On the Internal Stability of Two-Dimensional Filters

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***Abstract*—The internal stability concept for state-space representations of two-dimensional filters is introduced and an algebraic stability criterion is presented. The connections among internal stability, input–output stability, and coprimeness of the realization are also clarified.**

I. Introduction

The stability problem for two-dimensional filters in input–output form has been investigated by several authors [1]–[6]. The aim of this correspondence is to provide a first insight into the "internal" stability problem which arises when we consider state-space realizations of the filters.

State-space models of two-dimensional discrete-time filters constitute a recent field of investigation [7]–[14]. It has been shown [14] that all state-space models so far considered can be viewed as special cases of the following "doubly indexed dynamical system" $\Sigma=(A_1,A_2,B_1,B_2,C)$:

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$$(1)\qquad x(h+1,k+1)=A_1x(h+1,k)+A_2x(h,k+1)+B_1u(h+1,k)+B_2u(h,k+1)$$
$$y(h,k)=Cx(h,k)$$

where $u(h,k)$, the input value at (h,k), and $y(h,k)$, the output value at (h,k), are in $\mathbb{R}$, and $h,\ k\in Z$, $A_i\in\mathbb{R}^{n\times n}$, $B_i\in\mathbb{R}^{n\times 1}$, $C\in\mathbb{R}^{1\times n}$, $i=1,2$, and $x\in X=\mathbb{R}^n$ (local state space).

A first attempt to investigate the internal stability was made by Attasi [8] in the special case of separable filters. The class of filters we deal with is the whole class of filters having rational transfer function, so we shall consider the stability of dynamical systems represented by (1).

II. Stability Criterion

Introduce the following notation:

$$\mathfrak{X}_r=\{x(h,k):x(h,k)\in X,\ h+k=r\}.$$

Let $\|x\|$ denote the Euclidean norm of x in X and let

$$\|\mathfrak{X}_r\|=\sup_{n\in Z}\|x(r-n,n)\|.$$

We therefore have the following definition.

Definition: Let Σ be described by (1). The system Σ is asymptotically stable if, assuming $u=0$ and $\|\mathfrak{X}_0\|$ finite, $\|\mathfrak{X}_i\|\to 0$ as $i\to+\infty$.

As is well known, the asymptotic stability analysis of discrete-time linear systems is reduced to investigate the zero's position of the characteristic polynomial of the matrix A.

The asymptotic stability of Σ is related to the algebraic curve defined in $\mathbb{C}\times\mathbb{C}$ by the equation

$$\det(I-z_1A_1-z_2A_2)=0$$

as stated in the following proposition.

Proposition 1: Let Σ be as in (1). Then Σ is asymptotically stable if and only if the polynomial $\det(I-A_1z_1-A_2z_2)$ is devoid of zeros in the closed polydisk:

$$\mathbb{P}_1=\{(z_1,z_2)\in\mathbb{C}\times\mathbb{C}:|z_1|\leqslant 1,\ |z_2|\leqslant 1\}$$

sufficiency. Let $\det(I-z_1A_1-z_2A_2)\neq 0$ in $\mathcal{P}_1$ and call V the algebraic curve defined by $\det(I-A_1z_1-A_2z_2)=0$. Since V and $\mathcal{P}_1$ are closed, $V\cap\mathcal{P}_1=\varnothing$ implies that there exists $\epsilon>0$ such that the polydisk

$$\mathcal{P}_{1+\epsilon}=\{(z_1,z_2)\in\mathbb{C}\times\mathbb{C}:|z_1|\leqslant 1+\epsilon,\ |z_2|\leqslant 1+\epsilon\}$$

does not intersect V.

Then the rational matrix $(I-A_1z_1-A_2z_2)$ can be inverted in $\mathcal{P}_{1+\epsilon}$ and its McLaurin series expansion, given by

$$(I-A_1z_1-A_2z_2)^{-1}=\Sigma_{ij}M_{ij}z_1^iz_2^j$$

converges normally in the interior of $\mathcal{P}_{1+\epsilon}$ [15].

It follows that the series $\Sigma_{ij}\|M_{ij}\|$ converges. Consequently, $\Sigma_{i+j=r}\|M_{ij}\|\to 0$ as $r\to\infty$, [16]. This implies the asymptotic stability of Σ. For, assume $\|\mathfrak{X}_0\|$ finite and pick in $\mathfrak{X}_r$, $r>0$ any local state $x(m,r-m)$, then

$$\|x(m,r-m)\|=\left\|\sum_{i+j=r}M_{ij}x(m-i,r-m-j)\right\|$$
$$\leqslant\sum_{i+j=r}\|M_{ij}\|\,\|x(m-i,r-m-j)\|\leqslant\|\mathfrak{X}_0\|\sum_{i+j=r}\|M_{ij}\|$$

necessity. Assume Σ be asymptotically stable. Then for any $x\in X$, $M_{ij}x\to 0$ as $i+j\to\infty$. This fact and

$$\|M_{ij}\|\leqslant\sum_1^n{}_k\|M_{ij}e_k\|,$$

(with $\{e_k\}_1^n$ the standard basis in $X=\mathbb{R}^n$) imply

$$\lim_{i+j\to\infty} \|M_{ij}\| \le \lim_{i+j\to\infty} \sum_{1}^{n} \|M_{ij}e_k\| = 0.$$

By Abel's lemma, the series $\Sigma_{ij} M_{ij} z_1^i z_2^j$ converges in the interior of $\mathcal{P}_1$. Then $(I - A_1 z_1 - A_2 z_2)$ is invertible in the interior of $\mathcal{P}_1$.

