Stability Analysis of 2-D Systems

ETTORE FORNASINI AND GIOVANNI MARCHESINI

Abstract—A polynomial stability criterion for 2-D systems is taken as a
starting point for introducing a frequency dependent Lyapunov equation.
The Fourier analysis of its matrix solution leads to an infinite dimen-
sional quadratic form which provides a Lyapunov function for the global
state of the system.
The Fourier coefficients are explicitly obtained as the sum of series
involving the system matrices.
The convergence of these series constitutes a necessary and sufficient
stability condition, which generalizes the analogous condition for 1-D
systems.

I. INTRODUCTION AND PRELIMINARY DEFINITIONS

The stability problem of 2-D filters is currently receiving wide attention [1]–[10]. Yet almost all results are concerned with bounded input, bounded output (BIBO)

stability and disregard the internal dynamics of the systems which realize 2-D filters.

In this paper we are concerned with an “internal stability” theory which takes into account the dynamical model point of view.

When we try to extend to 2-D systems the 1-D state
space stability theory, we are faced with some natural difficulties due to the existence of an infinite dimensional “global” state and of updating equations on a finite di-

mensional “local” state space [6], [10]. Since the local state
dynamics are not uniquely determined by the updating of
the Nerode classes, which are the canonical global states,
different local state recursive equations arise. Thus different
2-D models, realizing the same class of input/output
maps, are admissible. Whatever 2-D models we may as-
sume, the basic stability definitions and criteria remain essen-
tially the same, and no particular advantage can be
reached by choosing a particular local dynamics.

*Rebus sic stantibus*, we shall refer to the recursive model presented in [6] which is up to now the most general and flexible.

From now on a 2-D System $\Sigma = (A_1, A_2, B_1, B_2, C)$ is identified by the following pair of equations:

$$
x(h+1, k+1) = A_1(x(h, k+1)) + A_2 x(h+1, k)
+ B_1 u(h, k+1) + B_2 u(h+1, k)
$$

$$
y(h, k) = C x(h, k)
$$

(1)

where

$$(h, k)$$ are elements of $\mathbb{Z} \times \mathbb{Z}$ ("time set"), partially ordered by the product of the orderings, $x(\cdot): \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}^n$ is a map whose value at time $(h, k)$ is called the "local state at time $(h, k)$,"

$u(\cdot): \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ and $y(\cdot): \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ are the input and output maps respectively, and $u(h, k)$, $y(h, k)$ are the input and output values at time $(h, k)$,

$A_1, A_2 \in \mathbb{R}^{n \times n}$, $B_1, B_2 \in \mathbb{R}^{n \times 1}$, $C \in \mathbb{R}^{1 \times n}$ are suitable matrices which completely characterize the 2-D system $\Sigma$.

When an input function $u(\cdot)$ is given the solution of (1) requires a complete information about a suitable set of local states called "initial global state."

A "separation set" in $\mathbb{Z} \times \mathbb{Z}$ is a nonempty set $\mathcal{C}$, defined by the following characteristic properties:

i) if $h > i$, $k > j$, $(h, k)$ and $(i, j)$ cannot belong simultaneously to $\mathcal{C}$;

ii) if $(h, k)$ belongs to $\mathcal{C}$, then $\mathcal{C}$ intersects the sets $((h-1), k), (h, k+1), (h+1, k+1)$ and $((h+1), k), (h, k+1), (h, k-1)$ and does not contain the set $((h+1), k), (h, k-1))$;

iii) for any $(i, j)$ in $\mathbb{Z} \times \mathbb{Z}$, the relation $(h, k) \preceq (i, j)$ cannot be satisfied by infinitely many elements $(h, k)$ in $\mathcal{C}$.

The "future of $\mathcal{C}$" is the set

$$
\mathcal{F}_c = \{ (i, j); (h, k) \preceq (i, j), \text{ for some } (h, k) \text{ in } \mathcal{C} \}.
$$

There are infinitely many possibilities of shaping the set $\mathcal{C}$. Examples of separation sets are given in Fig. 1(a), 1(b), 1(c). For instance, image processing usually refers to the separation set in Fig. 1(a).

The "global state" $\mathcal{X}_c$ on the separation set $\mathcal{C}$ is defined as

$$
\mathcal{X}_c = \{ x(h, k); (h, k) \in \mathcal{C} \}
$$

(2)

and the computation of a local state $x(i, j), (i, j)$ in $\mathcal{X}_c$, can be performed starting from $\mathcal{X}_c$, whenever $u$ is known in $\mathcal{X}_c$. In particular if $u$ is identically zero on $\mathcal{X}_c$ the local states in $\mathcal{X}_c$ depend only on $\mathcal{X}_c$.

