A PROGRESS REPORT ON OUTPUT FEEDBACK STABILIZABILITY OF MULTIVARIABLE 2D SYSTEMS

E. Fornasini

Istituto di Elettrotecnica e di Elettronica - Facoltà di Ingegneria
via Gradenigo, 6/a - PADOVA
Italy

ABSTRACT

The output feedback stabilizability conditions of 2D systems are expressed in terms of structural properties of a pair of commuting linear transformations. An algorithm is given for obtaining a stable closed loop 2D characteristic polynomial.

1. Introduction

A peculiar aspect of the synthesis of a stabilizing 2D compensator is that, in general, even when the plant is given by a factor coprime matrix fraction description ND^{-1}, it is not possible to freely assign the variety of the characteristic closed loop polynomial. In fact this variety is constrained to include the set of points where the minors of maximal order of $[D'\ N']$ vanish simultaneously |1|.

Since this set is not explicitly known, one would like to get rid of its calculation in the synthesis procedure and to assign the characteristic polynomial of the closed loop system in such a way that the above constraints are automatically satisfied.

The idea we pursue in this paper is that of deriving finite linear tests for checking feedback stabilizability without any explicit computation of the set above.

The results we present here, based on a matrix version of the Gröbner basis algorithm |3,5| are not complete and constitute a progress report on the state of the art on the subject.

2. Characteristic polynomials of the closed loop 2D systems

Let $W(z_1,z_2)$ be a strictly proper transfer matrix of dimension pxm and let

$$N(z_1,z_2)D^{-1}(z_1,z_2) = W(z_1,z_2)$$

be a right factor coprime Matrix Function Description (MFD).

Consider the ideal $\mathcal{I}$ generated by the minors of maximal order $m_1(z_1,z_2),\dots,m_\nu(z_1,z_2)$ of the matrix

$$\begin{bmatrix} D(z_1,z_2) \\ N(z_1,z_2) \end{bmatrix} \qquad (1)$$

The coprimeness condition on N and D corresponds to assume that the variety $\mathcal{V}(\mathcal{I})$ is a finite subset of $\mathbb{C}\times\mathbb{C}$ or, equivalently, that the quotient $\mathbb{R}[z_1,z_2]/\mathcal{I}$ is a finite dimensional vector space over $\mathbb{R}$.

Let $\Sigma = (A_1,A_2,B_1,B_2,C)$ be a 2D realization of $W(z_1,z_2)$ |2|, where $(I-A_1z_1-A_2z_2,\ B_1z_1+B_2z_2)$ are left factor coprime and $(I-A_1z_1-A_2z_2, C)$ are right factor coprime. Under these assumptions, we have |1|

$$\det(I-A_1z_1-A_2z_2) = \det D(z_1,z_2)$$

Consider an output feedback compensator represented by a proper MFD

$$W_c(z_1,z_2) = R^{-1}(z_1,z_2)S(z_1,z_2)$$

of dimension m x p and let $\Sigma_c = (F_1,F_2,G_1,G_2,H)$ be a realization of W_c satisfying the relation

$$\det(I-F_1z_1-F_2z_2) = \det R(z_1,z_2)$$

Then the characteristic polynomial Δ of the closed loop system obtained by the output feedback connection of Σ and Σ_c is given by

$$\Delta = \det(RD+SN)\ .$$

Using Binet-Cauchy formula, det(RD+SN) is expressed as the sum of the products of all possible minors of maximal order, q_i, $i=1,2,\dots,\nu$ of $[R\ S]$ into the corresponding minors of the same order m_i, $i=1,2,\dots,\nu$ of $[D'N']$, that is

$$\det(RD+SN) = \sum_{1}^{\nu} q_i m_i$$

Hence det(RD+SN) belongs to the ideal $\mathcal{I}$ for any choice of the compensator.

