}_{\mathscr{C}}\|<\infty$, there exists a positive integer m such that $\|x(i,j)\|<\epsilon$ when (i,j) is in the future of $\mathscr{C}$ and $d((i,j),\mathscr{C}) > m$.

The internal stability depends on the pair (A_1,A_2) and one could expect that it depends also on the separation set $\mathscr{C}$. Actually the dependence on $\mathscr{C}$ does not subsist [12] and the following Proposition gives an algebraic criterion for checking asymptotic stability:

PROPOSITION. A 2-D system $\Sigma = (A_1,A_2,B_1,B_2,C)$ is asymptotically stable if and only if the polynomial $\det(I-z_1A_1-z_2A_2)$ is devoid of zeros in the closed polydisc:

$$\mathscr{P}_1 = \{(z_1,z_2)\in\mathbb{C}\times\mathbb{C}:\ |z_1|\le 1,\ |z_2|\le 1\}.$$

The result presented in Proposition above makes suitable for asymptotic stability analysis those tests elaborated for input-output stability [13-15]. In fact for a two-dimensional filter, with transfer function $p(z_1,z_2)/q(z_1,z_2)$, $q(0,0)=1$, to be input-output stable it is necessary and sufficient that $q(z_1,z_2)$ not be zero in $\mathscr{P}_1$.

Coprimeness properties are relevant in analyzing the relations between input-output (BIBO) stability and asymptotic stability of 2-D systems. For this it is important to note [7] that if $\Sigma = (A_1,A_2,B_1,B_2,C)$ is a realization of a transfer function $p(z_1,z_2)/q(z_1,z_2)$ with p and q relatively prime and

(i) $(C, I-A_1z_1-A_2z_2)$ are left-coprime

(ii) $(I-A_1z_1-A_2z_2, B_1z_1+B_2z_2)$ are right-coprime

then $\det(I-A_1z_1-A_2z_2) = q(z_1,z_2)$.

Realizations satisfying (i) and (ii) will be called "coprime".

For 2-D systems input-output stability and internal stability are related as shown in the following Corollary:

COROLLARY. Let $\Sigma=(A_1,A_2,B_1,B_2,C)$. Then we have the following implications:

Σ asymptotically stable $\rightarrow$ Σ input-output stable

Σ asymptotically stable $\leftarrow$ Σ input-output stable + Σ coprime

E.Fornasini, G.Marchesini
Dept. of Electrical Eng., Univ. of Padua
Via Gradenigo 6/A, 35100 Padova, Italy

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