II. POLYNOMIAL CONDITIONS FOR INTERNAL STABILITY

The notion of internal stability of a 2-D system is related to the behavior of the free evolution (i.e., zero input) of local states resulting from a bounded global state assignment on a separation set $\mathcal{C}$. Let us assume in $\mathbb{Z} \times \mathbb{Z}$ the distance function

$$
d((i, j), (h, k)) = |i-h| + |j-k|
$$

and denote by

$$
d((i, j), \mathcal{C}) = \min_{(h, k) \in \mathcal{C}} d((i, j), (h, k))
$$

the distance between $(i, j)$ and the set $\mathcal{C}$. Introduce the following notation

$$
\| x \| = \sup_{x \in \mathcal{X}_c} \| x \|
$$

where $\| x \|$ denotes the Euclidean norm of $x$.

**Definition 1:** Let $\mathcal{C}$ be a separation set in $\mathbb{Z} \times \mathbb{Z}$, and assume $u=0$. The 2-D system (1) is internally stable with respect to $\mathcal{C}$ if given $\epsilon > 0$, for every $\mathcal{X}_c$ with $\| x \| < \infty$, there exists a positive integer $m$ such that $\| x(i, j) \| < \epsilon$ when $(i, j)$ is in the future of $\mathcal{C}$ and $d((i, j), \mathcal{C}) > m$. 
The internal stability depends on the pair \((A_1, A_2)\), and one could expect that it depends also on the separation set \(\mathcal{C}\).

Theorem 1 states that the internal stability does not depend on \(\mathcal{C}\) and allows us to discuss the internal stability without referring to any specific separation set \(\mathcal{C}\).

**Theorem 1** [7]: Internal stability with respect to any separation set \(\mathcal{C}\) implies internal stability with respect to every separation set.

At this point the construction of internal stability criteria is perhaps the most natural topic of investigation. It is well known that internal stability of a discrete linear system

\[
x(h+1) = Ax(h) + Bu(h)
y(h) = Cx(h)
\]

(3)
can be checked (i) by evaluating the distribution of the roots of the characteristic polynomial of \(A\) with respect to the unit circle in the Gauss plane, or (ii) by solving the Lyapunov equation. The following theorem provides the 2-D counterpart of the 1-D stability criterion based on the characteristic polynomial.

**Theorem 2** [6], [10]: A 2-D system \(\Sigma = (A_1, A_2, B_1, B_2, C)\) is internally stable if and only if the polynomial \(\det(I - A_1z_1 - A_2z_2)\) is devoid of zeros in the closed polydisc

\[
\mathcal{P}_1 = \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} : |z_1| < 1, |z_2| < 1\}.
\]

Several tests have been proposed in the literature [Shanks, Huang, Jury, Anderson, etc.] to check if \(\mathcal{P}_1\) intersects the complex variety of a polynomial \(q \in \mathbb{R}[z_1, z_2]\). The original field of application of these tests has been the external (BIBO) stability analysis, since Shanks' theorem [1] states that a 2-D filter with transfer function \(1/q\) is externally stable if and only if \(q(z_1, z_2)\neq 0\) in \(\mathcal{P}_1\). Of course, the condition \(q \neq 0\) in \(\mathcal{P}_1\) guarantees also the external stability when the transfer function is \(p/q\), with \(p\) and \(q\) coprime.

In 1978 Goodman [5] showed that a 2-D filter with transfer function \(p/q\) can be BIBO stable even when the denominator \(q\) vanishes in some points of the torus

\[
T_1 = \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} : |z_1| = |z_2| = 1\}.
\]

**Theorem 3** [5]: Let \(G(z_1, z_2) = p(z_1, z_2)/q(z_1, z_2)\) denote the transfer function of a BIBO stable filter. Then \(G(z_1, z_2)\) has no poles in the closed unit polydisc \(\mathcal{P}_1\) and no nonessential singularities of the second kind on \(\mathcal{T}_1\) except possibly on \(T_1\).

Then the mutual implications between root distribution of \(q\) and BIBO stability are as follows [13]:

- BIBO stability \(\Rightarrow q(z_1, z_2) \neq 0\) in \(\mathcal{P}_1 \setminus T_1\)
- BIBO stability \(\Rightarrow q(z_1, z_2) \neq 0\) in \(\mathcal{P}_1\).

This shows that the absence of intersections between \(\mathcal{P}_1\) and the variety of \(q\), which can be checked by the above-mentioned tests, ensures BIBO stability. Nevertheless we can have BIBO stability even when the intersection is nonempty, that is when the tests would give a negative result.

On the other hand, by Theorem 2 these tests fit perfectly to the internal stability analysis. In particular the following corollary of Theorem 2, derived from Huang’s criterion [2], will provide (see Section IV) a frequency dependent Lyapunov equation.

**Corollary 1**: A 2-D system \(\Sigma = (A_1, A_2, \cdots)\) is internally stable if and only if the complex matrix

\[
A_1 + e^{j\omega}A_2
\]

is stable (i.e., the magnitudes of its eigenvalues are less than 1) for any real \(\omega\).