Conversely, given any polynomial $p\in\mathcal{I}$, there exists a compensator R^{-1} such that |1,3|

$$A = \det(RD+SN) = p$$

This implies that $\mathcal{V}(\Delta)$ is freely assignable, except that it must include $\mathcal{V}(\mathcal{I})$ and does not contain (0,0). We summarize our conclusions in the following

Proposition 1 The System Σ admits a stabilizing compensator if and only if $\mathcal{V}(\mathcal{I})$ does not intersect the closed unit polydisc $\mathcal{P}_1\subset\mathbb{C}\times\mathbb{C}$.

3. Existence of stable closed loop polynomials

The above proposition allows us to establish some criteria for testing closed loop stabilizability of 2D systems.

Let $\mathcal{G}=(g_1,g_2,\dots,g_h)$ denote a Gröbner basis |5| of the ideal $\mathcal{I}$. Then the set $\{p_1=1, p_2,\dots,p_\nu\}$ of monic monomials in z_1 and z_2 that are not multiple of the leading power products of any of the polynomials in $\mathcal{G}$ is finite. The corresponding residue classes modulo $\mathcal{I}$, $\bar p_1,\bar p_2,\dots,\bar p_\nu$, constitute a basis for the vector space $\mathbb{R}[z_1,z_2]/\mathcal{I}$.

Example 1 Consider the transfer function

$$W(z_1,z_2) = \frac{z_1^3 - \frac{5}{2} z_1^2 - \frac{5}{2} z_1}{z_2 + z_1^2 - \frac{3}{2} z_1 - 3} = D^{-1} N$$

The ideal $\mathcal{J}$ is generated by the maximal order minors in (1), namely N and D, and it is easy to check that N and D constitute a Gröbner basis w.r.t. the lexicographical ordering of the power products.

The only monomials that are not multiple of the leading power products z_1^3 and z_2 of N and D are

$$p_1 = 1, \quad p_2 = z_1, \quad p_3 = z_1^2$$

Hence $\{p_i + \mathcal{J} \overset{\Delta}{=} \bar{p}_i,\ i = 1,2,3\}$ is a basis of $K[z_1,z_2]/\mathcal{J}$.

Consider now the following maps

$$\mathcal{X}_1 : \mathbb{R}[z_1,z_2]/\mathcal{J} \to \mathbb{R}[z_1,z_2]/\mathcal{J} \ : q + \mathcal{J} \mapsto z_1 q + \mathcal{J}$$

$$\mathcal{X}_2 : \mathbb{R}[z_1,z_2]/\mathcal{J} \to \mathbb{R}[z_1,z_2]/\mathcal{J} \ : q + \mathcal{J} \mapsto z_2 q + \mathcal{J}$$

They are both well defined, commuting linear transformations on the $\mathbb{R}$-vector space $R[z_1,z_2]/\mathcal{J}$. This implies that when $\mathbb{R}[z_1,z_2]/\mathcal{J}$ is represented onto $\mathbb{R}^\nu$, $\mathcal{X}_1$ and $\mathcal{X}_2$ are represented by a pair of commuting matrices M_1 and M_2 in $\mathbb{R}^{\nu \times \nu}$.

Note that the smallest $\mathcal{X}_1$ and $\mathcal{X}_2$ - invariant subspace generated by $\bar{p}_1 = \bar{1}$ is the whole space $\mathbb{R}[z_1,z_2]/\mathcal{J}$ Thus $\{M_1^i M_2^j\, e_1,\ i,j \in \mathbb{Z}\}$ is a set of generators for the space $\mathbb{R}^\nu$.

The construction of M_1 and M_2 is performed along the following lines:

i) associate with $\bar{p}_1, \bar{p}_2, \dots, \bar{p}_\nu$ the standard basis vectors in R^ν

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots\dots \quad e_\nu = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

ii) for each $\bar{p}_j$, represent $\overline{z_1 p_j}$ as a linear combination of $\bar{p}_1, \bar{p}_2, \dots, \bar{p}_\nu$. This requires a strightforward application of the so called "normal form algorithm" |5| w.r. to the Gröbner basis $(g_1, g_2, \dots, g_h)$.

iii) the coefficients m'_{ij} in

$$\overline{z_1 p_j} = \sum_j m'_{ij} \bar{p}_i \qquad i = 1,2,\dots,\nu$$

are the entries of the matrix M_1 we are looking for

$$M_1 = [m'_{ij}]$$

iv) and v) refer to the representation of $\overline{z_2 p_j}$ for obtaining the columns of M_2, and are analogous to ii) and iii).

