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STATE SPACE TECHNIQUES IN STABILIZING TWO-DIMENSIONAL FILTERS

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State and output feedback techniques for stabilizing two-dimensional filters are analysed and compared. Stabilizability depends both on structural properties of state space models and on the existence of non essential singularities of the second kind of transfer functions.

1. INTRODUCTION

Recently several papers have appeared in the literature [1,2], dealing with the synthesis of dynamic feedback compensators whose purpose is to improve the performance of two-dimensional filters and in particular to provide satisfactory stability behaviours.
The stabilization problem exhibits different aspects according to whether we deal with input-output descriptions or internal (state space) representations. As in the 1D case, internal stabilization guarantees external, while in general the viceversa does not hold.
The aim of this paper is to give a short account on state and output feedback stabilizing techniques and to compare these results with those obtainable using transfer function methods.

As known [3], a 2D filter, having transfer function

$$W(z_1,z_2) = \frac{n(z_1,z_2)}{d(z_1,z_2)} \qquad (1)$$

with n and d factor coprime polynomials and $d(0,0) = 1$, is BIBO stable if d is devoid of zeros in the unit closed polydisc

$$\bar{P} = \{(z_1,z_2) \in \mathbb{C} \times \mathbb{C}: |z_1| \leq 1, \ |z_2| \leq 1\}$$

In [4], Goodman gave some examples of BIBO stable 2D transfer functions whose denominator d vanishes at some points of the distinguished boundary

$$T = \{(z_1,z_2) \in \bar{P} : |z_1| = 1, \ |z_2| = 1\}$$

However, in these cases, the zeros of d in T are also zeros of n, i.e. they are nonessential singularities of the second kind.
It is worthwhile to point out that the condition that the zeros of d do not belong to $\bar{P}$-T and those in T are zeros of n is not sufficient to guarantee BIBO stability of (1). As far as we know, a characterization of BIBO stable transfer functions exhibiting Goodman's pathology is not yet available.

Internal stability refers to state space models (or 2D systems) given by the following equations [5]

$$\begin{aligned} x(h+1,k+1) &= A_1x(h,k+1)+A_2x(h+1,k)+ \\ &\quad +B_1u(h,k+1)+B_2u(h+1,k) \qquad (2) \\ y(h,k) &= Cx(h,k) \end{aligned}$$

The system (2) is internally stable if, for any initial "global state"

$$\mathcal{X}_o = \{x(i,-i), i \in \mathbb{Z}\}$$

with $\sup \| x(i,-i) \| < \infty$, the free state evolution goes to zero

$$\lim_{h+k \to +\infty} x(h,k) = 0$$

A necessary and sufficient condition for internal stability of system (2) is that the characteristic polynomial

$$\det(I-A_1z_1-A_2z_2)$$

is devoid of zeros in $\bar{P}$.

In the sequel we will be concerned with two different stabilization techniques.

The first one applies to input-output models (1) and is based on dynamic output feedback compensators represented by their transfer functions $r(z_1,z_2)/s(z_1,z_2)$. The overall transfer function is given by

$$\frac{ns}{nr+ds} \tag{3}$$

so that stabilizability essentially depends on the zeros location of the polynomial nr+ds as r and s vary.
The second stabilizing technique consists in constructing compensators in state space form, whose inputs are given by the output or the state variables of system (2).

2. INPUT-OUTPUT AND STATE SPACE STABILIZATION APPROACHES

If we look at the structure of the overall transfer function (3), it is clear that BIBO stabilizability depends on the possibility of choosing r and s so that the variety V(rn+ds) does not intersect the unitary polydisc. As r and s run over the ring of 2D polynomials, the varieties associated with rn+ds have a common intersection, given by the finite set

$$S \overset{\Delta}{=} V(n) \cap V(d)$$

There are not further constraints, besides this, on the structure of the variety V(nr+ds), except that it does not cross the origin.
Therefore, if $S \cap \bar{P} = \phi$, there exist polynomials r and s that make (3) stable, while if $S \cap (\bar{P}-T) \neq \phi$, the transfer function (3) cannot be stabilized. Due to the fact that a characterization of stable transfer functions having Goodman's pathology is not available, the case when

$$S \cap (P-T) = \phi \quad \text{and} \quad S \cap T = \phi$$

is critical in the sense that we don't know whether BIBO feedback stabiling techniques do exist.

When the stabilization problem is solvable, the procedure for obtaining r and s consists in selecting an appropriate polynomial $\bar{d}$ with the constraints

$$V(\bar{d}) \supset S \quad \text{and} \quad V(\bar{d}) \cap \bar{P} = \phi$$

and solving the Bézout equation

$$nr+ds = \bar{d}^{i} \tag{4}$$

which admits solution for sufficiently large values of i (Hilbert's Nullstellensatz). For the computational aspects involved in solving (4), the reader is referred to [6].