**Proof**: First recall that, by Huang’s criterion, a polynomial \(q \in \mathbb{R}[z_1, z_2]\) is devoid of zeros in \(\mathcal{P}_1\) if and only if i) \(q(z_1, 0) \neq 0\) for \(|z_1| < 1\) and ii) \(q(z_2, z_2) \neq 0\) for \(|z_1|=1\) and \(|z_2| < 1\).

Assume now the matrix \(A_1 + e^{j\omega}A_2\) to be stable for any real \(\omega\). The images of the unit polydisc given by the polynomial functions

\[
q_1(z_1, \eta) = \det(I - A_1z_1 - A_2z_1\eta)
\]

and

\[
q_2(\eta, z_2) = \det(I - A_1z_2\eta - A_2z_2)
\]

coincide with the images of the polynomial function \(\det(I - A_1z_1 - A_2z_2)\) when acting on the sets

\[
\mathcal{P}_1 \cap \{(z_1, z_2) : |z_1| > |z_2|\}
\]

and

\[
\mathcal{P}_1 \cap \{(z_1, z_2) : |z_2| > |z_1|\}.
\]

Since \(q_1(0, \eta) \neq 0\) for \(|\eta| < 1\) and, by stability assumption on \(A_1 + e^{j\omega}A_2\), \(q_1(z_1, e^{j\omega}) \neq 0\) for \(|z_1| < 1\). \(q(z_1, \eta)\) is devoid of zeros in \(\mathcal{P}_1\) by Huang’s criterion. The same property holds for \(q_2(\eta, z_2)\). Therefore, \(\det(I - A_1z_1 - A_2z_2) \neq 0\) in \(\mathcal{P}_1\) and \(\Sigma\) is internally stable.

Conversely internal stability implies \(\det(I - A_1z_1 - A_2z_2) \neq 0\) in \(\mathcal{P}_1\). Hence \(\det(I - A_1z_1 - A_2z_2e^{j\omega}) \neq 0\) for \(|z_1| < 1\) which gives the stability of \(A_1 + e^{j\omega}A_2\).

**Remark**: An obvious necessary 2-D stability condition resulting from Corollary 1 is that \(A_1 \pm A_2\) must both be stable.

### III Matrix Conditions for Internal Stability

In this section we are concerned with the extension to the 2-D case of some properties characterizing internally stable 1-D systems. These properties are recalled in Theorem 4.

**Theorem 4**: The following propositions are equivalent:

i) the system (3) is internally stable;

ii) the series \(\Sigma_{\eta=0}^\infty ||A_1^\eta||\) converges;

iii) \(||A_k|| < 1\) for some integer \(k > 0\);

iv) the series \(\Sigma_{\eta=0}^\infty (A_1^\eta)T(A_1^\eta)\) converges.
Note that $P = \sum_{\gamma=0}^{\infty} A^\gamma$ is the solution of the Lyapunov equation

$$X = I + A^T X A.$$  

As we shall see in Theorem 5, the family of matrices $\{A_1^r \cup \cup A_2, r, s \in \mathbb{N}\}$, defined as

$$A_1^r \cup \cup A_2 = A_1^r,$$

$$A_1^0 \cup \cup A_2 = A_2$$

$$A_1^r \cup \cup A_2 = A_1^r (A_1^{r-1} \cup \cup A_2) + A_2 (A_1^{r-1} \cup \cup A_2),$$

if $r, s > 0$ (4)

plays an essential role in extending points i–iv) of Theorem 4.

The matrix $A_1^r \cup \cup A_2$ is explicitly given by

$$A_1^r \cup \cup A_2 = \sum_{i, j, r > 0} A_{i, j, r} A_{i, j, r},$$

where the summation runs over all products including $r$ times the matrix $A_1$ and $s$ times the matrix $A_2$. This justifies the use of the symbol "\cup" borrowed from the shuffle product in formal languages theory.

**Theorem 5**: The following propositions are equivalent:

i) the 2-D system (1) is internally stable;

ii) the series $\sum_{r, s > 0} \| A_1^r \cup \cup A_2^s \|$ converges;

iii) $\sum_{r, s > 0} | \| A_1^r \cup \cup A_2^s \| - 1 > 1$ for some positive integer $k$;

iv) the series $\sum_{r, s > 0} (A_1^r \cup \cup A_2^s)(A_1^r \cup \cup A_2^s)$ converges.

**Proof**: (i)⇒(ii). By Theorem 2 internal stability implies det$(I - A_1 z_1 - A_2 z_2)$≠0 in $C^1$, hence in $C^1 = \{ |z_1| < 1+\epsilon, |z_2| < 1+\epsilon \}$ for some positive real $\epsilon$. Therefore, $(I - A_1 z_1 - A_2 z_2)^{-1}$ admits a normally convergent power series expansion in $C^1$.