Example 2 Consider again the ideal of the Example 1 and associate e_1 with $\bar{1}$, e_2 with $\bar{z}_1$ and e_3 with $\bar{z}_1^2$. Clearly

$$\mathcal{X}_1 \bar{1} = \bar{z}_1, \qquad \mathcal{X}_1 \bar{z}_1 = \bar{z}_1^2$$

Moreover

$$\begin{aligned} z_1^3 &= (z_1^3 - \frac{5}{2} z_1^2 - \frac{5}{2} z_1) + (\frac{5}{2} z_1^2 + \frac{5}{2} z_1) \\ &= \frac{5}{2} z_1^2 + \frac{5}{2} z_1 \qquad \pmod{\mathcal{J}} \end{aligned}$$

so that

$$\mathcal{X}_1 \bar{z}_1^2 = \frac{5}{2} \bar{z}_1^2 + \frac{5}{2} \bar{z}_1$$

Hence we have

$$M_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 5/2 \\ 0 & 1 & 5/2 \end{bmatrix}$$

The computation of M_2 is a little bit more involved. Note that

$$\begin{aligned} z_2 &= (z_2 + z_1^2 - \frac{3}{2} z_1 - 3) + (-z_1^2 + \frac{3}{2} z_1 + 3) \\ &= -z_1^2 + \frac{3}{2} z_1 + 3 \qquad \pmod{\mathcal{J}} \end{aligned}$$

$$\begin{aligned} z_1 z_2 &= (z_1 z_2 + z_1^3 - \frac{3}{2} z_1^2 - 3 z_1) + (-z_1^3 + \frac{3}{2} z_1^2 + 3 z_1) \\ &= -z_1^3 + \frac{3}{2} z_1^2 + 3 z_1 \qquad \pmod{\mathcal{J}} \\ &= (-z_1^3 + \frac{5}{2} z_1^2 + \frac{5}{2} z_1) + (-z_1^2 + \frac{1}{2} z_1) \\ &= -z_1^2 + \frac{1}{2} z_1 \qquad \pmod{\mathcal{J}} \end{aligned}$$

$$\begin{aligned} z_1^2 z_2 &= (z_1^2 z_2 + z_1^4 - \frac{3}{2} z_1^3 - 3 z_1^2) + (-z_1^4 + \frac{3}{2} z_1^3 + 3 z_1^2) \\ &= -z_1^4 + \frac{3}{2} z_1^3 + 3 z_1^2 \qquad \pmod{\mathcal{J}} \\ &= (-z_1^4 + \frac{5}{2} z_1^3 + \frac{5}{2} z_1^2) + (-z_1^3 + \frac{1}{2} z_1^2) \\ &= -z_1^3 + \frac{1}{2} z_1^2 \qquad \pmod{\mathcal{J}} \\ &= (-z_1^3 + \frac{5}{2} z_1^2 + \frac{5}{2} z_1) + (-2 z_1^2 - \frac{5}{2} z_1) \\ &= -2 z_1^2 - \frac{5}{2} z_1 \qquad \pmod{\mathcal{J}} \end{aligned}$$

This implies

$$\mathcal{X}_2 \bar{1} = -\bar{z}_1^2 - \frac{3}{2} \bar{z}_1 + 3 \cdot \bar{1}$$

$$\mathcal{X}_2 \bar{z}_1 = -\bar{z}_1^2 + \frac{1}{2} \bar{z}_1$$

$$\mathcal{X}_2 \bar{z}_1^2 = -2 \bar{z}_1^2 - \frac{5}{2} \bar{z}_1$$

and finally

$$M_2 = \begin{bmatrix} 3 & 0 & 0 \\ \frac{3}{2} & \frac{1}{2} & -\frac{5}{2} \\ -1 & -1 & -2 \end{bmatrix}$$

It is easy to check that M_1 and M_2 commute.