If we assume $\bar{d}=1$, we obtain a dead-beat compensator, that leads to an overall system whose transfer function is polynomial, so that its impulse response is finite. A necessary and sufficient condition for the solvability of (4) with $\bar{d}=1$ is that $S=\phi$, that corresponds to zero coprimeness of n and d.

If we refer now to the state model (2), stabilizability does not depend on the existence of Goodman's pathologies in the transfer function, as it actually does for input-output models. In fact stabilizability is fully characterized on the basis of the structural properties of the state equations, where input-state and state-output maps play an essential role.
More precisely, we are concerned with the rank of the following matrices

$$\left[I-A_1z_1-A_2z_2 \,\middle|\, B_1z_1+B_2z_2\right] \tag{5}$$

and

$$\begin{bmatrix} I-A_1z_1-A_2z_2 \\ C \end{bmatrix} \tag{6}$$

which constitute the 2D analogue of the matrices appearing in PBH tests of controllability and reconstructibility.

The state feedback dynamic compensators are represented by the following state model

$$\begin{aligned} \bar{x}(h+1,k+1) &= F_1\bar{x}(h,k+1)+F_2\bar{x}(h+1,k)+ \\ &\quad + G_1x(h,k+1)+G_2x(h+1,k) \\ u(h,k) &= H\bar{x}(h,k)+Jx(h,k) \end{aligned} \tag{7}$$

We say that (7) is a stabilizing compensator if the overall system resulting from the connection of (2) and (7) is internally stable.
We have the following Theorem:

Theorem 1 [1] *System (2) can be made internally stable by a state feedback compensator (7) if and only if (5) is full rank for any* (z_1,z_2) *in* $\bar{P}$.

If (5) is full rank in $\mathbb{C} \times \mathbb{C}$, there exists a 2D compensator such that the free state evolution of the resulting overall system goes to zero in a finite number of steps, for any initial global

state of (2) and (7).
From a structural point of view, in this case the rank condition on (5) corresponds to controllability of system (2), that is there exists an input function $u(\cdot,\cdot)$ that forces the state of (2) to go to zero in a finite number of steps. As a consequence of Theorem 1 such an input $u(\cdot,\cdot)$ can be obtained as the output of a multivariable 2D dynamic compensator, driven by the state of (2).

When the output of system (2) is assumed as the input to the compensator, the state equations of the compensator become

$$\begin{aligned} \bar{x}(h+1,k+1) &= F_1\bar{x}(h,k+1)+F_2\bar{x}(h+1,k)+ \\ &\quad + G_1y(h,k+1)+G_2y(h+1,k) \qquad (8) \\ u(h,k) &= H\bar{x}(h,k)+Jy(h,k) \end{aligned}$$

Here stabilizability of the overall system depends on the rank of both matrices (5) and (6), as stated in the following Theorem.

Theorem 2 [1] *System (2) can be made internally stable by an output feedback compensator (8) if and only if (5) and (6) are full rank for any* (z_1,z_2) *in* $\bar{P}$.

Similarly to what happens with the state feedback compensator, the existence of a dead-beat output feedback compensator is equivalent to the full rank condition in $\mathbb{C}\times\mathbb{C}$ of (5) and (6).

We shall now analyse the connections between BIBO stabilizability of a transfer function $W(z_1,z_2)$ and internal stabilizability of its realizations, namely of 2D systems which satisfy

$$C(I-A_1z_1-A_2z_2)^{-1}(B_1z_1+B_2z_2) = W(z_1,z_2)$$

Let K and R denote the subsets of $\mathbb{C}\times\mathbb{C}$ where (5) and (6) are not full rank. When K(R) is a finite set, the polynomial matrices $I-A_1z_1-A_2z_2$, $B_1z_1+B_2z_2$ ($I-A_1z_1-A_2z_2$, C) are left (right) factor coprime. For a proof of this fact, see |7|. A preliminary result in this framework is given by the following Lemma:

Lemma 1 [7] *Let (2) be a realization of the transfer function (1), where n and d are factor coprime. Then* $S \subseteq K\cup R$, $d|\det(I-A_1z_1-A_2z_2)$ *and the following facts are equivalent:*

i) $d = \det(I-A_1z_1-A_2z_2)$

ii) $S = K\cup R$

iii) $I-A_1z_1-A_2z_2$, $B_1z_1+B_2z_2$ *are left factor coprime and* $I-A_1z_1-A_2z_2$, C *are right factor coprime.*

For sake of conciseness, it seems convenient to discuss separately the two different situations that arise in the stabilization problem