Conversely, let $\sum_{r, s > 0} \| A_1^r \cup \cup A_2^s \| < \infty$, and let $\sum$ start from a bounded global state $x_0 \in C_0 = \{ (i, j): i+j=0 \}$, $\| X_0 \| = M < \infty$, $u = 0$. If $(h, k) \in C_0$ and $d((h, k), X_0) = t$, we obtain

$$\| x(h, k) \| = \sum_{i+j=t} A_1^i \cup \cup A_2^j x(h-i, k-j) \leq \sum_{i+j=t} \| A_1^i \cup \cup A_2^j \| M.$$  

This proves that $\| x(h, k) \| = 0$ as $h+k=t \rightarrow \infty$.

(ii)⇒(iii) is obvious. To prove the converse we need the following Lemma:

**Lemma**: For any pair of nonnegative integers $p$ and $q$ we have

$$\sum_{\mu, \nu > 0} \sum_{\nu = p+q} \| A_1^\mu \cup \cup A_2^\nu \| \leq \sum_{i, j > 0} \| A_{i, j} \cup \cup A_{i, j} \|.$$  

**Proof**: Let $\mu, \nu, p \in \mathbb{N}$ and $\mu + \nu > p$. An easy inductive argument shows that

$$A_1^{\mu+\nu} \cup \cup A_2 = \sum_{i, j > 0} \sum_{i+j=p} \| A_{i, j} \cup \cup A_{i, j} \|.$$

Equation (5) and standard norm inequalities give

$$\sum_{\mu, \nu > 0} \sum_{\nu = p+q} \| A_1^\mu \cup \cup A_2^\nu \| \leq \sum_{i, j > 0} \| A_{i, j} \cup \cup A_{i, j} \|.$$

We now return to the implication (iii)⇒(ii).

By the Euclidean algorithm any integer $h$ can be written

$$h = kq + \rho, \quad 0 < \rho < k.$$  

Upon setting $N = \max_{0 < \rho < k} \{ \sum_{r, s = 0} \| A_1^r \cup \cup A_2^s \| \}$, the lemma above gives

$$\sum_{r, s = 0} \| A_1^r \cup \cup A_2^s \| \leq \sum_{r, s = 0} \| A_1^r \cup \cup A_2^s \|.$$  

Consequently,

$$\sum_{r, s = 0} \| A_1^r \cup \cup A_2^s \| = \sum_{q = 0}^{\infty} \sum_{p = 0}^{k-1} \| A_{i, j} \cup \cup A_{i, j} \|.$$  

We have

$$N = \sum_{r, s = 0} \| A_1^r \cup \cup A_2^s \| \leq \sum_{r, s = 0} \| A_1^r \cup \cup A_2^s \|. $$  

(ii)⇒(iv) The convergence of $\sum_{r, s = 0} \| A_1^r \cup \cup A_2^s \|$ implies the convergence of $\sum_{r, s = 0} \| A_1^r \cup \cup A_2^s \|^2$, of
\[ \Sigma_{r,s=0}^{\infty} \|(A_{1} \otimes A_{2})^T (A_{1} \otimes A_{2})\| \] and, finally, of
\[ \Sigma_{r,s=0}^{\infty} \|(A_{1} \otimes A_{2})^T (A_{1} \otimes A_{2})\| \]
(iv) \implies (iii) Assume by contradiction \( \Sigma_{r,s=0}^{\infty} \|(A_{1} \otimes A_{2})^T (A_{1} \otimes A_{2})\| > 1 \) for every \( k > 0 \).

Then
\[ \sum_{r,s=0}^{\infty} \|(A_{1} \otimes A_{2})^T (A_{1} \otimes A_{2})\| > \frac{1}{k+1}, \quad k = 0, 1, 2, \ldots \]
because the minimum value of \( \Sigma_{r,s=0}^{k} \chi_{r,s}^2 \) in the region defined by \( \Sigma_{r,s=0}^{k} |x_{r,s}| > 1 \) is \((1/k+1)\). Therefore, the divergence of the harmonic series implies the divergence of \( \Sigma_{r,s=0}^{\infty} \|(A_{1} \otimes A_{2})^T (A_{1} \otimes A_{2})\| \) and \( \Sigma_{r,s=0}^{\infty} \|(A_{1} \otimes A_{2})^T (A_{1} \otimes A_{2})\| \) would not converge, contrary to the assumption.

IV. Stability Conditions by Fourier Analysis

By Corollary 1, \( \Sigma = (A_{1}, A_{2}, \ldots) \) is internally stable if and only if the Lyapunov equation
\[ P(\omega) = I + (A_{1} + e^{-j\omega} A_{2})^T P(\omega) (A_{1} + e^{j\omega} A_{2}) \]  
(6)
admits a positive definite Hermitian solution \( P(\omega) \) for every real \( \omega \).