The proof will be complete by showing that $\det(I - A_1 z_1 - A_2 z_2) \neq 0$ on the boundary $\rho\mathcal{P}_1$ of $\mathcal{P}_1$. For, let (a_1, a_2) belong to $\rho\mathcal{P}_1$ and assume

$$\det(I - A_1 a_1 - A_2 a_2) = 0.$$

Hence, there exists a nonzero vector $v \in \mathbb{C}^n$ which satisfies $v = A_1 a_1 v + A_2 a_2 v$. It is not restrictive to assume $|a_1| = 1$, so that it makes sense to consider $\mathcal{X}_0 = \{x_{n,-n}\}$ with

$$x_{n,-n} \begin{cases} = 0, & \text{if } n < 0 \\ = \alpha a_1^{-n} a_2^n v + \bar{\alpha} a_1^{-n} \bar{a}_2^n \bar{v}, & \text{if } n \geq 0, \alpha \in \mathbb{C}. \end{cases}$$

Assume now $\alpha = e^{j\psi}$ and write $a_1 = e^{-j\phi}$ and $v = r + jw$. Then the state values on $(0,k)$, $k = 0, 1, 2, \cdots$ are given by the sequence

$$x(k,0) = 2r\cos(k\phi + \psi) - 2w\sin(k\phi + \psi), \qquad k = 0, 1, 2, \cdots$$

and it is always possible to select a phase ψ which makes the sequence not infinitesimal as k goes to infinity.

As far as stability criteria are concerned, the result presented in Proposition 1 makes suitable for asymptotic stability analysis those tests elaborated for input–output stability [16]–[21]. In fact, for a two-dimensional filter with transfer function $p(z_1,z_2)/q(z_1,z_2)$, $q(0,0) = 1$, to be input–output stable it is necessary and sufficient that $q(z_1,z_2)$ not be zero in $\mathcal{P}_1$.

Coprimeness properties [13] are relevant in analyzing the relations between input–output stability and asymptotic stability of doubly indexed dynamical systems. For this it is important to note that if $\Sigma = (A_1, A_2, B_1, B_2, C)$ is a realization of a transfer function $p(z_1,z_2)/q(z_1,z_2)$ with p and q relatively prime and

i) $(C, I - A_1 z_1 - A_2 z_2)$ are left-coprime

ii) $(I - A_1 z_1 - A_2 z_2, B_1 z_1 + B_2 z_2)$ are right-coprime

then $\det(I - A_1 z_1 - A_2 z_2) = q(z_1, z_2)$.

Realizations satisfying i) and ii) will be called "coprime."

Then input–output stability and internal stability are related as shown in the following corollary.

Corollary: Let $\Sigma = (A_1, A_2, B_1, B_2, C)$. Then we have the following implications:

Σ asymptotically stable$\rightarrow\Sigma$ input–output stable

Σ asymptotically stable$\leftarrow\Sigma$ input–output stable + Σ coprime.

Any transfer function $p(z_1,z_2)/q(z_1,z_2)$ admits coprime realizations [14], so it is always possible to construct asymptotically stable realizations starting from stable transfer functions.

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Correction to "On the Internal Stability of Two-Dimensional Filters"

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The last corollary in the above note[1] is based on Shank's stability theorem [1]. The sufficiency part of Shank's theorem is not completely true, as pointed out by Goodman in [2] and the correct statement is the following

"Let $G(z_1,z_2)=P(z_1,z_2)/Q(z_1,z_2)$ be the transfer function of a BIBO stable filter. Then $G(z_1,z_2)$ has no poles in the closed unit polydisk $\mathcal{P}_1$ and no nonessential singularities of the second kind on $\mathcal{P}_1$ except possibly on the set $T^2=\{(z_1,z_2): |z_1|=|z_2|=1\}$."

As a consequence, the second implication in the last corollary, i.e.,

$$\Sigma \text{ asymptotically stable} \leftarrow \Sigma \text{i/o stable} + \Sigma \text{coprime}$$

should be strengthened as follows:

Σ asymptotically stable $\leftarrow \Sigma$i/o stable and devoid of nonessential singularities on $\mathcal{P}_1$ (T^2 included) + Σ coprime.

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