Several properties pf the ideal $\mathcal{J}$ and of the (finite) variety $\mathcal{V}(\mathcal{J})$ are strictly related to the structure of the commuting matrices M_1 and M_2 introduced above.

Property 1. <u>A polynomial $q \in \mathbb{R}[z_1,z_2]$ belongs to the ideal $\mathcal{J}$ if and only if $q(M_1,M_2) = 0$</u>

proof. Let $q(z_1,z_2) = \sum_{ij} q_{ij}\, z_1^i z_2^j \in \mathcal{J}$. This implies

$$0 = \sum_{ij} q_{ij} \bar{z}_1^i \bar{z}_2^j = \sum_{ij} q_{ij} \mathscr{Z}_1^i \mathscr{Z}_2^i \bar{1} \qquad (3)$$

and equivalently

$$0 = \sum_{ij} q_{ij} M_1^i M_2^j e_1 \qquad (4)$$

Multiplying (4) on the left by $M_1^r M_2^s$ and recalling the matrix commutativity we have

$$0 = (\sum q_{ij} M_1^i M_2^j)(M_1^r M_2^s e_1) \qquad r,s = 0,1,2,\dots,$$

This proves that $q(M_1,M_2) = 0$

The viceversa is easily obtained by following backword the lines of the proof above.

Corollary <u>Let $\psi_i(\xi)$, $i = 1,2$, denote the minimum polynomial of M_i. Then $\psi_i(z)$ is the minimum degree polynomial in $R|z_i| \cap \mathscr{I}$.</u>

A classical result |6| in the theory of commutative matrices is the existence of common eigenvector for M_1 and M_2. Property 2 clarifies how the pairs of eigenvalues that correspond to common eigenvectors are related to the structure of the variety $\mathscr{V}(\mathscr{I})$.

Property 2. <u>Let $(\alpha_1,\alpha_2) \in \mathbb{C} \times \mathbb{C}$. Then $(\alpha_1,\alpha_2) \in \mathscr{V}(\mathscr{I})$ if and only if M_1 and M_2 have a common eigenvector v and</u>

$$M_1 v = \alpha_1 v \qquad M_2 v = \alpha_2 v \qquad (5)$$

proof. Assume that (5) holds and consider any polynomial $q(z_1,z_2) = \sum q_{ij} z_1^i z_2^j$ in $\mathscr{I}$. By property 1

$$0 = q(M_1,M_2) = \sum q_{ij} M_1^i M_2^j$$

$$0 = \sum q_{ij} M_1^i M_2^j v = \sum q_{ij} \alpha_1^i \alpha_2^j v$$

$$0 = \sum q_{ij} \alpha_1^i \alpha_2^j = q(\alpha_1,\alpha_2)$$

Since $q(z_1,z_2)$ is arbitrary in $\mathscr{I}$, $(\alpha_1,\alpha_2) \in \mathscr{V}(\mathscr{I})$. Vice - versa, assume that (α_1,α_2) belongs to $\mathscr{V}(\mathscr{I})$ and denote by k_1 and k_2 the algebraic multiplicities of $z_1-\alpha_1$ and $z_2-\alpha_2$ in ψ_1 and ψ_2 respectively

$$\psi_1(z_1) = h_1(z_1)(z_1-\alpha_1)^{k_1}, \qquad h_1(\alpha_2) \neq 0$$

$$\psi_2(z_2) = h_2(z_2)(z_2-\alpha_2)^{k_2}, \qquad h_2(\alpha_2) \neq 0$$

Note that $h_1(z_1)h_2(z_2) \notin \mathscr{I}$, since $h_1(\alpha_1)h_2(\alpha_2) \neq 0$. Let t, $0 \leq t < k_1$, be the largest integer such that

$$h_1(z_1)h_2(z_2)(z_1-\alpha_1)^t \notin \mathscr{I}$$

and let r, $0 \leq r < \nu$, be the largest integer such that

$$s(z_1,z_2) \triangleq h_1(z_1)h_2(z_2)(z_1-\alpha_1)^t (z_2-\alpha_2)^r \notin \mathscr{I}$$