The entries of \( P(\omega) \) are rational functions of \( e^{j\omega} \), periodic in \( \omega \) with period \( 2\pi \). The positive definite character of \( P(\omega) \) can be checked by applying Sturm’s test to the principal minors of \( P(\omega) \).

The Fourier coefficients \( P_{k} \) of the expansion
\[ P(\omega) = \sum_{k=-\infty}^{\infty} P_{k} e^{j\omega k} \]
satisfy the following properties:

i) since the entries of \( P(\omega) \) are in \( L^{2}[-\pi, \pi] \), the sequence \( \{P_{k}\} \) belongs to \( L^{2}(\mathbb{R}^{n \times n}) \);

ii) for any integer \( k, P_{k} = P_{k}^T \), and the following set of equalities holds:
\[ P_{0} = I + A_{1}^{T} P_{0} A_{1} + A_{2}^{T} P_{0} A_{2} + A_{1}^{T} P_{0} A_{2} + A_{2}^{T} P_{0} A_{1} \]
\[ P_{1} = A_{1}^{T} P_{1} A_{1} + A_{2}^{T} P_{1} A_{2} + A_{1}^{T} P_{1} A_{2} + A_{2}^{T} P_{1} A_{1} \]
\[ \vdots \]
\[ P_{k} = A_{1}^{T} P_{k} A_{1} + A_{2}^{T} P_{k} A_{2} + A_{1}^{T} P_{k} A_{2} + A_{2}^{T} P_{k} A_{1} \]
(7)

\[ \cdots \]

iii) the doubly infinite block Toeplitz matrix
\[ \mathcal{T} = \begin{bmatrix}
P_{-1} & P_{0} & P_{1} & \ldots \\
& P_{-1} & P_{0} & P_{1} & \ldots \\
& & P_{-1} & P_{0} & P_{1} & \ldots \\
& & & P_{-1} & P_{0} & P_{1} & \ldots \\
& & & & P_{-1} & P_{0} & P_{1} & \ldots \\
& & & & & P_{-1} & P_{0} & P_{1} & \ldots \\
\end{bmatrix} \]  
(8)
induces a positive definite scalar product in the space \( L^{2}(C^{n}) \).

Property iii) relies on the fact that for every nonzero vector function \( v(\cdot) : [-\pi, \pi] \rightarrow C^{n} \), with elements in \( L^{2}[-\pi, \pi] \) we have
\[ \int_{-\pi}^{\pi} (P(\omega)) v(\omega) d\omega > 0. \]

Then
\[ \sum_{h,k} \tilde{v}_{h}^{*} P_{h-k} v_{k} > 0 \]
where \( v_{k} \) are the Fourier coefficients of \( v(\cdot) \):
\[ v(\omega) = \sum_{k=-\infty}^{\infty} v_{k} e^{j\omega k}. \]

By Riesz–Fischer theorem, there is a bijection between \( L^{2}(C^{n}) \) and the space of \( n \)-tuples with elements in \( L^{2}[-\pi, \pi] \). Therefore, the relation
\[ (u, w) = \sum_{h,k} \tilde{u}_{h}^{*} P_{h-k} \tilde{w}_{k} \]
defines a positive definite scalar product in \( L^{2}(C^{n}) \).

The converse is also true. For assume that there exists a sequence \( \{P_{k}\} \) of matrices in \( \mathbb{R}^{n \times n} \) satisfying conditions i), ii), iii). Then the series \( \sum_{k=-\infty}^{\infty} P_{k} e^{j\omega k} \) defines a.e. a matrix function \( P(\omega) \) with elements in \( L^{2}[-\pi, \pi] \) which solves (6). In fact, using (7), we obtain
\[ I + A_{1}^{T} P(\omega) A_{1} + A_{2}^{T} P(\omega) A_{2} + A_{1}^{T} e^{j\omega} P(\omega) A_{2} \]
\[ + A_{2}^{T} e^{-j\omega} P(\omega) A_{1} \]
\[ = I + \sum_{k} A_{1}^{T} P_{k} A_{1} e^{j\omega k} + \sum_{k} A_{2}^{T} P_{k} A_{2} e^{j\omega k} \]
\[ + \sum_{k} A_{1}^{T} P_{k} A_{2} e^{j\omega(k+1)} + \sum_{k} A_{2}^{T} P_{k} A_{1} e^{j\omega(k-1)} \]
\[ = \sum_{k<0} e^{j\omega k} (A_{1}^{T} P_{k} A_{1} + A_{2}^{T} P_{k} A_{2} + A_{1}^{T} P_{k-1} A_{2} + A_{2}^{T} P_{k+1} A_{1}) \]
\[ + (I + A_{1}^{T} P_{0} A_{1} + A_{2}^{T} P_{0} A_{2} + A_{1}^{T} P_{-1} A_{2} + A_{2}^{T} P_{1} A_{1}) \]
\[ + \sum_{k>0} e^{j\omega k} (A_{1}^{T} P_{k} A_{1} + A_{2}^{T} P_{k} A_{2} \]
\[ + A_{1}^{T} P_{k-1} A_{2} + A_{2}^{T} P_{k+1} A_{1}) \]
\[ \Rightarrow \sum_{k} P_{k} e^{j\omega k} = P(\omega). \]

Hence the entries of \( P(\omega) \) are a.e. real rational functions of \( e^{j\omega} \). Since
\[ P_{0} = \int_{-\pi}^{\pi} P(\omega) d\omega \]
is finite, and \( P(\omega) \) is a.e. positive definite, the analytic extension \( Q(\omega) \) of \( Q(e^{j\omega}) \) is \( P(\omega) \) has no poles on the unit circle.