1. $S\cap\bar{P} = \phi$

In this case, the realizations that satisfy

$$d = \det(I-A_1z_1-A_2z_2)$$

are stabilizable both by state and output feedback. In fact, because of Lemma 1, the sets K and R do not intersect the unit polydisc, so that Theorems 1 and 2 apply.
When we consider realizations where d is a proper factor of $\det(I-A_1z_1-A_2z_2)$, i.e.

$$\det(I-A_1z_1-A_2z_2) = dh \quad , \quad h \neq \text{const}$$

we have

$$K\cup R = S\cup V(h)$$

Consequently output feedback stabilization is possible if and only if $V(h)\cap\bar{P} = \phi$.
If $V(h)\cap\bar{P}\neq\phi$, stabilization can only be achieved by state feedback, provided that $V(h)\cap\bar{P}$ does not intersect K.

2. $S\cap\bar{P} \neq \phi$

Since (5) and/or (6) are not full rank in $\bar{P}$, there are no realizations of $W(z_1,z_2)$ which are stabilizable by output feedback. However those realizations which satisfy

i) $S\cap\bar{P}\cap K = \phi$

ii) $V(h)\cap\bar{P}\cap K = \phi$

are stabilizable by state feedback.
It is interesting to notice that since transfer functions having Goodman's property have non essential singularities of the second kind in T, they satisfy $S\cap\bar{P}\neq\phi$. Whether these transfer functions are BIBO stable or not, every realization is internally unstable and cannot be stabilized by output feedback.

If we look at the same kind of problems in the case of 1D systems, we easily see that some important differences arise due to the fact that 1D transfer functions do not exhibit non essential singularities of the second kind.
Actually any 1D transfer function admits realizations which are stabilizable by dynamic output feedback: it is enough to consider minimal realizations and proceed to complete pole alloca-

tion via output dynamic compensation.

On the other side, consider a transfer function $W(z_1,z_2)$ and let (ξ_1,ξ_2) be in S. Then, for every realization of W at least one of matrices (5) and (6) is not full rank in (ξ_1,ξ_2).
Consequently, the characteristic polynomial of any system obtained by output feedback compensation vanishes in (ξ_1,ξ_2). For this, denote by $\bar{A}_1$ and $\bar{A}_2$ the state matrices of the overall system and assume that (5) is not full rank in (ξ_1,ξ_2). Then there exists a non zero vector v such that

$$v^T[I-A_1\xi_1-A_2\xi_2 \mid B_1\xi_1+B_2\xi_2] = 0$$

and

$$[v^T \mid 0]\ [I-\bar{A}_1\xi_1-\bar{A}_2\xi_2]$$
$$= [v^T \mid 0]\begin{bmatrix} I-(A_1+B_1JC_1)\xi_1 & -B_1H\xi_1-B_2H\xi_2 \\ -G_1C\xi_1-G_2C\xi_2 & I-F_1\xi_1-F_2\xi_2 \end{bmatrix} = 0$$

Similar reasonings apply when (6) is not full rank in (ξ_1,ξ_2).
So, no output feedback stabilization applies to the realizations of transfer functions satisfying $S \cap \bar{P} \neq \phi$. However, state feedback stabilization is feasible when we deal with realizations that satisfy $K \cap P = \phi$.
As shown in [7], it is always possible to construct realizations having this property.

3. CONCLUDING REMARKS

The synthesis of state feedback compensators is based on the solution of the following 2D Bézout matrix equation

$$(B_1z_1+B_2z_2)N(z_1,z_2)+(I-A_1z_1-A_2z_2)M(z_1,z_2) = D(z_1,z_2) \qquad (9)$$

A necessary condition for (9) to be salvable is det $D \neq 0$ in K and stabilization is possible if det $D \neq 0$ in $\bar{P}$. Once M and N have been computed, a state equation of the compensator is obtained by constructing any controllable and detectable realization of NM^{-1} [1].
In particular, assuming D=I in (9), the synthesis procedure leads to dead-beat compensators.

One way to construct an output feedback compensator is to connect an asymptotic observer and a state feedback compensator driven by the state estimates. In this case, we need to solve equation (9) and the following Bézout equation [8]

$$P(z_1,z_2)C+Q(z_1,z_2)(I-A_1z_1-A_2z_2) = E(z_1,z_2) \qquad (10)$$

with det $E \neq 0$ in $\bar{P}$ and det $E = 0$ in R.
Once we have computed $P(z_1,z_2)$ and $Q(z_1,z_2)$, an asymptotic observer is given by any internally stable realization of

$$[Q(B_1z_1+B_2z_2) \mid P]$$

In particular, assuming D=I and E=I in (9) and (10), leads to a dead-beat output feedback compensator.

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