We therefore have that

$$s(z_1,z_2) \notin \mathscr{I} \qquad (6)$$

$$s(z_1,z_2)(z_1-\alpha_2) \in \mathscr{I} \qquad (7)$$

$$s(z_1,z_2)(z_2-\alpha_2) \in \mathscr{I} \qquad (8)$$

Hence

$$v \triangleq s(M_1,M_2)e_1 \neq 0 \qquad (9)$$

and

$$(M_1-\alpha_1 I)v = 0$$

$$(M_2-\alpha_2 I)v = 0$$

The last two equations show that the vector v defined in (9) is a common eigenvector.

A different characterization of the variety $\mathscr{V}(\mathscr{I})$ is based on the Frobenius theorem |7| on simultaneous triangularization of commutative matrices.

Theorem |FROBENIUS| <u>Let M_1 and M_2 be a pair of real commutative matrices. Then M_1 and M_2 can be simultaneously reduced to triangular form over C by a similarity transformation.</u>

Property 3. <u>Let $T_1 = |t_{ij}^1|$ and $T_2 = |t_{ij}^2|$ be a pair of common triangular forms of the matrices M_1 and M_2. Then (α_1,α_2) in $\mathbb{C} \times \mathbb{C}$ belongs to $\mathscr{V}(\mathscr{I})$ if and only if there exists an integer i such that</u>

$$t_{ii}^1 = \alpha_1, \qquad t_{ii}^2 = \alpha_2$$

proof. Since T_1 and M_1 as well as T_2 and M_2 are connected by a common similarity transformation, property 1 holds for matrices T_1 and T_2 too. Therefore, $q(z_1,z_2)$ belongs to $\mathscr{I}$ if and only if $q(T_1,T_2) = 0$.

Let $q(z_1,z_2) \in \mathscr{I}$. Then

$$0 = q(T_1,T_2) = \begin{bmatrix} q(t_{11}^1,t_{11}^2) & & * \\ & q(t_{22}^1,t_{22}^2) & \\ 0 & & q(t_{\nu\nu}^1,t_{\nu\nu}^2) \end{bmatrix}$$

Since $q(z_1,z_2)$ is arbitrary in $\mathscr{I}$, $q(t_{ii}^1,t_{ii}^2) = 0$ implies $(t_{ii}^1,t_{ii}^2) \in \mathscr{V}(\mathscr{I})$.

Viceversa, let $(\alpha_1,\alpha_2) \in \mathscr{V}(\mathscr{I})$ and suppose, by contradiction,

$$(\alpha_1,\alpha_2) \neq (t_{ii}^1,t_{ii}^2) \qquad i = 1,2,\dots,\nu$$

Then there exists a polynomial $q(z_1,z_2)$ vanishing in (t_{ii}^1,t_{ii}^2), $i = 1,2,\dots,\nu$, and different from zero in (α_1,α_2). We therefore have

$$q(T_1,T_2) = \begin{bmatrix} 0 & & * \\ & 0 & \\ & & \ddots & \\ & & & 0 \end{bmatrix}$$

so that $q^\nu(T_1,T_2) = 0$ and $q^\nu(z_1,z_2) \in \mathscr{I}$. Since $q^\nu(\alpha_1,\alpha_2)$ is different from zero, $(\alpha_1,\alpha_2) \in \mathscr{V}(\mathscr{I})$, contrary to the assumption.