Since by continuity \( P(\omega) \) is at least positive semidefinite and satisfies (6) for all real \( \omega \), then it is positive definite for all real \( \omega \).

Assume \( \Sigma = (A_{1}, A_{2}, \ldots) \) to be internally stable and let \( P(\omega) = \sum_{k=-\infty}^{\infty} P_{k} e^{j\omega k} \) be the solution of (6). Then (7) can be used to investigate the structure of matrices \( P_{k} \) in terms of the family of matrices (4).

First notice that the Toeplitz matrix \( \mathcal{T} \) satisfies the equation
\[ \mathcal{T} = I + \mathcal{T}^{*} \mathcal{T} \]
(9)
where $\mathcal{J}$ is the (infinite) identity matrix and $\mathcal{A}$ and $\mathcal{A}^*$ are the doubly infinite block Toeplitz matrices:

$$
\mathcal{A} = \begin{bmatrix}
0 & A_1 & A_2 & 0 & 0 & A_1 & A_2 & 0 & 0 & A_1 & A_2 & 0 \\
0 & A_1 & A_2 & 0 & 0 & A_1 & A_2 & 0 & 0 & A_1 & A_2 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{bmatrix}
$$

$$
\mathcal{A}^* = \begin{bmatrix}
0 & A_1^* & A_2^* & 0 & 0 & A_1^* & A_2^* & 0 & 0 & A_1^* & A_2^* & 0 \\
0 & A_1^* & A_2^* & 0 & 0 & A_1^* & A_2^* & 0 & 0 & A_1^* & A_2^* & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{bmatrix}
$$

We therefore have the following Theorem.

**Theorem 6:** Let $\mathcal{B}$ as in (7). Then $\mathcal{B}$ is the sum of the series

$$
\mathcal{J} + \sum_{i=1}^{\infty} \mathcal{A}^i \mathcal{A}^i
$$

of Toeplitz matrices and the blocks $P_k$ are expressed as

$$
P_0 = \sum_{r,s=0}^{\infty} (A_1^r W^s A_2^r)^T (A_1^s W^r A_2^s)
$$

$$
P_i = \sum_{r,s=0}^{\infty} (A_1^{r+1} W^s A_2^{r+1})^T (A_1^s W^{r+1} A_2^s)
$$

$$
\vdots
$$

$$
P_k = \sum_{r,s=0}^{\infty} (A_1^{r+k} W^s A_2^{r+k})^T (A_1^s W^{r+k} A_2^s)
$$

**Proof:** Since the $P_i$'s are the Fourier coefficients of the continuous bounded function $P(\omega)$, the operator $\mathcal{B}: l^2(C^*) \rightarrow l^2(C^*)$ is continuous [12]. Hence there exists a positive integer $M$ such that $\mathcal{B}^T \mathcal{B} < \mathcal{M}^2 \mathcal{B}^T \mathcal{B}$ for every $v$ in $l^2(C^*)$. Then (9) we have $v^T \mathcal{B} v + \mathcal{B}^T \mathcal{A}^* \mathcal{A} v + \cdots + \mathcal{B}^T \mathcal{A}^m \mathcal{A}^m v < M \mathcal{B}^T \mathcal{B} v$

$$
\sigma = \sup_{\|v\|=1} \min_{1 \leq i \leq \infty} (\mathcal{B}^T \mathcal{A}^i v, \cdots, \mathcal{B}^T \mathcal{A}^m \mathcal{A}^m v) < 1.
$$

Let $t$ be a positive integer such that $\sigma^{t(M-1)} < 1/2M$. For any $r > t$ and $v$ in $l^2(C^*)$, $\|v\|=1$, there exists a partition of $v$

$$
v = v_1 + v_2 + \cdots + v_{r+1},
$$

such that

$$
\mathcal{B}^T \mathcal{A}^r v_1 < \sigma
$$

$$
\mathcal{B}^T \mathcal{A}^r \mathcal{A}^r v_2 < \sigma^2
$$

$$
\vdots
$$

$$
\mathcal{B}^T \mathcal{A}^r \mathcal{A}^r \cdots \mathcal{A}^r v_r < \sigma^r
$$

and

$$
\mathcal{B}^T \mathcal{A}^r \mathcal{A}^r \cdots \mathcal{A}^r \mathcal{A}^r v_{r+1} < \sigma^r M < \sigma^{(r/M)-1} M < 1/2.
$$