The condition for output feedback stabilizability, given in Proposition 1, can be restated in terms of structural properties of the commutative matrices M_1 and M_2. The following proposition is a strightforward consequence of Properties 2 and 3 above

Proposition 2. <u>The following facts are equivalent</u>

i) <u>Σ is output feedback stabilizable</u>

ii) any common eigenvector of M_1 and M_2 refers to a pair of eigenvalues (α_1,α_2) such that $|\alpha_1| > 1$ and/or $|\alpha_2| > 1$

iii) any pair (t^1_{ii}, t^2_{ii}) in the triangular form of M_1 and M_2 satisfies $|t^1_{ii}| > 1$ and/or $t^2_{ii} > 1$.

Remark The proposition above does not provide an efficient algorithm for testing output feedback stabilizability of Σ. In fact, simultaneous triangularization of M_1 and M_2 as well as the computation of common eigenvectors cannot be performed in a finite number of steps. However properties 2 and 3 have a theoretical intrinsic interest, in the sense that they could constitute a good starting point for obtaining linear stabilizability criteria in the style of Lyapunov equation.

In some particular cases, stabilizability conditions are easy to check. For instance, all the eigenvalues of M_1 have modulus greater than one if and only if there exists a negative definite matrix P satisfying the linear matrix equation

$$M_1^T P M_1 - P = -Q \qquad \text{(Q positive definite)}$$

Since $\psi_1(z_1)$ belongs to $\mathcal{J}$ and 1 is devoid of zeros in the closed unit disk $\{z_1 : |z_1| \le 1\}$, there exists a stabilizing compensator such that the closed loop 2D characteristic polynomial of the system is given by $\psi_1(z_1)$.

Analogous considerations hold if all the eigenvalues of M_2 have modulus greater than one.

A more general situation comes out when some product $M_1^i M_2^j$ is devoid of eigenvalues in the closed unit disk. This happens if and only if the equation

$$(M_1^i)^T (M_2^j)^T P M_2^j M_1^i - P = Q \qquad \text{(Q positive definite)} \quad (10)$$

admits a negative definite solution P.

Referring to triangular forms, it is easy to convince ourselves that condition (iii) in Proposition 2 is satisfied.

In this case the minimum polynomial $\psi_{ij}(\xi)$ of $M_1^i M_2^j$ is devoid of zeros in the closed unit disk, and the variety of

$$\psi_{ij}(z_1^i z_2^j) = (z_1^i z_2^j - \gamma_1)(z_1^i z_2^j - \gamma_2)\dots(z_1^i z_2^j - \gamma_t)$$

does not intersect $\mathcal{P}_1$, since $|\gamma_i| > 1 \quad i = 1,2,\dots,\nu$.

4. Stabilizing compensator design

As remarked at the end of section 3, in some cases it is possible to construct directly a 2D stable polynomial in $\mathcal{J}$: whenever equation (10) admits a negative definite solution P, the minimum polynomial of $M_1^i M_2^j$ can be used. Suppose, however, we only have a criterion for deciding whether $\mathcal{J}$ includes a stable polynomial, which does not provide any constructive technique. In this case an iterative procedure for obtaining a stable closed loop polynomial is based on the following propositions |8|

Proposition 3. For each positive integer k, there exists in $\mathcal{J}$ a monic polynomial $\psi_k(z_1^k+z_2^k)$ of minimal degree in $(z_1^k+z_2^k)$. The set $\mathbb{R}[z_1^k+z_2^k] \cap \mathcal{J}$ contains stable 2D polynomials if and only if $\psi_k(z_1^k+z_2^k)$ is 2D stable.

proof. Consider the matrix $M_1^k+M_2^k$ and denote by $\psi_k(\xi)$ its minimum polynomial. Then $\psi_k(z_1^k+z_2^k) \in \mathcal{J}$, since $\psi_k(M_1^k+M_2^k) = 0$ and minimality follows from the definition of ψ_k.

Since each polynomial in $\mathbb{R}[z_1^k+z_2^k] \cap \mathcal{J}$ is a multiple of ψ_k, then $\mathbb{R}[z_1^k+z_2^k] \cap \mathcal{J}$ contains 2D stable polynomials if and only if ψ_k is stable.

Proposition 4. Assume $\mathcal{V}(\mathcal{J}) \cap \mathcal{P}_1 = \emptyset$. Then there exist an integer k and a 2D stable polynomial in $\mathbb{R}[z_1^k+z_2^k] \cap \mathcal{J}$.

proof. Referring to commutative matrices in triangular form, any (complex) pair (t^1_{ii}, t^2_{ii}) satisfies $|t^1_{ii}| > 1$ and/or $|t^2_{ii}| > 1$. It is not difficult to show that there exists an integer h such that $|(t^1_{ii})^h + (t^2_{ii})^h| > 2$, for $i=1,2,..\nu$. Let $\psi_h(\xi)$ denote the minimum polynomial of $M_1^h+M_2^h$. The polynomial $\psi_h(z_1^h+z_2^h)$ belongs to $\mathcal{J}$ by proposition 2 and factorizes as

$$\psi_h(z_1^h+z_2^h) = (z_1^h+z_2^h - \delta_1)(z_1^h+z_2^h - \delta_2)\dots(z_1^h+z_2^h - \delta_t)$$

Since $\psi_h(\xi)$ is devoid of zeros in the disk $\{\xi : |\xi| \le 2\}$, it follows that $|\delta_i| > 2$, $i = 1,2,\dots,t$, which in turn implies 2D stability of all factors $z_1^h+z_2^h - \delta_i$.

The proof of Proposition 4 immediately suggests an algorithm for computing a 2D stable polynomial in $\mathcal{J}$.

1. Consider the sequence of matrices

$$S_1 = \frac{M_1+M_2}{2} \qquad S_2 = \frac{M_1^2+M_2^2}{2}, \qquad S_3 = \frac{M_1^3+M_2^3}{2}, \dots$$

and solve the matrix equations

$$S_i^T P_i S_i - P_i = -I \qquad i = 1,2,3,\dots$$

until a negative definite S_i is found. By Proposition 4, this procedure stops after a finite number of steps if and only if the system is stabilizable. Let h be the first integer such that S_h in negative definite.

2. Construct the minimum polynomial $\psi_h(\xi)$ of $M_1^h+M_2^h$. Then $\psi_h(z_1^h+z_2^h)$ is a stable 2D polynomial in $\mathcal{J}$.

References

1. M.Bisiacco, E.Fornasini, G.Marchesini: "Controller design for 2D systems", in "Frequency Domain and State Space Methods for Linear Systems", pp. 99-113, Elsevier (1986).

2. E.Fornasini, G.Marchesini: "Doubly indexed dynamical systems: state space models and structural properties", Math. Systems Theory, 12, 1, pp. 59-72, (1978).

3. J.P. Guiver and N.K. Bose: "Causal and weakly causal 2D filters with applications in stabilization", in "Multidimensional Systems Theory", pp. 52-100, Reidel (1985).

4. M.Bisiacco, E.Fornasini, G.Marchesini: "On some connections between BIBO and internal stability of two-dimensional filters", IEEE Trans., CAS-32, n. 9, pp. 948-53, (1985).

5. B.Buchberger: "Gröbner bases: an algorithmic method in polynomial ideal theory", in "Multidimensional Systems Theory", pp. 184-232, Reidel (1985).

6. M.P. Drazin, J.W. Dungey, K.W. Grüenberg: "Some theorems on commutative matrices", J.London Math.Soc., vol. 26, pp. 221-228 (1951).

7. D.A. Suprunenko, R.I. Tyshkevich: "Commutative matrices", Acad. Press (1968).

8. M.Bisiacco, E.Fornasini, G.Marchesini: "Causal 2D compensators: stabilization algorithms for multivariable 2D systems", 25th CDC, Athens (1986).