Thus the sequence of operators $\mathcal{B}_0 = \mathcal{J}$, $\mathcal{B}_i = \mathcal{B}_{i-1} + \mathcal{A}^i \mathcal{A}^i$, $i > 0$, is a Cauchy sequence, as the following inequalities:

$$
\|\mathcal{B}_n - \mathcal{B}_m\| < \sum_{i=m+1}^{n} \|\mathcal{A}^i \mathcal{A}^i\| < \sum_{i=m+1}^{n} \frac{2M^2}{\sigma^i}
$$

hold for $n > m > t$ and $\|v\|=1$. Since $l^2(C^*)$ is complete, the operator $\sum_{i=1}^{\infty} \mathcal{A}^i \mathcal{A}^i T + \mathcal{J}$ is well defined and solves (8). Finally $\mathcal{B}$ coincides with the series $\mathcal{J} + \sum_{i=1}^{\infty} \mathcal{A}^i \mathcal{A}^i$.

In fact

$$
\Delta = \mathcal{B} - (\mathcal{J} + \sum \mathcal{A}^i \mathcal{A}^i)
$$

satisfies the equations chain

$$
\Delta = \mathcal{B} \Delta = \cdots = \mathcal{A}^i \Delta \mathcal{A}^i
$$

and $\Delta = 0$, as $\lim_{i \to \infty} \mathcal{A}^i = 0$.

**Remark:** Assume $\mathcal{X}$ as separation sets

$$
\mathcal{X} = \{(h, k): h + k = i\}, \quad i = 0, 1, \ldots
$$

and let $\mathcal{X}_0$ belong to $l^2(R^d)$. Then the quadratic form $\mathcal{V}(\mathcal{X}_0) = \mathcal{X}_0^T \mathcal{B} \mathcal{X}_0$ can be viewed as a Lyapunov function, and the relation

$$
\mathcal{X}_0^T \mathcal{B} \mathcal{X}_0 = \mathcal{X}_0^T \mathcal{A} \mathcal{X}_0 + \sum_{i=1}^{\infty} \mathcal{A}^i \mathcal{A}^i
$$

shows that $\mathcal{V}$ is a decreasing function of $i$.

**Example** (Scalar Case): If the local state space is 1-D, the matrices $A_1$ and $A_2$ become scalars $a_1$ and $a_2$, respectively, and a closed-form computation of the Fourier coefficients $P_k$ is possible when the 2-D system is internally stable.

The solution (5), given by

$$
P(\omega) = (1 - a_1^2 - a_2^2 - 2a_1a_2 \cos \omega)^{-1}
$$

is positive for every real $\omega$ if and only if

$$
1 - a_1^2 - a_3^2 < 2a_1a_2.
$$

Incidentally note that condition (10) can be restated in the following equivalent forms:

i) $|a_1| + |a_2| < 1$;

ii) $q(z_1, z_2) = 1 - a_1z_1 - a_2z_2 \neq 0$ in $\mathcal{H}_1$;

iii) $\begin{vmatrix}
1 - a_1^2 - a_2^2 & 2a_1a_2 \\
2a_1a_2 & 1 - a_1^2 - a_2^2
\end{vmatrix} > 0$.

When condition (10) is satisfied, we obtain

$$
P_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} (1 - a_1^2 - a_2^2 - 2a_1a_2 \cos \omega)^{-1} d\omega
$$

$$
= \left[ (1 - a_1^2 - a_2^2)^2 - 4a_1^2a_2^2 \right]^{1/2}.
$$

When once $P_0$ is computed, the coefficients $P_k$, $k = 1, 2, \ldots$ are obtained recursively from (6).

Observe that in general the sum of the series (9) is not a rational function of the matrices $A_1$ and $A_2$.

**V. Concluding Remarks**

1) Equations (6) and (9) provide a generalization of the 1-D Lyapunov equation. Equation (6) depends on the real
parameter $\omega$, so that the check of the positive definiteness of its solution requires to test the variations in sign of the polynomial principal minors of $P(\omega)$.

On the other hand, (9) is infinite dimensional, and does not give any direct finite procedure for checking internal stability.

The $P_0$ block in $\mathcal{P}$ is the sum of the series

$$
\sum_{r,s=0}^{\infty} (A_1 \mathcal{W} A_2)^T (A_1 \mathcal{W} A_2)
$$

which appears in point iv) of Theorem 5 as the counterpart of the series $\sum_{t=0}^{\infty} (A^T)^T A^t$ of Theorem 4. However, while the latter series is the solution of a finite dimensional Lyapunov equation, $P_0$ is only a finite portion of the solution of an infinite dimensional Lyapunov equation.

2) A way to obtain a finite dimensional Lyapunov equation could consist in reducing a 2-D dynamics to a one dimensional, by assuming a periodic initial global state, and then in applying 1-D theory. As we shall see in the sequel this procedure is not fruitful, since the existence of stability criteria under periodic initial conditions does not imply a stable behavior under generic initial conditions.

Let call a global state $\mathcal{X}_{G_2}$ "H periodical" if

$$
x(h,k) \in \mathcal{X}_{G_2} \Rightarrow x(h,k) = x(h+H, k-H)
$$

for any $(h,k)$ in $\mathcal{C}_f$.

Clearly the stability check for a $H$-periodical global initial state reduces to solve a standard Lyapunov equation of dimension $nH \times nH$.

The 2-D system considered in the following example is unstable. However, it shows a stable behavior corresponding to any periodic global initial state.

$$
\begin{bmatrix}
\dot{x}_1(h,-h+H) \\
\dot{x}_1(h+1,-h+1+H) \\
\vdots \\
\dot{x}_1(h+H,-h-H)
\end{bmatrix} =
\begin{bmatrix}
(H-1) & 0 & (H-1) & 0 & \cdots & (H-1) \\
0 & \mu_1^{H-1} & 0 & \cdots & (H-1) \\
(H-1) & \mu_1 & (H-1) & \mu_2 & \cdots & (H-1) \\
0 & \mu_1^{H-2} & 0 & \cdots & (H-1) \\
(H-1) & \mu_1 & (H-1) & \mu_2 & \cdots & (H-1) \\
0 & \mu_1^{H-1} & 0 & \cdots & (H-1) \\
(H-1) & \mu_1 & (H-1) & \mu_2 & \cdots & (H-1) \\
0 & \mu_1 & (H-1) & \mu_2 & \cdots & (H-1) \\
\end{bmatrix}
\begin{bmatrix}
x_1(h,-h) \\
x_1(h+1,-h-1) \\
\vdots \\
x_1(h+H,-h-H)
\end{bmatrix}
$$

Since $G$ is a circulant matrix [11], its eigenvalues are

$$
\lambda = (H-1) \mu_1^{H-1} + (H-1) \mu_1^{H-2} \mu_1 r_1 + (H-1) \mu_1^{H-3} \mu_1^2 r_1^2 + \cdots + (H-1) \mu_1^{H-1} r_1^{H-1}
$$

and

$$
\lambda = (p_1 + \mu_1 r_1)^{H-1} = 0.5 \left( e^{i\sqrt{0.5}} + r_1 e^{i\sqrt{0.3}} \right)^{H-1}
$$

where $r_1$, $l=1,2,\cdots, H$ denote the $H$th roots of 1.

Due to the fact that $|r_1 + \mu_1 e^{i\omega}|$ is less than 1 for any real $\omega$, except $\omega = \sqrt{0.5} - \sqrt{0.3}$ (mod $2\pi$), and that $r_l = e^{i2\pi/l}$, $l=1,2,\cdots, H$, $|p_1 + r_1 \mu_1|$ is always less than 1.
Consequently $G$ is a stable matrix and the components $\dot{x}_i$ of local states in $\mathcal{H}_{0}$ eventually decay to 0 as $i \to +\infty$. A similar argument holds for $\dot{x}_2$ components.

This shows that any periodic global state $\mathcal{H}_{0}$ determines a stable free evolution.

REFERENCES


Giovanni Marchesini received the Laurea in electrical engineering in 1961 from the University of Padua, Italy, and the Libera Docenza in automatic control in 1966. From March 1961 to October 1962 he worked on research of plasma physics at the Plasma Center, and from November 1962 to October 1964 he was Professor Incaricato of Numerical Analysis at the Department of Mathematics, University of Padua. Since November 1964 he has been at the Department of Electrical Engineering, University of Padua, where he is currently Professor of Mathematical System Theory. He was Consultant with Olivetti, Ivrea, Italy, from 1967 to 1972. He received a NATO-CNR grant to spend part of 1972 and 1974 in the Department of Electrical Engineering, Stanford University, CA. From November 1972 to January 1973 he was Visiting Professor in the Department of Mathematics, University of Florida, Gainesville, and from December 1974 to January 1975 he was Consultant in the Center for Mathematical System Theory, Department of Electrical Engineering, University of Florida, Gainesville. From July 1976 to October 1976 he was Visiting Professor in the Department of Applied Mathematics and Physics, University of Kyoto, Japan.

G. Marchesini is a member of the Scientific Council of the Italian Group of Automatics and System Research, and member of the IFAC's Technical Committee on Mathematics of Control.

Etore Fornasini received the laurea in electrical engineering and the laurea in mathematics from the University of Padova, Italy, in 1969 and 1973, respectively. Since October 1971 he has been with the Department of Electrical Engineering, University of Padua, where he is presently Professor Incaricato of System Theory. His current interests are in the fields of 2-D filters and nonlinear systems.